A DEGREE PHYSICS

Part 1

The General Properties of Matter

BY

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This book is dedicated with filial gratitude to the memory of my Father, the late Mr. Harry Smith, of Malvern, who by his help in the early days and by making many sacrifices enabled me to have an academic career.

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PREFACE

In presenting to my readers this second edition of the first volume of a series of text books to be published under the title 'A Degree Physics', the opportunity has been taken of removing much material, now considered out-of-date, and giving a full account of modern work in connexion with gravitation. Also, in dealing with the motion of rigid bodies and with oscillations more use has been made of the principle of the conservation of energy. In the chapter on surface tension new methods for the accurate determination of this quantity have been discussed and some fundamental formulae derived from the energy principle. The introduction of some work in hydraulics has made it desirable to replace the old chapter on viscosity by two; one on viscosity, which has been thoroughly revised, and the other on hydraulics and non-Newtonian liquids. In this last chapter emphasis has been laid on the use of a Pitot tube.

The ground covered is just a little more than that usually required by students reading physics for a B.Sc. General Degree. In fact, the book is written to meet the needs of those who wish to gain a thorough insight into the fundamentals of physics and, as such, it should be found to cover most of the work in this branch of physics which is normally dealt with in a first year special honours course.

This first volume deals with the fundamental parts of an ill-defined group of subjects usually met with under the title of 'General Physics', but some of the very elementary parts, including the theory of dimensions, are omitted, since they are discussed in my *Intermediate Physics*. In the present text this is referred to as I.P.

As regards the spelling of words, I have tried to follow the *Oxford Dictionary* and favour the spelling that is correct etymologically. On coming to the choice of symbols, it is at once found that there are not sufficient convenient symbols to enable one to be kept for each quantity that appears; lists of symbols which are recommended from time to time seem to vary with the decade in which they are published and, primarily, they are for use in connexion with papers for publication in scientific journals. In the teaching world it is found that many more symbols are required and while one particular symbol may be required for a certain purpose in a journal, the same symbol is, in the classroom, often much better used to denote an entirely different quantity. The thing that really matters is that whenever a symbol is used, it should be defined explicitly,

and then the choice of the symbol must be left to the writer, be he an author, an examiner, or a candidate in an examination hall.

In particular, while dealing with the subject of notation, I must point out that after long consideration I have decided to use γ for the gravitational constant in preference to G, a practice generally acceptable to geophysicists and mathematicians but not always to physicists. Such a procedure leaves $\vec{G}$ to define the strength of a gravitational field, just as $\vec{E}$ and $\vec{H}$ are used in connexion with electric and magnetic fields. Also, gravitational potential is defined in terms of work done against the field; this agrees with the methods used in electrostatics and in magnetism, and there seems no fundamental reason why the practice should be reversed in gravitational field theory.

Throughout this series of books I have not hesitated to use vectors and for this reason I have introduced them to the reader in a mathematical introduction. I have, however, used a notation which can be written as well as printed.

In dealing with the various parts of the subject I have endeavoured to retain an historical background; the facts stated therein may not be considered of much immediate value, yet they will render apparent some of the difficulties which the early workers in physics had to surmount. Coming to modern work I have not always given an account of the very latest methods, since a knowledge of them is often not essential for understanding basic principles and an acquaintance with modern work is perhaps best left until the reader has reached a more mature stage. The choice of the actual matter that appears is somewhat regretfully fixed in part by the examination syllabuses of the different universities but in several instances I have dealt very fully with those parts of the subject which are of especial interest today. No two people will ever agree in detail as to the actual topics which should be discussed in a book of this nature and I can only hope that the selection now found will be approved by many.

At the end of each chapter a set of questions, mainly numerical, has been appended. Some of these have been set at University of London examinations and I do appreciate the ready permission given to me by the Senate to publish them. These particular questions are denoted by the letters G and S to indicate those from papers for general and special honours respectively. I regard the solving of numerical riders by students as an essential part of any physics course and in order to help them solutions are given. From bitter experience I know only too well how errors creep in, but in this instance I almost dare to hope that a few such errors do remain for then students who endeavour to obtain solutions merely by

substitution in a formula will be often frustrated. By working through many such examples and doing each one quite honestly, it is hoped that a student will acquire an attitude of mind which says 'If this solution isn't correct, none is.'

The work has occupied me for many years; the process of writing the book has been to me most interesting and while the labour has been much greater than anything I could have anticipated, yet if the result gives satisfaction to my readers I shall be more than content.

Finally I have to thank Prof. S. Tolansky, F.R.S., for certain facilities which he has placed at my disposal in connexion with the writing of this book and for friendly advice so freely given. Also, I wish to place on record my appreciation of the help I have received from both past and present colleagues at the Royal Holloway College and from co-examiners. In particular I must mention the late Prof. Ferguson, Dr. R. C. Brown, Dr. A. R. Stokes, Miss M. Pick, M.A., Dr. Mary Bradburn, J. W. Reed Esq., M.Sc., Prof. T. S. Moore, Prof. H. T. Flint, J. Pryce-Jones Esq., Dr. S. P. F. Humphreys-Owen and Dr. R. Mansfield. I should also like to place on record the help I have received from Dr. E. J. Irons and acknowledge the permission granted by Dr. C. A. Haywood to describe his experiment for determining the centre of percussion of a bar pendulum, and also that of Prof. Leo L. Baggerly who supplied me with a correct theory of the oscillations of a physical balance; Dr. W. Bullerwell helped me very considerably in connexion with gravity surveys and Dr. A. H. Cook supplied much detailed information concerning his method for determining gravity in absolute measure. C. Salter Esq., M.A., readily gave details concerning Pitot tubes. My best thanks are due to my wife, who has checked the proofs, and removed many obscurities and errors. Without her diligent aid the publication of the book would have been well-nigh impossible. For any errors of omission, or of commission, which may remain, I alone must be responsible but I hope it is not too much to ask my readers to draw my attention to any which may be found. In this connexion, amongst the readers of the first edition of this book, I must mention Wm. R. Torrance Esq., M.A. and M. W. McElhinny Esq.; their friendly criticisms have been most helpful.

C. J. Smith.

Royal Holloway College,
Englefield Green, Surrey.
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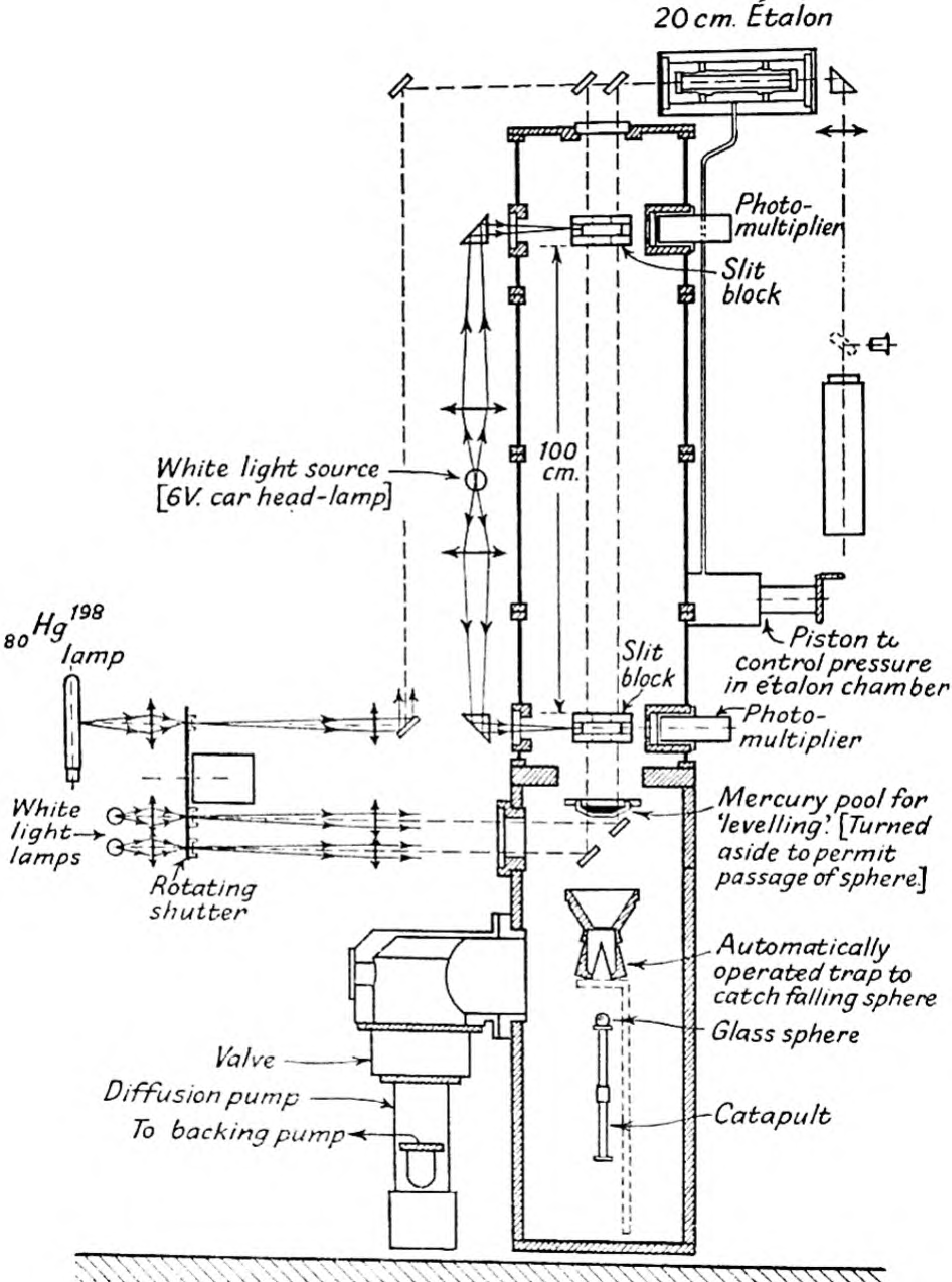


N.P.L. apparatus for an absolute determination of gravity (g).
[After Dr. A. H. Cook and Miss H. M. Richardson.]

PART I

THE GENERAL PROPERTIES OF MATTER

CHAPTER I

MATHEMATICAL INTRODUCTION

Scalars and vectors.—In all branches of mathematical physics a distinction has to be made between two kinds of quantity usually designated ***scalars*** and ***vectors***. Those magnitudes which have no directional properties, so that each one may be represented completely by a number, are termed ***scalars***; other magnitudes, having size in the accepted algebraic meaning of the word and, in addition, direction in space, are known as ***vectors***; the numerical value of a vector is described as its magnitude. A vector may be represented by the segment of a straight line whose length, on a definite scale, is equal to the magnitude of the vector and whose direction and sense (considered as drawn from one end to the other) are the same as those of the vector. The beginning of this representative straight line is known as the ***origin***; the other extremity is the ***terminus***. When necessary to distinguish between a vector and its magnitude, an essentially positive scalar, the latter will be represented by a letter—thus, A—while the vector is represented by $\vec{A}$. Sometimes, in order to avoid confusion, the magnitude of a vector will be denoted by $|A|$. This notation is particularly useful in such instances as the following. $|A|\ |B|$ is the product of the magnitudes of the two vectors $\vec{A}$ and $\vec{B}$. Very frequently a vector is represented in the literature of the subject by **A**, but it is difficult to do this in manuscript.

Thus, the displacement of a particle from one position to another is a vector, which is represented by a segment of a straight line drawn and directed from the initial position to the final position of the particle. Other examples of vectors are velocity, acceleration, force, strength of a magnetic field, etc. Mass (in classical mechanics—the mass of a moving electron is directional since its 'transversal' mass is different from its 'longitudinal' mass) work,

energy and potential (whether electric, magnetic or gravitational are all examples of scalars.

The geometrical space with which we deal will be referred to a system of rectangular axes with the right-handed screw convention, i.e. if a right-handed screw is being driven along the positive direction of the z-axis, the rotation is from the x-axis to the y-axis through the right angle between those axes.

Elementary vector operations.—(*a*) ***Addition and subtraction of vectors.***—Let $\vec{A}$ and $\vec{B}$ be two vectors. Let $\vec{A}$ be represented by the straight line PQ, Fig. 1·01(*a*). Then, if at the end of the line

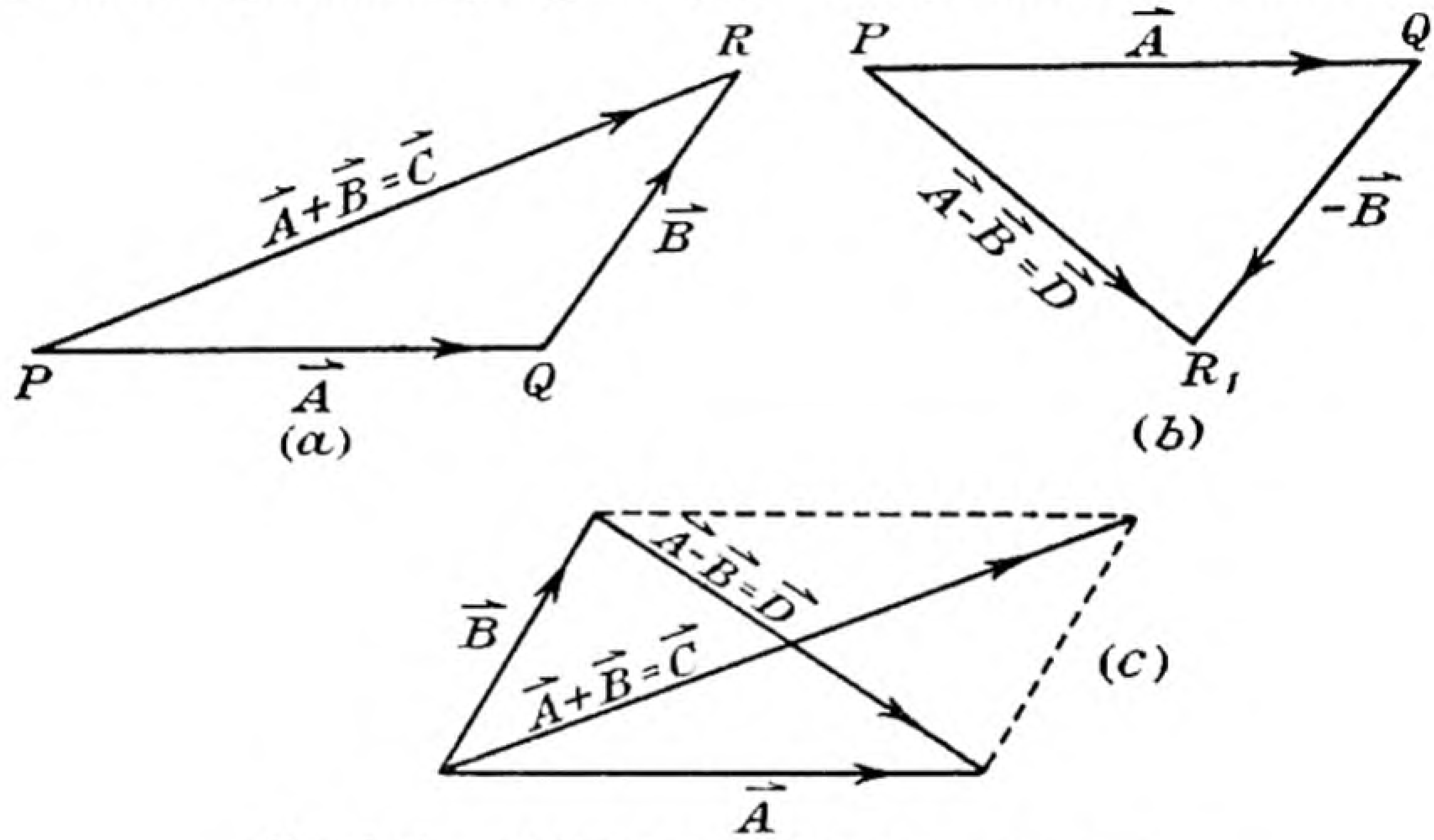


FIG. 1·01.—Addition and subtraction of vectors.

representing $\vec{A}$ we draw a straight line QR to represent $\vec{B}$, the vector represented by the straight line obtained by joining the origin of $\vec{A}$ to the terminus of $\vec{B}$, i.e. the straight line PR, is called the vector sum of the vectors $\vec{A}$ and $\vec{B}$. If this vector sum is denoted by $\vec{C}$, we have $\vec{A}+\vec{B}=\vec{C}$.

To obtain the vector difference between a pair of vectors $\vec{A}$ and $\vec{B}$, PQ, Fig. 1·01(*b*), is drawn to represent $\vec{A}$ and then at the terminus of this line QR_1 is drawn equal in magnitude to that of $\vec{B}$ but opposite in sense. The vector $\vec{D}$ represented by PR_1 is termed the vector difference of $\vec{A}$ and $\vec{B}$. Thus

$$\vec{A}-\vec{B}=\vec{D}.$$

From the above one concludes at once, that if two vectors $\vec{A}$ and $\vec{B}$, are represented by the two adjacent sides of a parallelogram,

then the diagonal drawn through the point of intersection of those two sides, cf. Fig. 1·01(*c*), represents the vector sum of the two vectors, whereas the other diagonal represents the vector difference of the two vectors.

The process of adding or of subtracting vectors may be extended from two to three or more vectors. The reason for this is that the

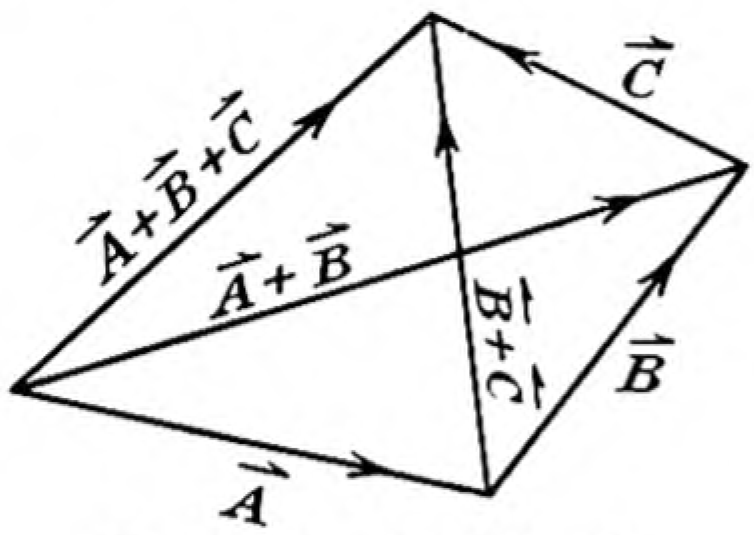


FIG. 1·02.—Addition of several vectors.

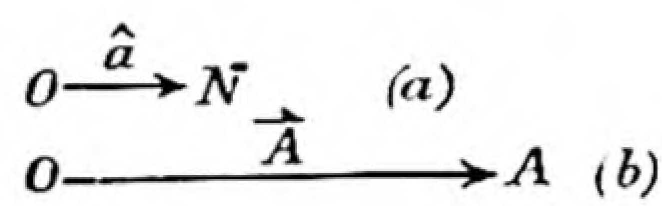


FIG. 1·03.—A unit vector.

sum or difference of two vectors is also a vector. From Fig. 1·02 it is seen that by arranging three vectors in a chain

$$\vec{A} + \vec{B} + \vec{C} = (\vec{A} + \vec{B}) + \vec{C} = \vec{A} + (\vec{B} + \vec{C}),$$

and any number of vectors may be treated in this manner.

Theorem I. Neither the grouping nor the order of the addends affects the sum of any number of vectors, i.e. the addition of vectors is commutative and associative.

The above theorem really asserts that the fundamental laws of ordinary algebraic summation may be applied without reservation to the summation of vectors

Definition of a unit vector.—A unit vector is one which is completely represented by a straight line, one unit in length and orientated in some definite direction, and sense, in space. Such a vector is represented by the symbol $\hat{a}$ or $\vec{a}$.

Derivation of a vector from a unit vector.—Starting with some unit vector $\hat{a}$, represented by $\overrightarrow{ON}$, Fig. 1·03(*a*), its length can be increased A times without altering its direction or sense, where A is any positive real number not necessarily an integer. The result is a vector $\overrightarrow{OA}$, Fig. 1·03(*b*), such that

$$\hat{a}A = \overrightarrow{OA} = \vec{A}.$$

Thus A is the numerical magnitude of A and it may be regarded as an operator the effect of which is to increase A times the length of $\hat{a}$ without altering its direction or sense.

The unit vectors $\hat{i}$, $\hat{j}$, $\hat{k}$. Let $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$ be three unit vectors in three given directions which are mutually perpendicular. To show that any vector $\vec{r}$ can be expressed as the sum of three others in the three given directions let $\overrightarrow{\mathrm{OP}} = \vec{r}$ [cf. Fig. 1·04]. On OP as

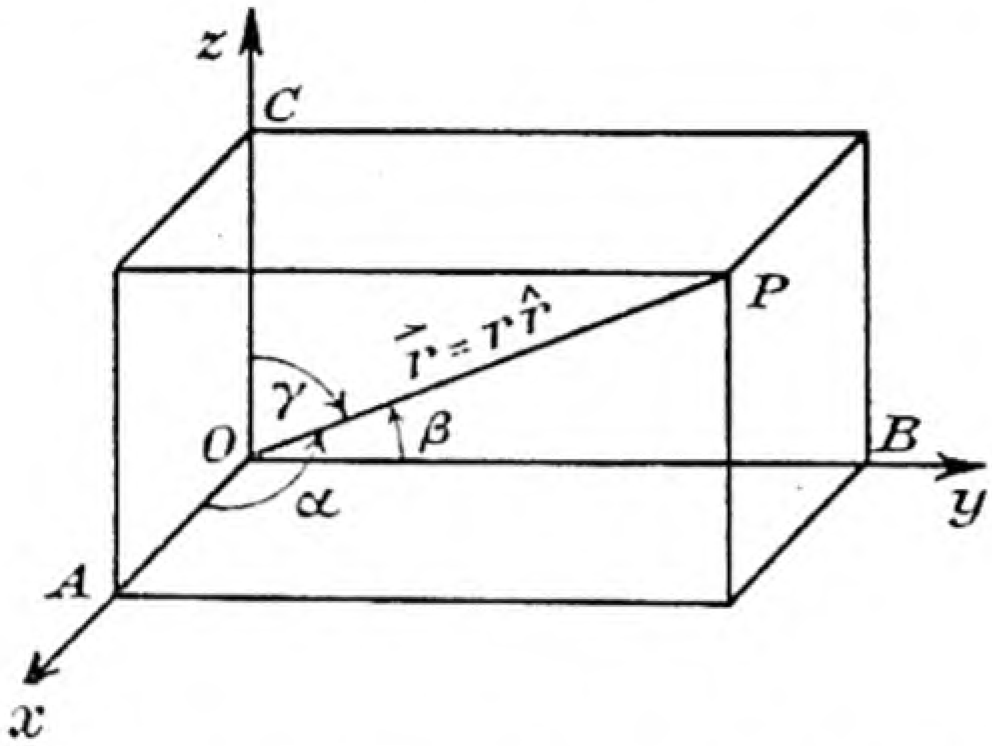


Fig. 1·04.—The three unit vectors $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$.

diagonal construct a rectangular parallelepiped with edges OA, OB, OC. Then if x, y, z are the measures of OA, OB, OC, we have

$$\vec{r} = \overrightarrow{\mathrm{OA}} + \overrightarrow{\mathrm{OB}} + \overrightarrow{\mathrm{OC}}$$

$$= \hat{\imath}x + \hat{\jmath}y + \hat{k}z.$$

Thus $\vec{r}$ is the resultant of the three vectors $\hat{\imath}x$, $\hat{\jmath}y$, $\hat{k}z$, which are termed the components of $\vec{r}$ in the three given directions. Since only one parallelepiped can be constructed in this way, the resolution of $\vec{r}$ is unique.

If α, β, γ are the angles which $\overrightarrow{\mathrm{OP}}$ makes with the axes, the direction cosines of $\overrightarrow{\mathrm{OP}}$ are $\cos\alpha$, $\cos\beta$, $\cos\gamma$, and these, as usual, are denoted by l, m, n.

Now $x = r\cos\alpha$, $y = r\cos\beta$ and $z = r\cos\gamma$. Hence if $\hat{r}$ denotes the unit vector along OP, since $\vec{r} = r\hat{r}$, then

$$\hat{r} = \frac{\vec{r}}{r} = \hat{\imath}\cos\alpha + \hat{\jmath}\cos\beta + \hat{k}\cos\gamma = \hat{\imath}l + \hat{\jmath}m + \hat{k}n.$$

(*b*) ***Multiplication of vectors. Scalar products.*** The ***scalar product*** of a pair of vectors $\vec{\mathrm{A}}$ and $\vec{\mathrm{B}}$ is defined to be the scalar AB $\cos\theta$, where θ is the angle included between the two vectors. This product is denoted by $\{\vec{\mathrm{A}}\cdot\vec{\mathrm{B}}\}$, or $(\vec{\mathrm{A}}\cdot\vec{\mathrm{B}})$, although the brackets may be dispensed with when no confusion is likely to arise.

From the above definition it follows that

$$\{\vec{A}\cdot\vec{B}\} = \{\vec{B}\cdot\vec{A}\}.$$

Also, if either $0 < \theta < \frac{\pi}{2}$, or $\frac{3\pi}{2} < \theta < 2\pi$, then $\{\vec{A}\cdot\vec{B}\}$ is positive; if $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$, then $\{\vec{A}\cdot\vec{B}\}$ is negative. If the two vectors are perpendicular to each other, then $\{\vec{A}\cdot\vec{B}\} = 0$, since $\cos\frac{\pi}{2} = 0$.

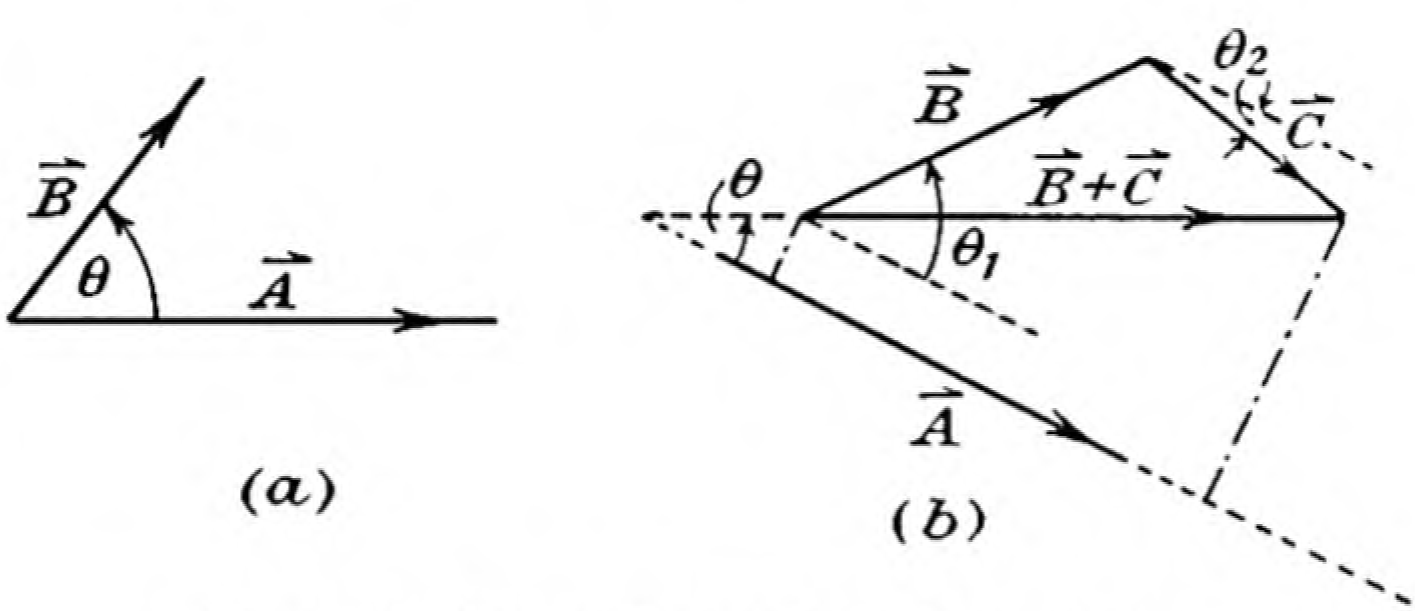


FIG. 1·05.—The scalar product of two vectors.

Again, from the definition of a scalar product it follows that the scalar product of a pair of vectors is equal to the magnitude of the first vector multiplied by the magnitude of the projection of the second vector along the direction of the first vector—cf. Fig. 1·05(a).

Let us now consider the scalar product of $\vec{A}$ and the vector sum $(\vec{B} + \vec{C})$. From the above, we have

$$\{\vec{A}\cdot(\vec{B} + \vec{C})\} = A \times \text{projection of } (\vec{B} + \vec{C}) \text{ on } \vec{A}.$$

Now the projection of the sum of two vectors on any other vector is the sum of the projections of the single vectors (addends) on that vector—cf. Fig. 1·05(b). Hence

$$\{\vec{A}\cdot(\vec{B} + \vec{C})\} = AB\cos\theta_1 + AC\cos\theta_2 = \{\vec{A}\cdot\vec{B}\} + \{\vec{A}\cdot\vec{C}\}.$$

Similarly,

$$\{\vec{A}\cdot(\vec{B} + \vec{C} + \vec{D}\ldots)\} = \{\vec{A}\cdot\vec{B}\} + \{\vec{A}\cdot\vec{C}\} + \{\vec{A}\cdot\vec{D}\} + \ldots$$

Theorem II. The scalar multiplication of pairs of vectors and of vector sums is commutative and distributive.

Hence the rules of ordinary algebra permit us to develop the scalar product of a pair of vector sums; thus

$$\{(\vec{A} + \vec{B})\cdot(\vec{C} + \vec{D})\} = \{\vec{A}\cdot\vec{C}\} + \{\vec{B}\cdot\vec{C}\} + \{\vec{A}\cdot\vec{D}\} + \{\vec{B}\cdot\vec{D}\}.$$

The above rule enables us to express the scalar product of two vectors in another form.

If $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$ are unit vectors along three mutually perpendicular axes,

$$\vec{A} = \hat{\imath}A_x + \hat{\jmath}A_y + \hat{k}A_z,$$

where A_x, A_y and A_z are the measures of the projections of $\vec{A}$ on the three axes. Similarly,

$$\vec{B} = \hat{\imath}B_x + \hat{\jmath}B_y + \hat{k}B_z.$$

$$\begin{aligned} \therefore\ \{\vec{A}\cdot\vec{B}\} &= (\hat{\imath}A_x + \hat{\jmath}A_y + \hat{k}A_z)\cdot(\hat{\imath}B_x + \hat{\jmath}B_y + \hat{k}B_z) \\ &= \hat{\imath}\cdot\hat{\imath}A_xB_x + \hat{\jmath}\cdot\hat{\jmath}A_yB_y + \hat{k}\cdot\hat{k}A_zB_z, \end{aligned}$$

the terms involving $\hat{\imath}\cdot\hat{\jmath}$, etc. disappearing since such scalar products are each zero. Moreover,

$$\hat{\imath}\cdot\hat{\imath} = \hat{\jmath}\cdot\hat{\jmath} = \hat{k}\cdot\hat{k} = 1,$$

since the angle between the vectors in each product is always zero and $\cos\theta = 1$

$$\therefore\ \{\vec{A}\cdot\vec{B}\} = A_xB_x + A_yB_y + A_zB_z.$$

If (l_1, m_1, n_1) and (l_2, m_2, n_2) are the direction cosines of $\vec{A}$ and $\vec{B}$ and θ is the included angle,

$$\begin{aligned} AB\cos\theta = \{\vec{A}.\vec{B}\} &= A_xB_x + A_yB_y + A_zB_z \\ &= l_1A.l_2B + m_1A.m_2B + n_1A.n_2B \\ &= (l_1l_2 + m_1m_2 + n_1n_2)\,AB. \\ \therefore\ \cos\theta &= l_1l_2 + m_1m_2 + n_1n_2. \end{aligned}$$

Vector products. Let $\vec{A}$ and $\vec{B}$ be two vectors inclined to one another at an angle θ; these are shown in Fig. 1·06. Then the ***vector product*** of this pair of vectors is defined as a third vector $\vec{C}$ whose magnitude is $AB\sin\theta$, and whose direction is such that a right-handed rotation about $\vec{C}$ as axis, and through an angle less than π, carries the vector $\vec{A}$ into a direction coinciding with that of $\vec{B}$. This vector product is denoted by $[\vec{A}\times\vec{B}]$ or $[\vec{A}_\wedge\vec{B}]$, so that

$$\vec{C} = [\vec{A}\times\vec{B}] = \hat{n}AB\sin\theta,$$

where $\hat{n}$ is a unit vector perpendicular to the plane containing $\vec{A}$ and $\vec{B}$, and having the same direction as the translation of a right-handed screw due to a rotation from $\vec{A}$ to $\vec{B}$.

From the definition of a vector product it is seen that on interchanging the factors the direction of the vector which is the vector product is reversed, its magnitude, however, is unaltered, i.e.

$$[\vec{B} \times \vec{A}] = -[\vec{A} \times \vec{B}].$$

Hence the vector product is not commutative, and the order of the factors has to be considered carefully.

Again, if $\theta = 0$, the vector product $[\vec{A} \times \vec{B}] = \hat{n}\,AB \sin\theta = 0$; the converse is also true, viz. if $[\vec{A} \times \vec{B}] = 0$ then the vectors $\vec{A}$ and $\vec{B}$ are parallel to one another.

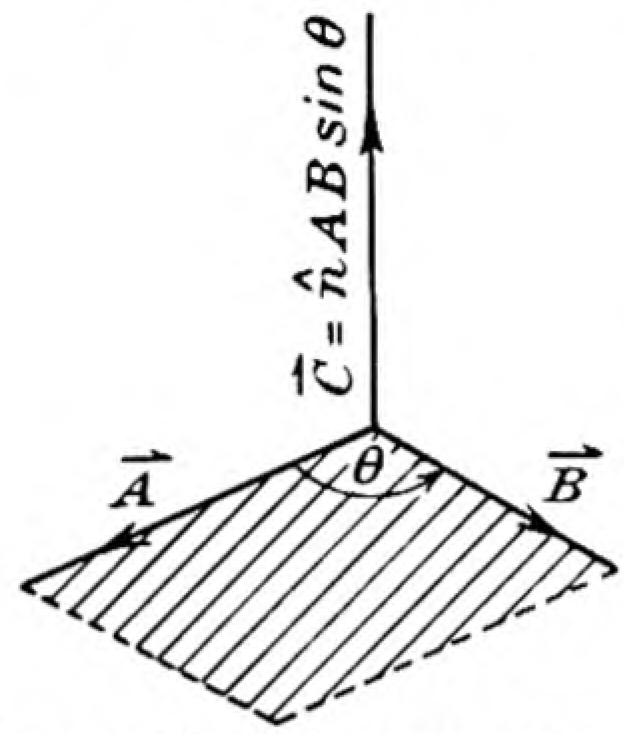


FIG. 1·06.—The vector product of two vectors.

For the unit vectors $\hat{\imath}, \hat{\jmath}, \hat{k}$, we have

$$\hat{\imath} \times \hat{\imath} = \hat{\jmath} \times \hat{\jmath} = \hat{k} \times \hat{k} = 0$$
$$\hat{\imath} \times \hat{\jmath} = \hat{k} = -\hat{\jmath} \times \hat{\imath},$$
$$\hat{\jmath} \times \hat{k} = \hat{\imath} = -\hat{k} \times \hat{\jmath},$$
$$\hat{k} \times \hat{\imath} = \hat{\jmath} = -\hat{\imath} \times \hat{k}.$$

It is sometimes desirable to express the vector product $[\vec{A} \times \vec{B}]$ in terms of the components of the vectors, viz. A_x, A_y, A_z and B_x, B_y, B_z, respectively. This can be done with the help of the theorems already established. Thus

$$[\vec{A} \times \vec{B}] = [(\hat{\imath}A_x + \hat{\jmath}A_y + \hat{k}A_z) \times (\hat{\imath}B_x + \hat{\jmath}B_y + \hat{k}B_z)]$$
$$= \hat{\imath}(A_yB_z - B_yA_z) + \hat{\jmath}(A_zB_x - B_zA_x) + \hat{k}(A_xB_y - B_xA_y).$$

Thus

$$[\vec{A} \times \vec{B}] = \begin{vmatrix} \hat{\imath}, & \hat{\jmath}, & \hat{k} \\ A_x, & A_y, & A_z \\ B_x, & B_y, & B_z \end{vmatrix}.$$

We have already seen how the scalar product of two vectors enables us to express the cosine of the angle between them in terms of the direction cosines of the vectors; the vector product gives us the value of the sine of the included angle. For since $\vec{A} \times \vec{B} = \hat{n}\,AB \sin\theta$, we have

$$A^2B^2 \sin^2\theta = \{[\vec{A} \times \vec{B}]\cdot[\vec{A} \times \vec{B}]\}$$
$$= \hat{\imath}\cdot\hat{\imath}(A_yB_z - B_yA_z)^2 + \hat{\jmath}\cdot\hat{\jmath}(A_zB_x - B_zA_x)^2 + \hat{k}\cdot\hat{k}(A_xB_y - B_xA_y)^2$$
$$= [(m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2 + (l_1m_2 - l_2m_1)^2]A^2B^2.$$
$$\therefore \sin^2\theta = (m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2 + (l_1m_2 - l_2m_1)^2.$$

Solid angles.—Let AB, Fig. 1·07(*a*), be the boundary of a finite portion of a surface and P a given point. If from P a sufficiently large number of straight lines is drawn each to pass through a point on the boundary they will generate a cone. Suppose that with P as centre a series of spherical surfaces is constructed, the above cone intercepting an area from each of them. Then it is found that the ratio obtained by dividing one of these areas by the square of the radius of the corresponding sphere is a constant for

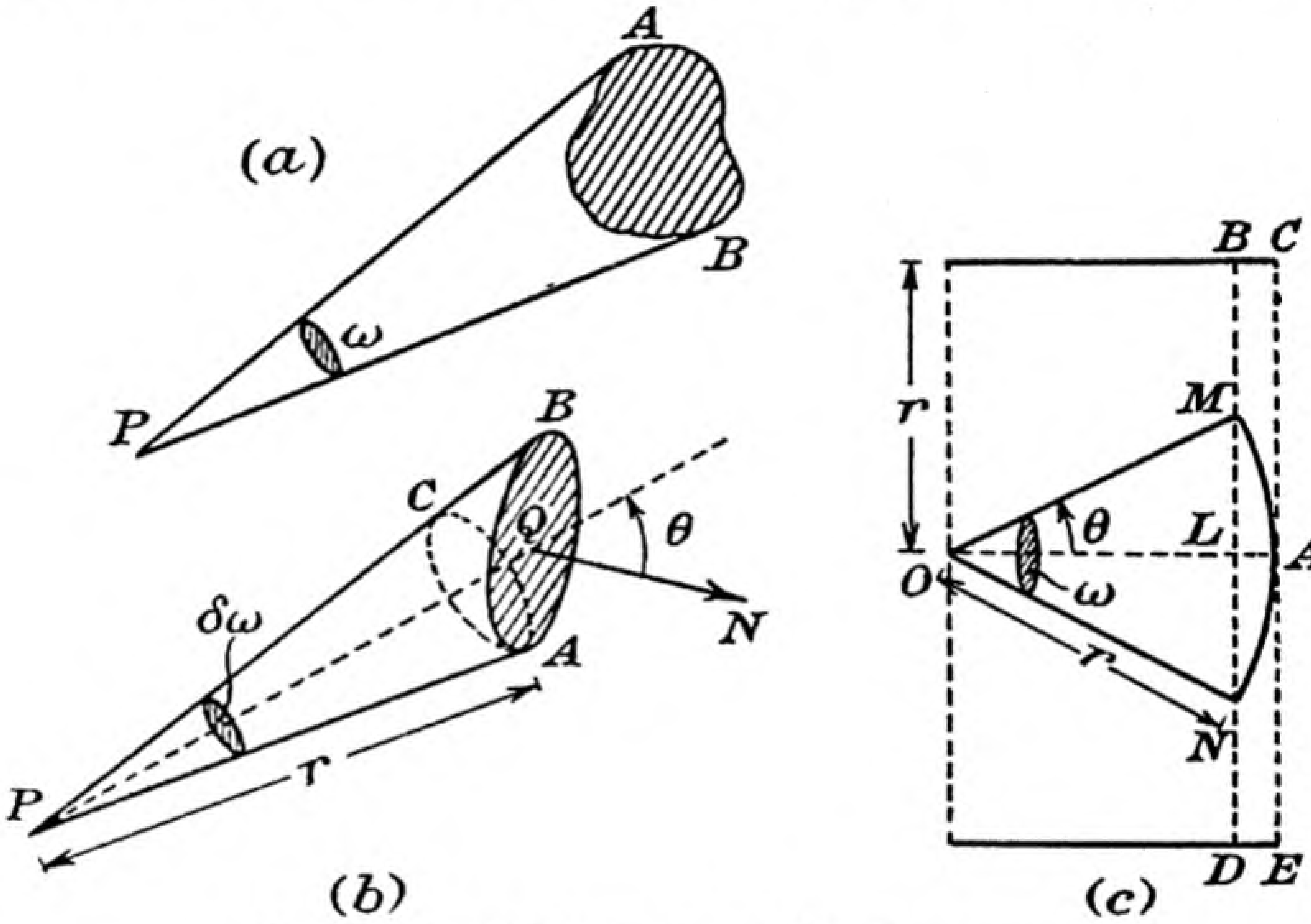


FIG. 1·07.—Solid angles and their measure.

the cone PAB. From analogy with the conventional method of measuring a plane angle, the above ratio is called ***the measure of the solid angle*** subtended at P by the surface AB. This angle is usually denoted by ω.

From the above definition it is seen that the measure of the solid angle depends only on the closed curve and not on the shape of the surface on which the closed curve may be considered to be drawn.

If now AB, Fig. 1·06(*b*), is a small closed curve, let $\delta\omega$ be the solid angle it subtends at P. With P as centre and radius PA describe a sphere to intersect the cone APB in a closed curve AC. Let PQ be the axis of the cone intersecting the elementary surface AB in Q, and let QN be the normal to this surface at Q. Let θ be the angle indicated. Then AC = AB cos θ. If PA = r, then

$$\delta\omega = \frac{\text{AC}}{r^2} = \frac{\text{AB}\cos\theta}{r^2}.$$

The measure of the solid angle subtended by a spherical cap AMN, Fig. 1·06(*c*), at the centre O of the sphere of which the cap

forms part in terms of θ, the semi-vertical angle of the cone formed by joining the boundary of the cap to the centre O, may be obtained as follows. It is a well-known theorem that the area of the spherical cap AMN is equal to the area of the curved surface BCED of the right circular cylinder having OA as its axis and its radius equal to the radius, r, of the sphere. Thus

Area $\text{AMN} = 2\pi\text{AC}.\text{BC} = 2\pi.r.(\text{OA} - \text{OL}) = 2\pi r^2(1 - \cos\theta)$.
Hence if ω is the solid angle required, we have

$$\omega = \frac{2\pi r^2(1 - \cos\theta)}{r^2} = 2\pi(1 - \cos\theta).$$

Note on spherical polar coordinates.—Let O, Fig. 1·08(a), be the origin of a system of rectangular coordinates. Let P be a point whose position is to be defined. With O as centre and radius

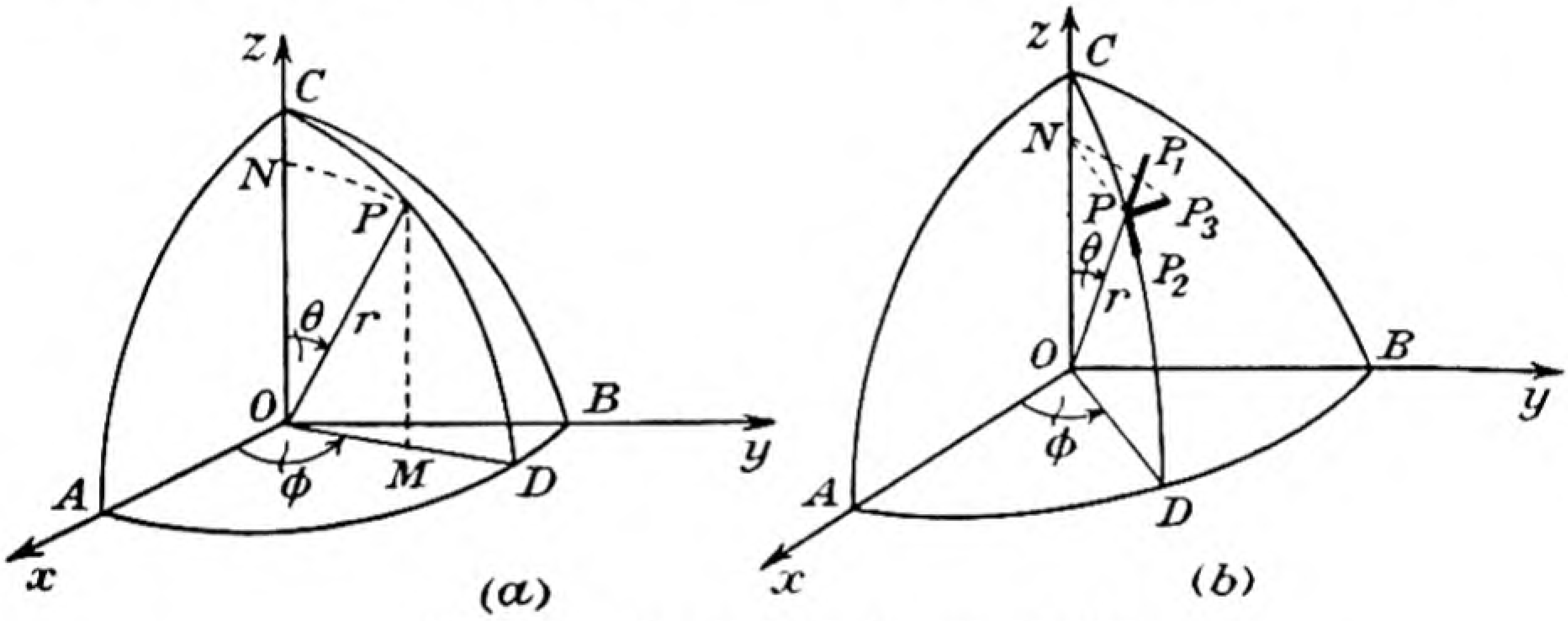


FIG. 1·08.—Spherical polar coordinates.

$\text{OP} = r$ describe a sphere to intersect the axes at A, B and C respectively; we have to show how the position of P on the surface of this sphere may be specified.

Let the plane AOC rotate about OC through an angle ϕ until it contains the point P. Let COD be the position of this plane. In this plane let the line OC rotate through an angle θ until it reaches OP. Then P is the point (r, θ, ϕ); these are the ***spherical polar coordinates*** of the point P.

If M and N are the projections of P on OD and OC respectively,

$$z = \text{ON} = r\cos\theta,$$

and
$$\text{OM} = r\sin\theta.$$

But M is the point $(x, y, 0)$, so that

$$\left.\begin{aligned} x &= \text{OM}\cos\phi = r\sin\theta\cos\phi \\ y &= \text{OM}\sin\phi = r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned}\right\},$$

and

as already proved.

2

To find an expression for an element of volume in spherical polar coordinates consider Fig. 1·08(*b*). If r becomes $r + \delta r$, the point P moves to P_1 in the plane COD, and $PP_1 = \delta r$. If θ becomes $\theta + \delta\theta$, P moves to P_2 and $PP_2 = r\,\delta\theta$, while if ϕ becomes $\phi + \delta\phi$ the point P moves to P_3, and $PP_3 = PN\,\delta\phi = r \sin\theta\,\delta\phi$. The rectangular element whose sides are PP_1, PP_2 and PP_3 has a volume $r^2 \sin\theta\,\delta r\,\delta\theta\,\delta\phi$.

Some important properties of an ellipse.

The radius of curvature at the point $x = 0$, $y = -b$ on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.—Differentiating with respect to x we get

$$\frac{x}{a^2} + \frac{y}{b^2}\left(\frac{dy}{dx}\right) = 0,$$

and differentiating again we find

$$\frac{1}{a^2} + \frac{1}{b^2}\left(\frac{dy}{dx}\right)^2 + \frac{y}{b^2} \cdot \frac{d^2y}{dx^2} = 0.$$

Hence

$$\left(\frac{dy}{dx}\right)_{\substack{x=0\\y=-b}} = 0,$$

and

$$\left(\frac{d^2y}{dx^2}\right)_{\substack{x=0\\y=-b}} = \left[-\frac{b^2}{a^2} \cdot \frac{1}{y}\right]_{\substack{x=0\\y=-b}} = \frac{b}{a^2}.$$

If R is the radius of curvature at $x = 0$, $y = -b$, we have

$$R = \left[\frac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}\right]_{\substack{x=0\\y=-b}} = \frac{a^2}{b}.$$

***To prove that the feet of the perpendiculars* (SY,S′Y′) *from the foci to the tangent at a point* P *on an ellipse lie on the auxiliary circle and* SY.S′Y′ $= b^2$.**—Let C be the centre of the ellipse which, together with its auxiliary circle, is shown in Fig. 1·09. Let S and S′ be its foci and produce S′P and SY to meet in Q. Since the radii vectors SP and S′P are equally inclined to the tangent at P it follows that

$$S\widehat{P}Y = Y\widehat{P}Q.$$

The Δ's SPY and QPY are congruent so that SP = PQ.

Also SY = YQ and since CS = S′C it follows that CY is parallel to S′Q. Hence

$$\begin{aligned} \mathrm{CY} &= \tfrac{1}{2}\mathrm{S'Q} = \tfrac{1}{2}(\mathrm{S'P} + \mathrm{PS}) \\ &= \mathrm{CA} = a. \end{aligned}$$

∴ Y lies on the auxiliary circle.
Similarly Y′ lies on this circle.

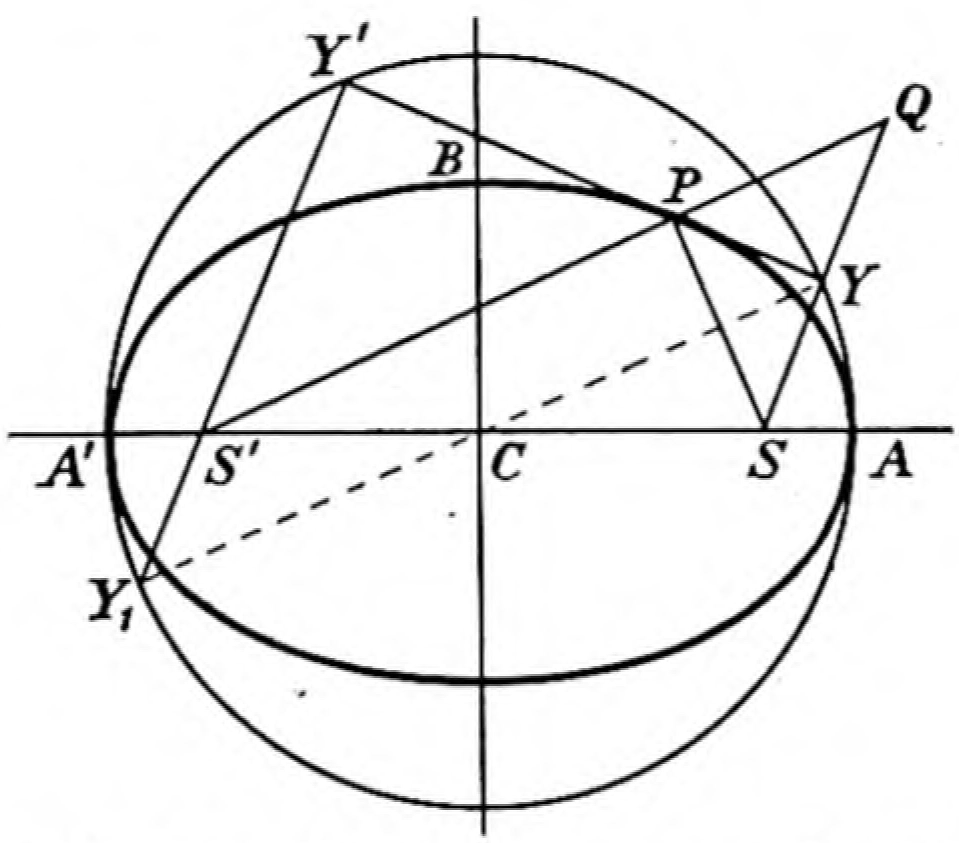


FIG. 1·09.—Some important properties of an ellipse.

Now produce Y′S′ to meet the auxiliary circle in $\mathrm{Y_1}$. Join $\mathrm{YY_1}$. Then since $\mathrm{Y\widehat{Y'}Y_1} = \frac{1}{2}\pi$, $\mathrm{YY_1}$ passes through C.
Also $\mathrm{S'Y_1} = \mathrm{SY}$.

$$\begin{aligned} \therefore\ \mathrm{SY.S'Y'} &= \mathrm{S'Y_1.S'Y'} = \mathrm{A'S'.AS'} \\ &= (a - \mathrm{S'C})(a + \mathrm{S'C}) = a^2 - \mathrm{CS}^2 = b^2. \end{aligned}$$

If we call SY and S′Y′ respectively p and p' the above equation becomes

$$pp' = b^2.$$

The polar equation to an ellipse.—Let S and S′, Fig. 1·10(a), be the foci of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, so that, with the usual notation, CA = a and CB = b. If SL is drawn through S and normal to the major axis of the ellipse to cut the latter in L, then SL is the semi-latus rectum, l. If e is the eccentricity of the ellipse, then

$$\mathrm{SL} = e.\mathrm{SZ},$$

where Z is the point in which the major axis of the ellipse, when produced, cuts the directrix.

Now let P, Fig. 1·10(*b*), the point (r, θ) with respect to S, be a point on the ellipse. Then

$$r = SP = e.PM = e[SZ - r\cos\theta]$$
$$= l - er\cos\theta.$$
$$\therefore\ r = \frac{l}{1 + e\cos\theta}$$

which is the polar equation to the ellipse.

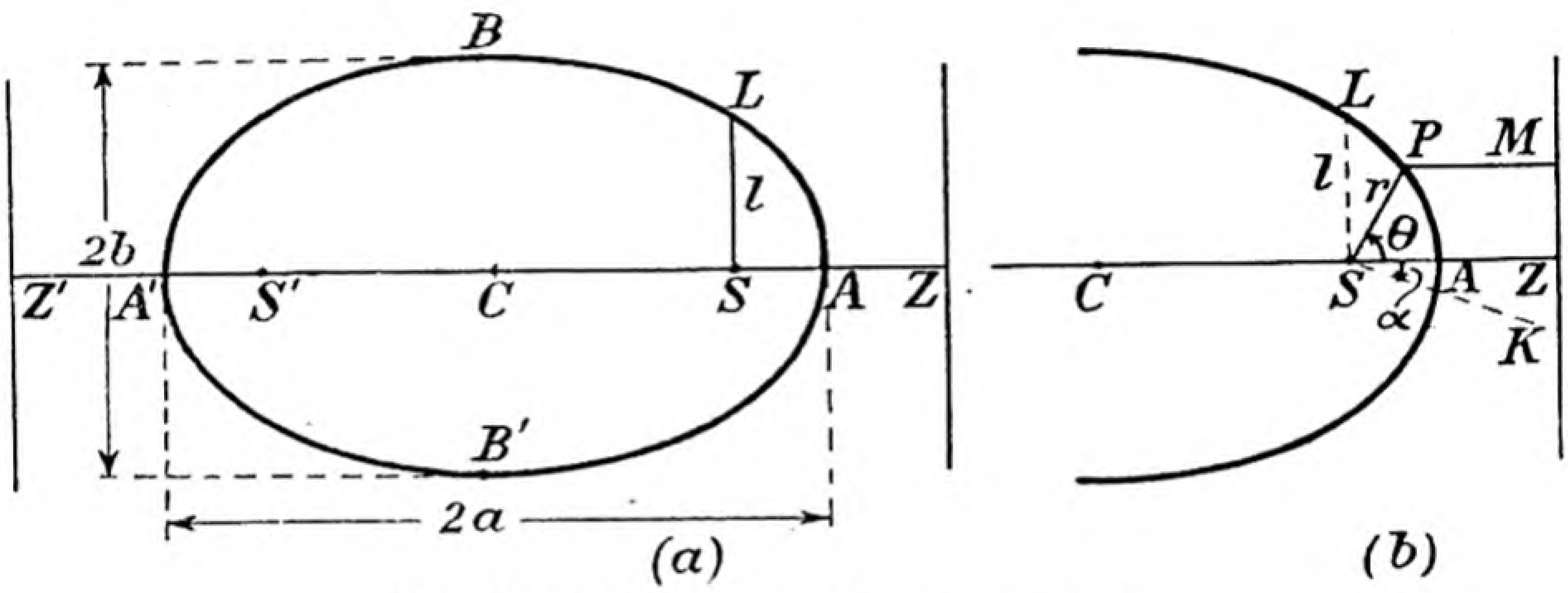


FIG. 1·10.—Polar equation to an ellipse.

If the initial line to which the orientation of the radius vector SP is referred is SK and this makes an angle α with SZ, then $\widehat{PSZ} = (\theta - \alpha)$, and the polar equation to the ellipse becomes

$$r = \frac{l}{1 + e\cos(\theta - \alpha)}.$$

The p,r or pedal equation to a curve.—An equation expressing the relation between the length of the perpendicular from a fixed point to the tangent to a curve, and the radius vector of the point of contact of the tangent with the curve and measured from the same fixed point, is known as the ***pedal equation*** to the curve.

To show that the pedal equation of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, ***with regard to a focus is*** $\frac{b^2}{p^2} = \frac{2a}{r} - 1$.

If P, Fig. 1·11(*a*), is a point on the ellipse so that $SP = r$, $S'P = r'$ and $SY = p$, $S'Y' = p'$, then since the Δ's SPY and S'PY' are similar

$$\frac{p}{r} = \frac{p'}{r'} = \sqrt{\frac{pp'}{rr'}}.$$

$$\therefore\ \frac{p}{r} = \frac{b}{\sqrt{r(2a - r)}}. \qquad [\because\ pp' = b^2 \text{ and } SP + PS' = 2a.]$$

Squaring both sides of this equation we find

$$\frac{b^2}{p^2} = \frac{2a}{r} - 1.$$

To obtain this equation by another method let us consider the polar equation to an ellipse, viz.

$$l = r(1 + e\cos\theta). \qquad \text{(i) [cf. p.12]}$$

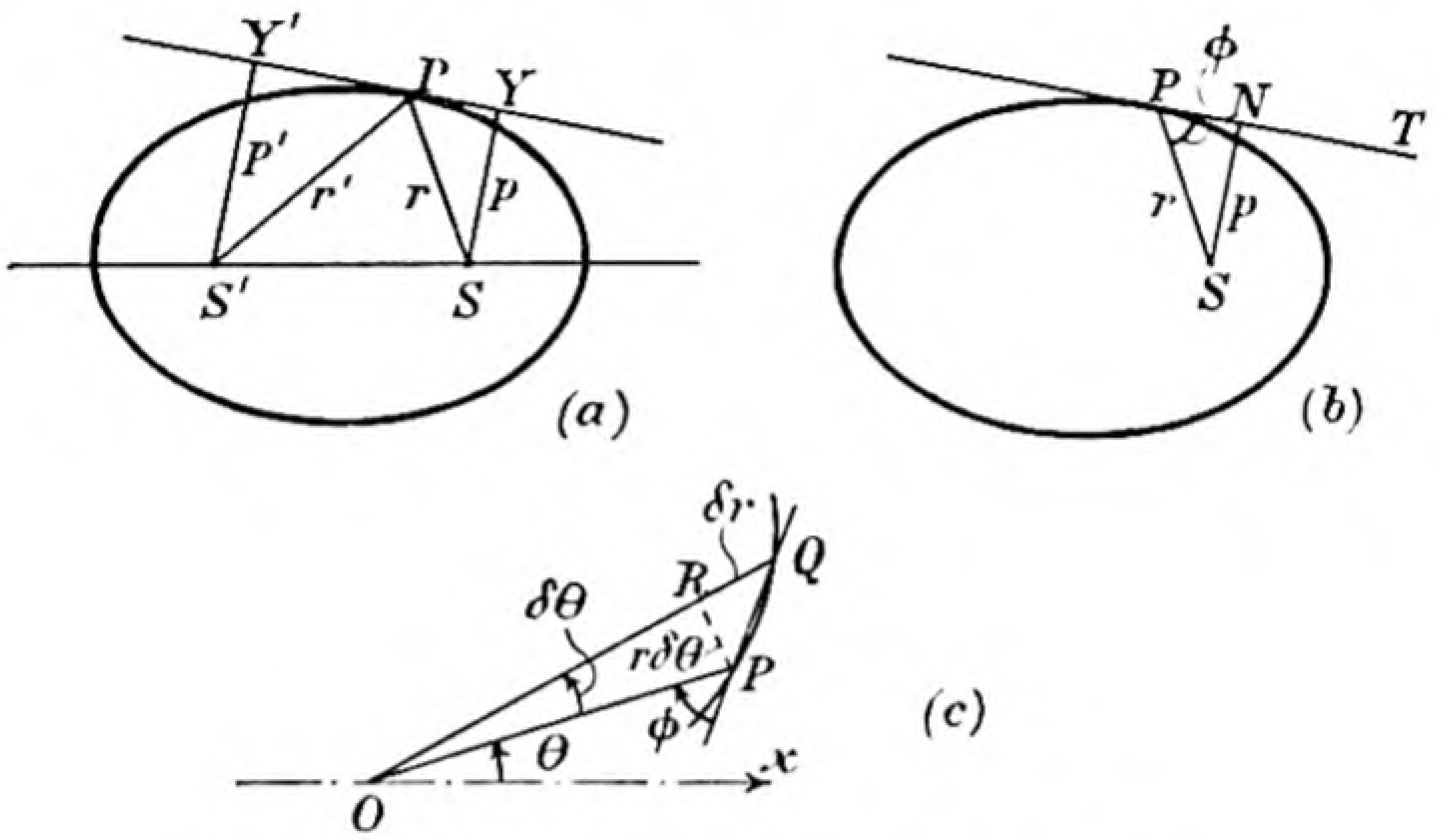


FIG. 1·11.—The pedal equation to an ellipse.

Let ϕ be the angle which SP makes with PT, the tangent at P. Then, cf. Fig. 1·11(b),

$$p = r\sin\phi, \qquad \text{(ii)}$$

and

$$\tan\phi = r\frac{d\theta}{dr}, \qquad \text{(iii)}$$

since, cf. Fig. 1·11(c),

$$\tan\phi = \lim\frac{\text{PR}}{\text{RQ}} = \lim\frac{r\,\delta\theta}{\delta r}$$

$$= r\frac{d\theta}{dr}.$$

Hence if θ and ϕ are eliminated from equations (i), (ii) and (iii) the required equation between p and r will be obtained. In the present instance

$$1 + e\cos\theta = \frac{l}{r} = \frac{b^2}{ar}, \quad \because\ l = \frac{b^2}{a}.$$

$$\therefore\ e\sin\theta\frac{d\theta}{dr} = -\frac{b^2}{ar^2}.$$

But since $\sin\phi = \dfrac{p}{r}$, $\tan\phi = \dfrac{p}{\sqrt{r^2 - p^2}}$.

$$\therefore\ \frac{p^2}{r^2 - p^2} = r^2\left(\frac{d\theta}{dr}\right)^2 = r^2 . \frac{b^4}{a^2 r^4} . \frac{1}{e^2 \sin^2\theta}$$

$$= \frac{b^4}{a^2 r^2} . \frac{1}{e^2\left[1 - \left(\dfrac{b^2}{ar} - 1\right)^2 \div e^2\right]},$$

which, by using $b^2 = a^2(1 - e^2)$, and gives, after some reduction,

$$\frac{b^2}{p^2} = \frac{2a}{r} - 1.$$

FOURIER ANALYSIS

General nature of the problem.—Let a given curve be represented by

$$y = f(x) \tag{i}$$

Then the constants a_0, a_1, in the equation

$$y = a_0 + a_1 \cos x \tag{ii}$$

may be determined so that the two curves (i) and (ii) intersect in any two given points over the range $0 < x < \pi$. Thus, let (x_1, y_1) and (x_2, y_2) be any two conjugate values in the equation $y = f(x)$ and such that $0 < x_1 < \pi$ and $0 < x_2 < \pi$. Substituting these values in (ii), we obtain

$$\left.\begin{aligned} y_1 &= a_0 + a_1 \cos x_1 \\ y_2 &= a_0 + a_1 \cos x_2 \end{aligned}\right\} \tag{iii}$$

The equations (iii) are sufficient to determine a_0 and a_1. In this way coincidence of the two curves is compelled in two selected places. Similarly, the two curves

$$y = f(x)$$

and

$$y = a_0 + \sum_{n=1}^{n} a_n \cos nx \tag{iv}$$

may be made to coincide in $(n + 1)$ selected places over the range $0 < x < \pi$.

Hence the question arises as to whether or not any function throughout the range $0 < x < \pi$ may be represented by an infinite series of the above type. It can be shown, that any function of x, say $f(x)$, which is single, finite, and continuous over the range $0 < x < \pi$, or, if, over that range, it has finite discontinuities each

of which is preceded and followed by continuous portions, can be represented by such an infinite trigonometrical series. The function and the series will be identical for all values of x between $x = 0$ and $x = \pi$ but not including the values $x = 0$ and $x = \pi$ unless the given function is zero for these particular values of x.

The cosine series.—Let

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx \tag{v}$$

over the range $0 < x < \pi$. We shall assume that this series may be integrated term by term. Let both sides of (v) be multiplied by dx and then integrated with respect to x between the limits $x = 0$ and $x = \pi$. Then

$$\int_0^{\pi} f(x)\, dx = a_0 \int_0^{\pi} dx + \text{terms which vanish} = a_0 \pi.$$

$$\therefore\ a_0 = \frac{1}{\pi} \int_0^{\pi} f(x)\, dx.$$

To determine the other coefficients we multiply throughout by $\cos mx\, dx$ and integrate as before. Then

$$\int_0^{\pi} f(x) \cos mx\, dx = a_0 \int_0^{\pi} \cos mx\, dx + \sum_{n=1}^{n=\infty} \int_0^{\pi} a_n \cos mx \cos nx\, dx.$$

Now $\int_0^{\pi} \cos mx\, dx = 0$, and if $m \neq n$

$$\int_0^{\pi} \cos mx \cos nx\, dx = \frac{1}{2} \int_0^{\pi} [\cos (m - n)x + \cos (m + n)x]\, dx = 0.$$

If $m = n$ and $n \neq 0$

$$\int_0^{\pi} \cos^2 nx\, dx = \tfrac{1}{2}\pi.$$

$$\therefore\ a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx\, dx.$$

The above process of determining the coefficients is equivalent to taking $(n + 1)$ terms of (v) and obtaining $(n + 1)$ equations by substituting in

$$y = a_0 + \sum_{n=1}^{n} a_n \cos nx$$

the coordinates of $(n + 1)$ points which lie on the curve $y = f(x)$ and whose abscissae are equidistant. The constants are determined by the equations thus obtained and the constants in the cosine series would be the limiting values of the constants thus determined when $n \to \infty$.

Example.—Let $f(x) = x$ over the range $0 < x < \pi$.

Now $$\int_0^\pi x\,dx = a_0\int_0^\pi dx, \quad \therefore\ a_0 = \frac{\pi}{2},$$

and $$\int_0^\pi x\cos nx\,dx = a_n\int_0^\pi \cos^2 nx\,dx = \frac{\pi}{2}\cdot a_n.$$

But $$\int_0^\pi x\cos nx\,dx = \left[\frac{x}{n}\sin nx + \frac{1}{n^2}\cos nx\right]_0^\pi$$
$$= \frac{1}{n^2}[\cos \pi n - \cos 0] = \frac{1}{n^2}[(-1)^n - 1].$$

$\therefore$ If n is odd $\qquad a_n = -\frac{4}{\pi}\cdot\frac{1}{n^2}$,

while if n is even $\qquad a_n = 0.$

$$\therefore\ x = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots + \frac{\cos(2n+1)x}{(2n+1)^2} + \dots\right] \qquad \text{(vi)}$$

Now (vi) is also true for values $x = 0$ and $x = \pi$ since

$$0 = \frac{\pi}{2} - \frac{4}{\pi}\left[\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\right],$$

and $$\pi = \frac{\pi}{2} - \frac{4}{\pi}\left[(-1)\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\right],$$

since the series under the $\sum\limits_{n=1}^{\infty}$ sign is $\frac{\pi^2}{8}$. [Cf. p. 25.]

The sine series.—In the same way, and under the same conditions, it can be shown that

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx,$$

where $$b_n = \frac{2}{\pi}\int_0^\pi f(x)\sin nx\,dx.$$

Example.—$f(x) = x$ over the range $0 < x < \pi$.
The series obtained in this instance is

$$x = 2\left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \dots\right]$$
$$= 2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nx}{n}.$$

The cosine series and sine series compared.—It has just been shown that any function of x under certain limitations may be expressed either as a cosine series or as a sine series. Now

although either series will be identical with $f(x)$ for all values of x such that $0 < x < \pi$, and possibly for $x = 0$ and $x = \pi$, there is a marked difference between the two series for other values of x.

Both series are periodic functions of x, the period being 2π. Let y denote the series under consideration. If the portion of the curve lying between $-\pi < x < \pi$ is constructed, the complete curve may be obtained by repetition of this portion.

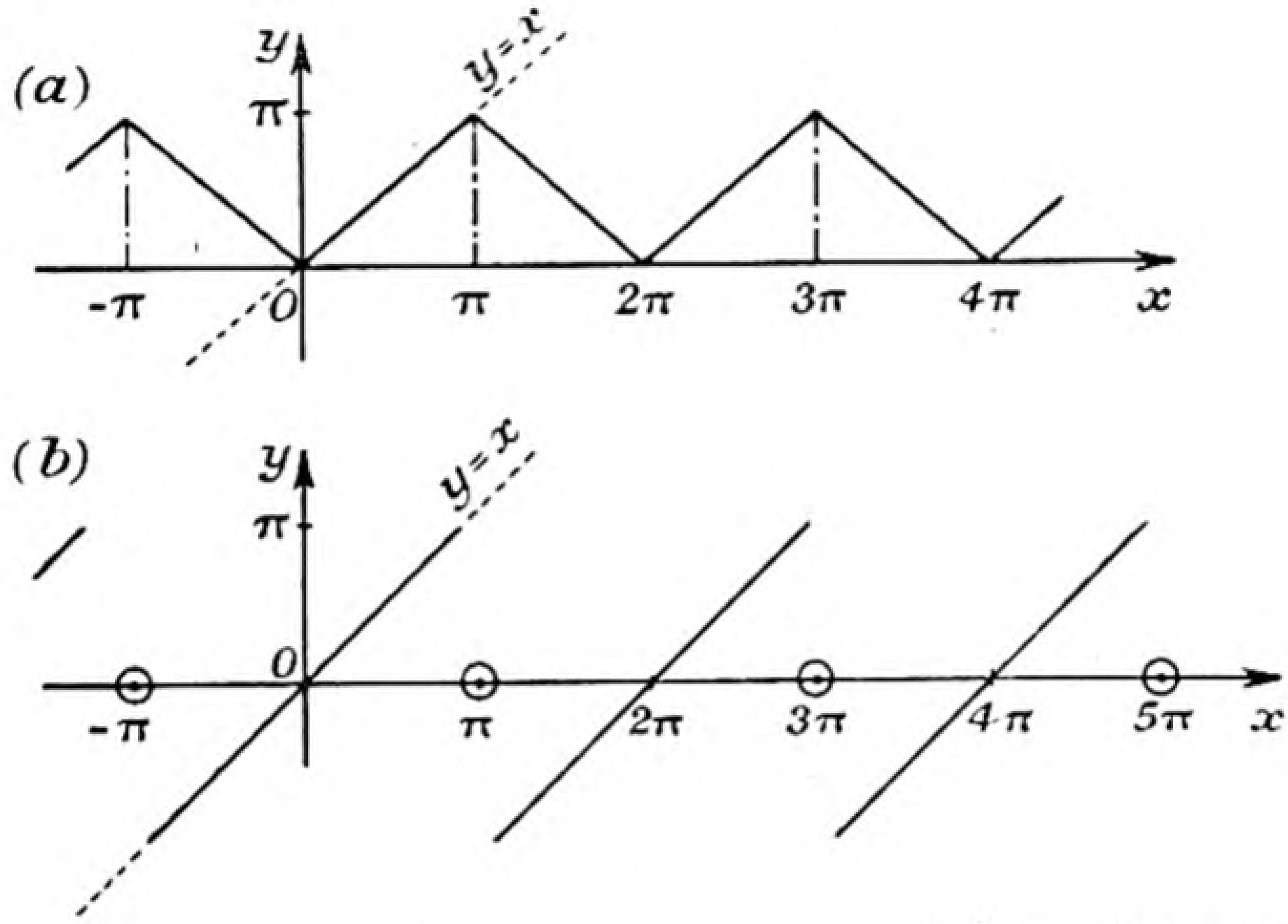


FIG. 1·12.—(a) The cosine series $y = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\cos (2n-1)x}{(2n-1)^2}$ and the curve $y = x$.

(b) The sine series $y = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin nx}{n}$ and the curve $y = x$.

Consider the cosine series first. Since $\cos nx = \cos (-nx)$, it follows that the ordinate in the cosine curve corresponding to any value of x over the range $-\pi < x < 0$ will be equal to the ordinate corresponding to the positive value of x, i.e. the curve $y = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$ is symmetrical with respect to the y-axis.

Returning to the sine series, since $-\sin nx = \sin (-nx)$, it follows that the curve will be symmetrical with respect to the origin, i.e. if (x_1, y_1) lies on the curve, $(-x_1, -y_1)$ also lies on the curve.

The curves corresponding to $y = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\cos (2n-1)x}{(2n-1)^2}$ and $y = 2\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin nx}{n}$, are shown in Figs. 1·12(a) and (b) respectively. It is seen that the sine series represents $y = x$ over the range $-\pi < x < \pi$, whereas the cosine series represents it only over the range $0 < x < \pi$. Both these curves coincide with $y = x$ over the range $0 < x < \pi$ and, in addition, the sine series coincides

from $-\pi < x < 0$, but no coincidence is obtained for values of x outside the range $-\pi < x < \pi$. Also, the sine series gives the isolated points $[\pm(2n-1)\pi, 0]$, where $n = 1, 2, 3, \ldots$

The above remarks illustrate the following general rules:

(*a*) If $f(x)$ is an even function, i.e. $f(x) = f(-x)$, the cosine series corresponding to it will be equal to $f(x)$ over the range $-\pi < x < \pi$, the value $x = 0$ not being excepted.

(*b*) If $f(x)$ is an odd function, i.e. $f(x) = -f(-x)$, the sine series will be equal to it over the range $-\pi < x < \pi$, except perhaps for the value $x = 0$ when the series is zero.

Fourier series.—The above rules permit us to develop a series which shall represent $f(x)$ over the range $-\pi < x < \pi$. For we have the identity

$$f(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2}$$

$$= \text{an even function of } x + \text{an odd function of } x$$

$$= (a_0 + \sum_{n=1}^{\infty} a_n \cos nx) + (\sum_{n=1}^{\infty} b_n \sin nx),$$

where $$a_n = \frac{2}{\pi}\int_0^\pi \frac{f(x) + f(-x)}{2} \cos nx\, dx,$$

$$= \frac{1}{\pi}\left[\int_0^\pi f(x) \cos nx\, dx + \int_0^\pi f(-x) \cos nx\, dx\right],$$

$$= \frac{1}{\pi}\left[\int_0^\pi f(x) \cos nx\, dx + \int_0^{-\pi} f(z) \cos(-nz)(-dz)\right],$$

if $z = -x$. Thus

$$a_n = \frac{1}{\pi}\left[\int_0^\pi f(x) \cos nx\, dx + \int_{-\pi}^0 f(z) \cos nz\, dz\right],$$

$$= \frac{1}{\pi}\int_{-\pi}^\pi f(x) \cos nx\, dx. \qquad \left[\because \int_1^2 \mathrm{F}(p)\, dp = \int_1^2 \mathrm{F}(q)\, dq.\right]$$

Similarly,

$$b_n = \frac{1}{\pi}\int_{-\pi}^\pi f(x) \sin nx\, dx,$$

and $$a_0 = \frac{1}{2\pi}\int_{-\pi}^\pi f(x)\, dx.$$

$$\therefore f(x) = a_0 + \sum_{n=0}^{n=\infty} a_n \cos nx + \sum_{n=1}^{n=\infty} b_n \sin nx.$$

This is known as a ***Fourier series.***

It is not essential that $f(x)$ should be expressed by one and the same equation over the whole range. If, for example, $y = f_1(x)$ over the range $-\pi < x < 0$ and $y = f_2(x)$ over the range $0 < x < \pi$, then in the integration each term must be integrated between the appropriate limits. Thus

$$a_0 = \frac{1}{2\pi}\left[\int_{-\pi}^{0} f_1(x)\,dx + \int_{0}^{\pi} f_2(x)\,dx\right].$$

Similarly for a_n and for b_n.

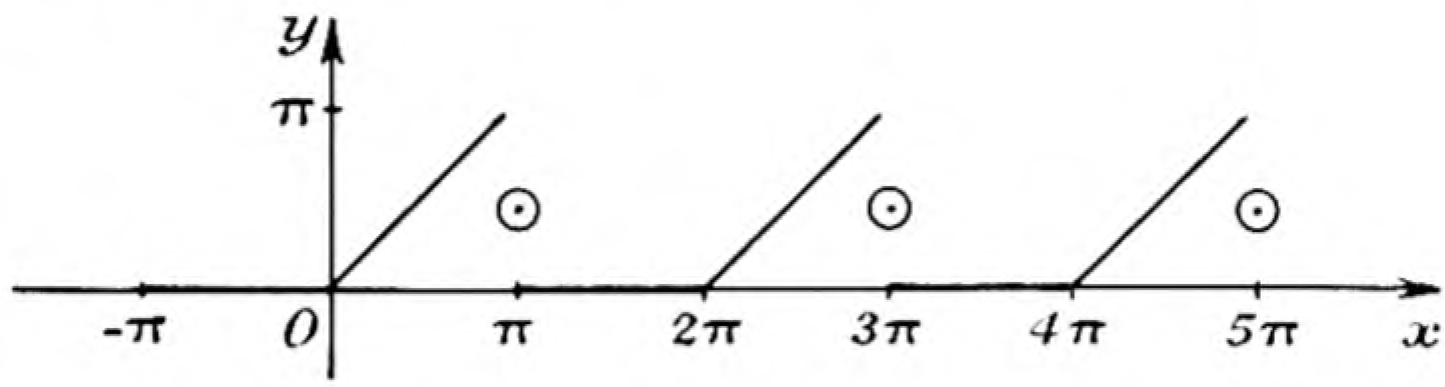


FIG. 1·13.—The series $y = \frac{\pi}{4} - \frac{2}{\pi}\left[\cos x + \frac{\cos 3x}{3^2} + \ldots\right] + \left[\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{2} - \ldots\right].$

Example of a Fourier expansion.—Let $f(x) = 0$ over the range $-\pi < x < 0$ and $f(x) = x$ over the range $0 < x < \pi$.

$$\therefore\ a_0 = \frac{1}{2\pi}\int_{-\pi}^{0} [0]\,dx + \frac{1}{2\pi}\int_{0}^{\pi} x\,dx = \frac{\pi}{4},$$

and
$$a_n = \frac{1}{\pi}\int_{-\pi}^{0} [0]\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} x\cos nx\,dx,$$
$$= \frac{1}{\pi}\left[\frac{x}{n}\sin nx + \frac{1}{n^2}\cos nx\right]_0^{\pi} = \frac{(-1)^n - 1}{\pi n^2}.$$

Similarly, $$b_n = \frac{1}{\pi}\int_0^{\pi} x\sin nx\,dx = \frac{(-1)^{n+1}}{n}.$$

$$\therefore\ f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left[\cos x + \frac{\cos 3x}{3^2} + \ldots\right] + \left[\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \ldots\right].$$

At $x = 0$, $$y = \frac{\pi}{4} - \frac{2}{\pi}\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right] + 0 = 0,$$
$$\left[\because \sum_{n=1}^{n=\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{8},\quad \text{cf. p. 25.}\right]$$

and at $x = \pi$, $$y = \frac{\pi}{4} + \frac{2}{\pi}\left[\sum_{n=1}^{n=\infty}\frac{1}{(2n-1)^2}\right] = \frac{\pi}{2}.$$

Hence the series also represents the isolated points $[(2n-1)\pi, \frac{1}{2}\pi]$; cf. Fig. 1·13.

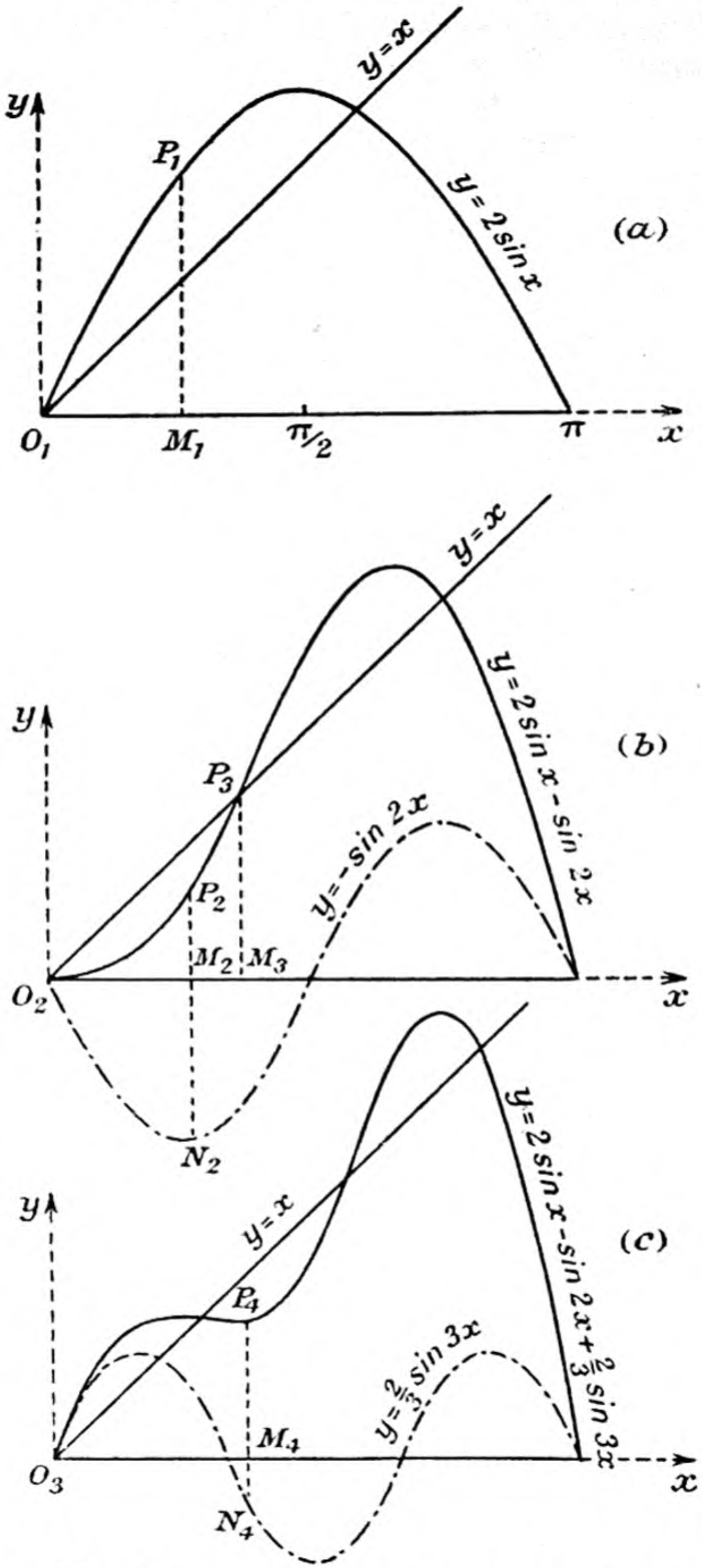


FIG. 1·14.—Successive approximations to the sine series representing $y = x \; [0 < x < \pi]$.

Successive approximations to a sine series.—It has already been shown that the curve

$$y = 2\left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \ldots\right] \quad . \quad . \quad \text{(vi)}$$

fits the curve $y = x$ over the range $0 < x < \pi$. The methods of analysis here developed will perhaps be elucidated more fully if we show how, by a series of successive approximations, the graph $y = x$ may be constructed graphically. Fig. 1·14(a) shows the graph of $y = 2 \sin x$. The dotted curve in Fig. 1·14(b) is $y = -\sin 2x$ and the curve $y = 2 \sin x - \sin 2x$ is obtained by constructing a series of points P_2 as follows. Ordinates are drawn at corresponding points M_1 and M_2 on the x-axis in each diagram. If the ordinate through M_2 cuts the curve $y = -\sin 2x$ in N_2, then, with N_2 as centre and radius $N_2P_2 = M_1P_1$, an arc is drawn to cut the ordinate in P_2. Fig. 1·14(c) shows the curve $y = 2 \sin x - \sin 2x + \frac{2}{3} \sin 3x$, a point P_4 on this curve being obtained by making N_4P_4 equal to M_3P_3, the notation used being self-explanatory. These three curves show how even a few terms of the sine series are sufficient to give a curve which rapidly approximates to the straight line $y = x$ over the given range. If, however, the curves are constructed outside this range the coincidence no longer persists.

Extension of the range.—In practical cases the range of x is seldom $-\pi < x < \pi$, so that the narrow boundaries must be widened if the analysis is to be of general application. Suppose

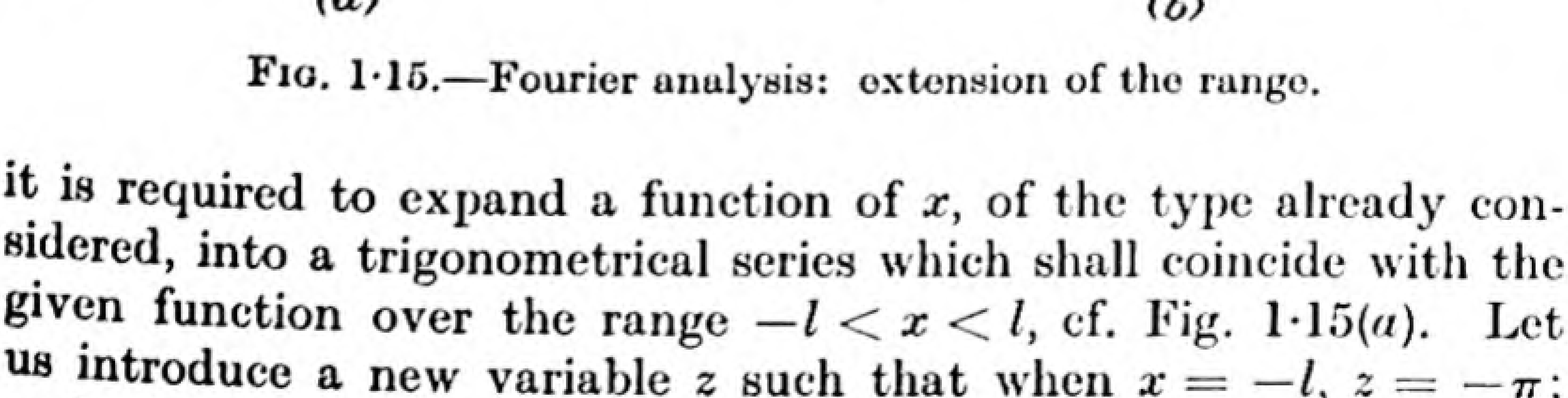


FIG. 1·15.—Fourier analysis: extension of the range.

it is required to expand a function of x, of the type already considered, into a trigonometrical series which shall coincide with the given function over the range $-l < x < l$, cf. Fig. 1·15(a). Let us introduce a new variable z such that when $x = -l$, $z = -\pi$; and when $x = l$, $z = \pi$. Then

$$\frac{x}{z} = \frac{l}{\pi}, \quad \text{or} \quad z = \frac{\pi x}{l}.$$

Let $\phi(z)$ represent the curve over the range—cf. Fig. 1·15(b).

Then if $f(x)$ is the equation to the curve when x is the variable,

$$\phi(z) = y = f(x),$$

where y is the same ordinate on either diagram.

Now if $\phi(z) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nz + b_n \sin nz)$, we have, from the methods already established,

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \phi(z)\, dz, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \phi(z) \cos nz\, dz,$$

and

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \phi(z) \sin nz\, dz.$$

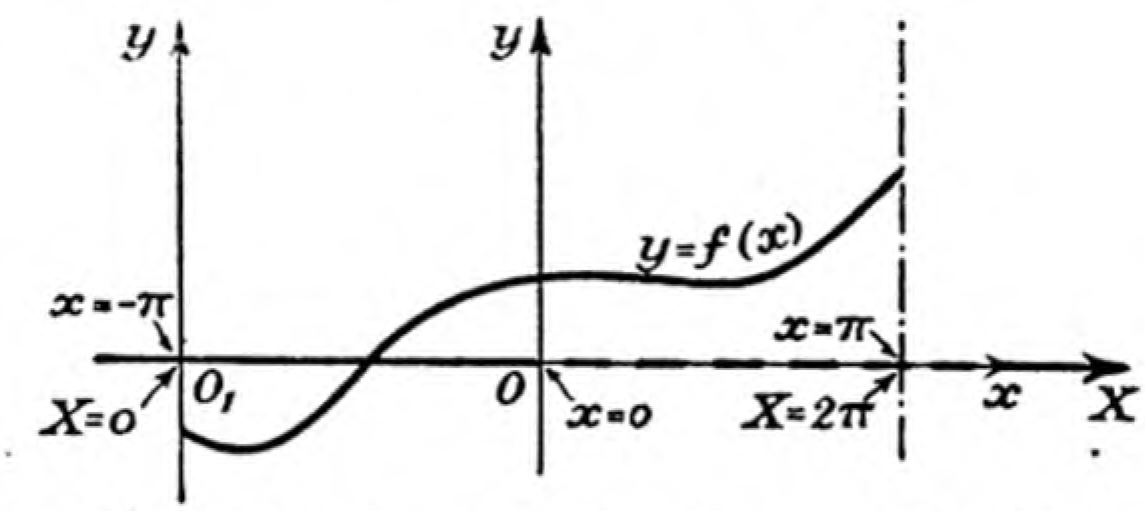


FIG. 1·16.—Fourier analysis: the range $0 < X < 2\pi$.

And since $\phi(z) = f(x)$, we have

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos \frac{n\pi}{l} x + b_n \sin \frac{n\pi}{l} x\right),$$

where, in order to calculate the a's and b's, we now use the facts $\phi(z) = f(x)$ and $\delta z = \frac{\pi}{l}\,\delta x$, so that

$$a_0 = \frac{1}{2\pi}\int_{-l}^{l} f(x)\frac{\pi}{l}\, dx = \frac{1}{2l}\int_{-l}^{l} f(x)\, dx,$$

and $$a_n = \frac{1}{\pi}\int_{-l}^{l} f(x) \cos \frac{n\pi}{l} x . \frac{\pi}{l}\, dx = \frac{1}{l}\int_{-l}^{l} f(x) \cos \frac{n\pi}{l} x\, dx.$$

Similarly,

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x) \sin \frac{n\pi}{l} x\, dx.$$

***Note on the range* $0 < X < 2\pi$.** Suppose that $y = f(x)$ over the range $-\pi < x < \pi$. Let us transfer the origin O, Fig. 1·16, to a new origin O_1, i.e. $x = -\pi$, $y = 0$ is the new origin referred to the old origin. Let (X, y) be the coordinates of a point (x, y) when it is referred to the new axes. Then the range is now $0 < X < 2\pi$, and $y = F(X)$.

If $$y = a_0 + a_1 \cos x + a_2 \cos 2x + \dots$$
$$+ b_1 \sin x + b_2 \sin 2x + \dots$$

then $$y = a_0 + a_1 \cos (X - \pi) + a_2 \cos 2(X - \pi) + \dots$$
$$+ b_1 \sin (X - \pi) + b_2 \sin 2(X - \pi) + \dots$$
$$= a_0 - a_1 \cos X + a_2 \cos 2X - \dots$$
$$+ (-b_1) \sin X + b_2 \sin 2X - \dots,$$

so that the form of the expansion is unchanged. To determine the a's and b's we have,

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_0^{2\pi} f(X - \pi)\,dX$$
$$= \frac{1}{2\pi}\int_0^{2\pi} F(X)\,dX = A_0,$$

if $$F(X) = A_0 + A_1 \cos X + A_2 \cos 2X + \dots$$
$$+ B_1 \sin X + B_2 \sin 2X + \dots$$

Also

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} f(X - \pi) \cos n(X - \pi)\,dx$$
$$= \frac{(-1)^n}{\pi}\int_0^{2\pi} F(X) \cos nX\,dX.$$

$$\therefore\ A_n = \frac{1}{\pi}\int_0^{2\pi} F(X) \cos nX\,dX.$$

Similarly for B_n.

Example.—Prove that if $-\frac{\pi}{2} < x < \frac{\pi}{2}$,

$$\frac{\pi}{4} = \cos x - \tfrac{1}{3}\cos 3x + \tfrac{1}{5}\cos 5x - \dots$$

First consider an expansion in series to represent $\frac{\pi}{4}$ over the range $0 < X < \pi$. Let

$$\frac{\pi}{4} = a_0 + \sum_{n=1}^{\infty} a_n \cos nX.$$

Then $$a_0 = \frac{1}{\pi}\int_0^{\pi} \frac{\pi}{4}\,dx = \tfrac{1}{4}\pi,$$

and this suggests at once that $a_n = 0$; this is easily verified. Let us therefore write

$$\frac{\pi}{4} = \sum_{n=1}^{\infty} b_n \sin nX.$$

Then

$$b_n = \frac{2}{\pi}\int_0^\pi \frac{\pi}{4}\sin nX\,dX = \frac{1}{2n}\Big[-\cos nX\Big]_0^\pi = -\frac{1}{2n}[\cos n\pi - \cos 0]$$

$$= \frac{1}{n}\sin \tfrac{1}{2}n\pi \sin \tfrac{1}{2}n\pi$$

$$= \frac{1}{n} \text{ if } n \text{ is odd, and } 0 \text{ if } n \text{ is even.}$$

$$\therefore \frac{\pi}{4} = \sin X + \tfrac{1}{3}\sin 3X + \tfrac{1}{5}\sin 5X + \dots$$

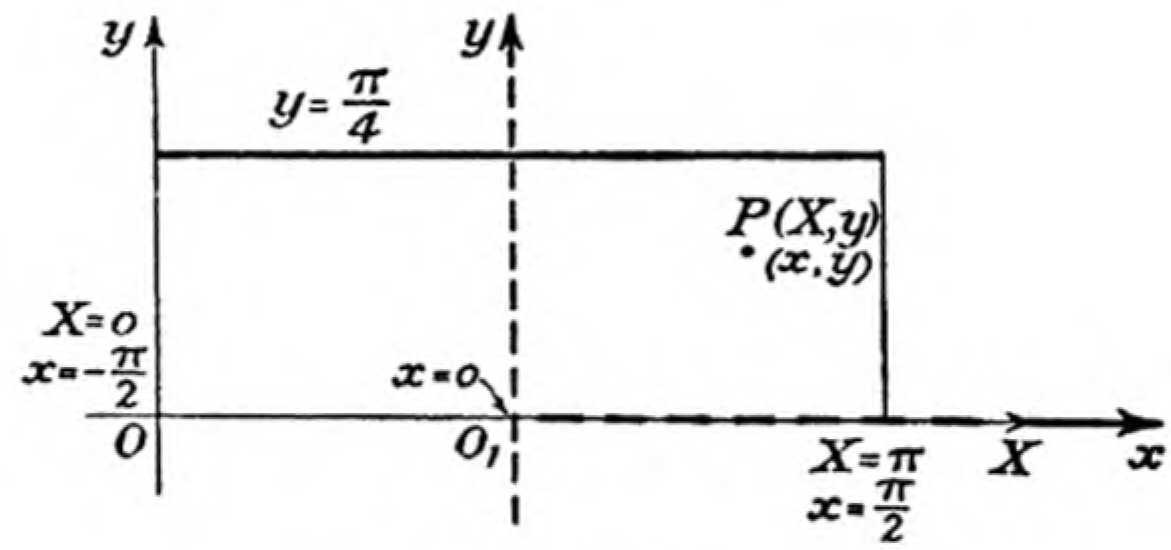


FIG. 1·17.

Now transfer the origin from O to O_1, cf. Fig. 1.17, and let (x, y) be the new coordinates of a point P which was (X, y) in the old system. Then

$$x + \frac{\pi}{2} = X.$$

$$\therefore \frac{\pi}{4} = \sin X + \tfrac{1}{3}\sin 3X + \tfrac{1}{5}\sin 5X + \dots$$

$$= \sin\left(x + \frac{\pi}{2}\right) + \tfrac{1}{3}\sin 3\left(x + \frac{\pi}{2}\right) + \tfrac{1}{5}\sin 5\left(x + \frac{\pi}{2}\right) + \dots$$

$$= \cos x - \tfrac{1}{3}\cos 3x + \tfrac{1}{5}\cos 5x - \dots$$

as required.

Some important series and their summations.—(*a*) The series $\sum_{n=1}^{\infty}\frac{1}{n^2}$ and $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$ are of frequent occurrence in physical problems and they may be evaluated as follows.

Integrating by parts it is readily shown that

$$\int_0^\pi x^2 \cos nx\,dx = 2\left[\frac{x\cos nx}{n^2}\right]_0^\pi = (-1)^n\frac{2\pi}{n^2},$$

if $n \neq 0$. Consider x^2 over the range $-\pi < x < \pi$. Then

$$f(x) = x^2 = a_0 + \sum_{n=1}^{\infty} a_n \cos nx.$$

By the rules already established for evaluating the coefficients in the above expression, we have

$$\int_{-\pi}^{\pi} x^2\,dx = a_0 . 2\pi.$$

$$\therefore\ a_0 = \frac{\pi^2}{3}.$$

Also $a_n = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} x^2 \cos nx\,dx = (-1)^n . \dfrac{4}{n^2}$.

$$\therefore\ x^2 = \frac{\pi^2}{3} + 4\left[-\frac{1}{1^2}\cos x + \frac{1}{2^2}\cos 2x - \frac{1}{3^2}\cos 3x + \ldots\right].$$

Putting $x = 0$, we have

$$0 = \frac{\pi^2}{3} - 4\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\right]$$

$$= \frac{\pi^2}{3} - 4\left[\left\{\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right\} - \left\{\frac{2}{2^2} + \frac{2}{4^2} + \frac{2}{6^2} + \ldots\right\}\right]$$

$$= \frac{\pi^2}{3} - 4\sum_{n=1}^{\infty}\frac{1}{n^2} + 4\left(\frac{2}{2^2}\right)\sum_{n=1}^{\infty}\frac{1}{n^2},$$

$$\therefore\ \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

Also, since $\displaystyle\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$

$$= \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$$

$$= \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots + \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \ldots - \frac{1}{2^2}\sum_{n=1}^{\infty}\left(\frac{1}{n^2}\right)$$

$$= \sum_{n=1}^{\infty}\frac{1}{n^2} - \frac{1}{4}\sum_{n=1}^{\infty}\left(\frac{1}{n^2}\right)$$

$$= \frac{3}{4}\sum_{n=1}^{\infty}\frac{1}{n^2},$$

we have

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.$$

(*b*) The series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^4}$ also occurs frequently in physics and the

evaluation of the summation supplies another useful example in the use of Fourier series. We have

$$\int_0^\pi x^4 \cos nx\, dx = \left[x^4 \frac{\sin nx}{n}\right]_0^\pi - \frac{4}{n}\int_0^\pi x^3 \sin nx\, dx$$

$$= (-1)^n\left[\frac{4\pi^3}{n^2} - \frac{24\pi}{n^4}\right], \ldots \quad [\text{if } n \neq 0]$$

after a few more reductions of a similar nature.

Since $$\int_0^\pi x^2 \cos nx\, dx = (-1)^n \frac{2\pi}{n^2}, \qquad [\text{if } n \neq 0]$$

we have $$\int_0^\pi (x^4 - 2\pi^2 x^2) \cos nx\, dx = (-1)^{n-1} \frac{24\pi}{n^4}.$$

Now expand $x^4 - 2\pi^2 x^2$ in a Fourier series for the range $-\pi < x < \pi$. We have, in the usual way,

$$x^4 - 2\pi^2 x^2 = a_0 + \sum_{n=1}^{\infty} a_n \cos nx,$$

the 'sine' terms being omitted since the function to be represented is an even one.

Now $$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} (x^4 - 2\pi^2 x^2)\, dx = -\frac{7}{15}\pi^4.$$

Also

$$\int_{-\pi}^{\pi} x^4 \cos nx\, dx = 2\int_0^\pi x^4 \cos nx\, dx = 2\left[(-1)^n\left(\frac{4\pi^3}{n^2} - \frac{24\pi}{n^4}\right)\right],$$

and $$-2\pi^2 \int_{-\pi}^{\pi} x^2 \cos nx\, dx = (-1)(-1)^n \frac{8\pi^3}{n^2}.$$

$$\therefore \int_{-\pi}^{\pi} (x^4 - 2\pi^2 x^2) \cos nx\, dx = (-1)^n\left(\frac{-48\pi}{n^4}\right),$$

$$\therefore a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (x^4 - 2\pi^2 x^2) \cos nx\, dx = (-1)^n\left(\frac{-48}{n^4}\right),$$

$$\therefore x^4 - 2\pi^2 x^2 = -\frac{7}{15}\pi^4 + 48\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\cos nx}{n^4}. \quad [-\pi < x < \pi].$$

Put $x = 0$. Then

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^4} = \frac{7}{15.48}\pi^4.$$

But

$$\sum_{n=1}^{\infty}\frac{1}{n^4}=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^4}+2\left[\frac{1}{2^4}+\frac{1}{4^4}+\frac{1}{6^4}+\ldots+\frac{1}{(2n)^4}+\ldots\right]$$

$$=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^4}+\frac{1}{2^3}\left[\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\ldots\frac{1}{n^4}+\ldots\right].$$

$$\therefore\ \frac{7}{8}\sum_{n=1}^{\infty}\frac{1}{n^4}=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^4}=\frac{7}{15.48}\pi^4,$$

$$\therefore\ \sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{\pi^4}{90}.$$

SOME DIFFERENTIAL EQUATIONS OF APPLIED PHYSICS

Equations of the First Order

Linear equations.—The equation

$$\frac{dy}{dx}+\mathrm{P}y=\mathrm{Q}\quad . \quad . \quad . \quad . \quad \text{(i)}$$

where P and Q are functions of x only, or constants, is a linear differential equation of the first order. If $\mathrm{Q} = 0$, (i) may be written

$$\frac{dy}{y}=-\mathrm{P}\,dx\quad . \quad . \quad . \quad . \quad \text{(ii)}$$

and this on integration leads to

$$\ln y=-\int \mathrm{P}\,dx+\mathrm{A},$$

where A is an arbitrary constant, say $\ln u$.

$$\therefore\ \frac{y}{u}=\exp\left(-\int \mathrm{P}\,dx\right),\quad \text{or}\quad y=u\exp\left(-\int \mathrm{P}\,dx\right).\quad \text{(iii)}$$

When Q differs from zero, we assume that (iii) still gives y, but now take u to be a function of x.

Now $$\frac{dy}{dx}=\frac{d}{dz}\{u\exp(-z)\}.\frac{dz}{dx}\qquad \left[\text{where } z=\int \mathrm{P}\,dx\right]$$

$$=\left[\frac{du}{dz}\exp(-z)-u\exp(-z)\right]\mathrm{P}\qquad \left[\because\ \frac{dz}{dx}=\mathrm{P}\right]$$

$$=\left[\frac{du}{dx}-u\mathrm{P}\right]\exp\left(-\int \mathrm{P}\,dx\right)\quad . \quad . \quad . \quad . \quad \text{(iv)}$$

Substituting for y and its first derivative in (i) we obtain

$$\left[\frac{du}{dx} - u\mathrm{P}\right] \exp\left(-\int \mathrm{P}\, dx\right) + \mathrm{P}u \exp\left(-\int \mathrm{P}\, dx\right) = \mathrm{Q}.$$

$$\therefore\ \frac{du}{dx} = \mathrm{Q} \exp\left(\int \mathrm{P}\, dx\right)$$

$$\therefore\ u = \mathrm{B} + \int \mathrm{Q} \exp\left(\int \mathrm{P}\, dx\right), \qquad . \qquad . \qquad \text{(v)}$$

where B is an arbitrary constant.

$$\therefore\ y = u \exp\left(-\int \mathrm{P}\, dx\right)$$

$$= \mathrm{B} \exp\left(-\int \mathrm{P}\, dx\right) + \exp\left(-\int \mathrm{P}\, dx\right) \int \mathrm{Q} \exp\left(\int \mathrm{P}\, dx\right) dx$$

is the primitive of the differential equation.

The preceding analysis shows that

$$\frac{dy}{dx} + \mathrm{P}y = \frac{du}{dx} \exp\left(-\int \mathrm{P}\, dx\right),$$

and therefore

$$\exp\left(\int \mathrm{P}\, dx\right)\left(\frac{dy}{dx} + \mathrm{P}y\right)$$

is a perfect differential, viz. $\dfrac{du}{dx}$. The quantity $\exp\left(\int \mathrm{P}\, dx\right)$ is called an ***integrating factor***, because when both sides of the equation $\dfrac{dy}{dx} + \mathrm{P}y = \mathrm{Q}$ are multiplied by this factor, we obtain an equation which can be solved by simple integration. It is important to remember that when an integrating factor (I.F.) has been used, equation (i) assumes the form

$$\frac{d}{dx}(y \times \text{I.F.}) = \mathrm{Q} \times \text{I.F.},$$

so that if $\int (\mathrm{Q} \times \text{I.F.})\, dx$ is integrable, a solution is at once obtainable.

Example.—

$$\frac{dy}{dx} + y \cos x = \tfrac{1}{2} \sin 2x.$$

Here $P = \cos x$, and since $\int \cos x \, dx = \sin x$, the integrating factor is $\exp \sin x$.

$$\therefore\ y \exp \sin x = B + \int \sin x \cos x \exp (\sin x) \, dx$$

$$= B + \int \theta \exp (\theta) \, d\theta \qquad [\text{where } \theta = \sin x]$$

$$= B + \theta \exp (\theta) - \int \exp (\theta) \, d\theta$$

$$= B + (\sin x - 1) \exp (\sin x).$$

Second Order Linear Differential Equations with Constant Coefficients

The operator D and its properties.—Before discussing the solution of second order linear differential equations with constant coefficients, e.g. equations of the type

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = \phi(x),$$

where a and b are constants and $\phi(x)$ is a function of x only, it is necessary to establish the validity of certain theorems concerning differentiation and integration.

Let D denote the operator $\dfrac{d}{dx}$, so that $Dy = \dfrac{dy}{dx}$. Then D^2 denotes the operator $\dfrac{d^2}{dx^2}$; thus $D^2y = \dfrac{d^2y}{dx^2}$ and proceeding in the same way, we have

$$D^ny = \frac{d^ny}{dx^n}.$$

If u and v are differentiable functions of x, we have

$$D(u + v) = Du + Dv,$$

$$D^mD^nu = D^{m+n}u = D^nD^mu,$$

if m and n are positive integers, and

$$D(\kappa u) = \kappa Du,$$

provided κ is a constant.

From these results it appears that the operator D obeys the fundamental laws of algebra, but it should be observed that D does not commute with functions of x. Thus $D(xy)$ is not equal to $x(Dy)$ but to $y + x\,Dy$.

Making use of these properties we may write

$$\frac{du}{dx} + \alpha u = (\mathrm{D} + \alpha)u,$$

$$\begin{aligned}\frac{d^2u}{dx^2} - \alpha^2 u &= (\mathrm{D}^2 - \alpha^2)u \\ &= (\mathrm{D} - \alpha)(\mathrm{D} + \alpha)u \\ &= (\mathrm{D} + \alpha)(\mathrm{D} - \alpha)u,\end{aligned}$$

and $$\frac{d^2u}{dx^2} - (\alpha + \beta)\frac{du}{dx} + \alpha\beta u = (\mathrm{D} - \alpha)(\mathrm{D} - \beta)u,$$

where α and β are constants. In interpreting these expressions it must be noted that a factor such as $(\mathrm{D} - \alpha)$ in the last equation operates on all the functions which follow it.

The inverse operator D^{-1}.—The symbol D^{-1} is equivalent to an integration, for if $v = \mathrm{D}u$ and we write $u = \mathrm{D}^{1}v$, we have

$$v = \mathrm{D}u = \mathrm{D}.\mathrm{D}^{-1}v.$$

Hence the symbol D^{-1} must be such that if D^{-1} operates upon v so that $\mathrm{D}^{-1}v$ is the result and this new quantity is operated upon by D, the original quantity v is obtained. Thus the operation D^{-1} is equivalent to an integration, but since the special object of these operators is to find an integral and not a complete integral, the arbitrary constant which arises in integration is made zero. We therefore have

$$\mathrm{D}^{-1}(2x) = x^2; \quad \mathrm{D}^{-1}\cos x = \sin x; \quad \mathrm{D}^{-1}\exp \alpha x = \alpha^{-1}\exp \alpha x.$$

When the inverse operation under review is repeated, we denote it by $\mathrm{D}^{-2}u$; it is the result of integrating u twice with respect to x, arbitrary constants being omitted. Hence

$$\mathrm{D}^{-2}(2x) = \mathrm{D}^{-1}(x^2) = \tfrac{1}{3}x^3;$$

and $$\mathrm{D}^{-2}\exp \alpha x = \mathrm{D}^{-1}\{\alpha^{-1}\exp \alpha x\} = \alpha^{-2}\exp \alpha x.$$

Similarly, $$\mathrm{D}^{-n}x = \frac{x^{n+1}}{(n+1)!}.$$

Hence, with the restrictions indicated above,

$$\mathrm{D}^m\mathrm{D}^n u = \mathrm{D}^{m+n}u,$$

for all integral values of m and n.

The operator $(D - \alpha)^{-1}$.—The operator denoted sybolically by $(D - \alpha)^{-1}$, where α is any constant, is one which is reversed by the subsequent application of the operator $(D - \alpha)$. Thus

$$(D - \alpha)[(D - \alpha)^{-1}u] = u,$$

which may be written

$$(D - \alpha)v = u,$$

where
$$v = (D - \alpha)^{-1}u.$$

Now if we regard D as an ordinary number and expand $(D - \alpha)^{-1}$, we get

$$(D - \alpha)^{-1} = \frac{1}{-\alpha\left(1 - \frac{D}{\alpha}\right)} = -\frac{1}{\alpha}\left[1 + \frac{D}{\alpha} + \frac{D^2}{\alpha^2} + \frac{D^3}{\alpha^3} + \ldots\right].$$

Thus

$$(D - \alpha)^{-1}x^2 = \frac{(-1)x^2}{\alpha\left(1 - \frac{D}{\alpha}\right)} = -\left[\frac{1}{\alpha} + \frac{D}{\alpha^2} + \frac{D^2}{\alpha^3} + \frac{D^3}{\alpha^4} + \ldots\right]x^2$$

$$= -\left[\frac{x^2}{\alpha} + \frac{2x}{\alpha^2} + \frac{2}{\alpha^3} + 0\right].$$

In carrying out operations of this kind the result is always finite for, in general, $D^n x^p = 0$ when $n > p$.

General theorems concerning the operator D.—(*a*) To show that

$$\phi(D) \exp \lambda x = \phi(\lambda) \exp \lambda x,$$

where $\phi(x)$ is a rational function of x, i.e. say,

$$\phi(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots$$
$$+ b_1x^{-1} + b_2x^{-2} + b_3x^{-3} + \ldots,$$

the a's and b's being independent of x.

Now $\quad D \exp \lambda x = \lambda \exp \lambda x, \quad$ and $\quad D^{-1} \exp \lambda x = \lambda^{-1} \exp \lambda x.$

$$\therefore\ D^n \exp \lambda x = \lambda^n \exp \lambda x,$$

whether n is a positive or negative integer. Hence

$$\begin{aligned}\phi(D) \exp \lambda x &= [a_0 + a_1D + a_2D^2 + a_3D^3 + \ldots \\ &\quad + b_1D^{-1} + b_2D^{-2} + b_3D^{-3} + \ldots] \exp \lambda x \\ &= (a_0 + a_1\lambda + a_2\lambda^2 + a_3\lambda^3 + \ldots \\ &\quad + b_1\lambda^{-1} + b_2\lambda^{-2} + b_3\lambda^{-3} + \ldots) \exp \lambda x \\ &= \phi(\lambda) \exp \lambda x. \qquad \text{(i)}\end{aligned}$$

(*b*) To show that

$$\phi(D)\{\exp \lambda x . X\} = \exp \lambda x . \phi(D + \lambda)X,$$

where X is any rational function of x.

Now $$D\{\exp \lambda x . X\} = \lambda \exp \lambda x . X + \exp \lambda x . DX$$
$$= \exp \lambda x . (D + \lambda)X,$$

which may be written

$$\exp(-\lambda x)D\{\exp \lambda x . X\} = (D + \lambda)X, \quad . \quad . \quad \text{(ii)}$$

so that the effect of operating on $\exp \lambda x . X$ with $[\exp(-\lambda x) . D]$ is to give $(D + \lambda)X$.

Again $$D^2(\exp \lambda x . X) = D\{\lambda \exp \lambda x . X + \exp \lambda x . DX\}$$
$$= \lambda^2 \exp \lambda x . X + \lambda \exp \lambda x . DX$$
$$+ \lambda \exp \lambda x . DX + \exp \lambda x . D^2X$$
$$= \exp \lambda x . (D + \lambda)^2 X.$$

These results suggest

$$D^n(\exp \lambda x . X) = \exp \lambda x . (D + \lambda)^n X, \quad . \quad . \quad \text{(iii)}$$

provided n is a positive integer. If this is assumed to be true we may operate again to get

$$D^{n+1}(\exp \lambda x . X) = D\{\exp \lambda x . (D + \lambda)^n X\}$$
$$= \exp \lambda x . (D + \lambda)\{(D + \lambda)^n X\}, \quad \text{using equation (ii)}$$
$$= \exp \lambda x . (D + \lambda)^{n+1} X.$$

Hence, if equation (iii) is true for any given positive value of n it is true for the next higher value. It has, however, been shown to be true for $n = 1$ and $n = 2$; it is therefore true for all positive integral values of n.

[In passing it should be noted that the operation expressed by

$$\phi(D) \exp \lambda x . X = \exp \lambda x . \phi(D + \lambda)X$$

is an illustration of the fact that D and functions of D do not commute with functions of x, since

$$\phi(D)\{\exp \lambda x . X\} \neq \exp(\lambda x) . \phi(D)\{X\}.$$

Second order linear differential equations with constant coefficients.—This class of differential equation is represented by

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = X, \quad . \quad . \quad . \quad \text{(i)}$$

where a and b are constants and X is a function of x only.

Let η be any particular value of y which satisfies (i). Let $y = \eta + Y$. Then substituting in (i) we obtain

$$\left(\frac{d^2\eta}{dx^2} + a\frac{d\eta}{dx} + b\eta\right) + \left(\frac{d^2Y}{dx^2} + a\frac{dY}{dx} + bY\right) = X \quad . \qquad \text{(ii)}$$

But since η satisfies (i), so that $\frac{d^2\eta}{dx^2} + a\frac{d\eta}{dx} + b\eta = X$, it follows

$$\frac{d^2Y}{dx^2} + a\frac{dY}{dx} + bY = 0,$$

and this equation must be solved before the complete solution of equation (i) is obtained.

The complete solution of (i) therefore consists of two parts:

(α) the quantity Y, the so-called ***complementary function***, which is the solution of the given equation when the right-hand side is made zero.

(β) the quantity η, the so-called ***particular integral***, which is any solution whatsoever of the original equation.

The sum of these two parts is the complete solution or primitive of the given differential equation.

Methods for finding the complementary function.—It has just been shown that

$$(D^2 + aD + b)y = 0 \quad . \quad . \quad . \qquad \text{(i)}$$

is the differential equation to be solved in order to obtain the complementary function for the differential equation

$$(D^2 + aD + b)y = X.$$

Now $\quad (D^2 + aD + b) \exp \lambda x = (\lambda^2 + a\lambda + b) \exp \lambda x,$

by the second theorem on p. 31. Let λ_1 and λ_2 be the roots of the equation $\lambda^2 + a\lambda + b = 0$. Then (i) becomes

$$(D - \lambda_1)(D - \lambda_2)y = 0.$$

Let $(D - \lambda_2)y = z$. Then $(D - \lambda_1)z = 0$.
Hence

$$\frac{dz}{dx} - \lambda_1 z = 0.$$

$\therefore\ z = A \exp \lambda_1 x,\quad$ where A is an arbitrary constant,

and therefore

$$(D - \lambda_2)y = A \exp \lambda_1 x,$$

which is $\qquad \dfrac{dy}{dx} - \lambda_2 y = A \exp \lambda_1 x.$

Multiplying this equation throughout by the integrating factor $\exp(-\lambda_2 x)$, and integrating with respect to x, we obtain

$$y \exp(-\lambda_2 x) = A \int \exp[(\lambda_1 - \lambda_2)x]\, dx$$

$$= B + \frac{A}{(\lambda_1 - \lambda_2)} \exp(\lambda_1 - \lambda_2)x, \quad \text{where B is a constant.}$$

$$\therefore\ y = B \exp \lambda_2 x + C \exp \lambda_1 x, \qquad \text{(ii)}$$

$$\text{where } C = \frac{A}{(\lambda_1 - \lambda_2)}.$$

If λ_1 and λ_2 are different, (ii) is the complete solution. If, however, $\lambda_1 = \lambda_2$, then $y = (B + C) \exp \lambda_2 x$, and since $B + C = C_1$ (say) is a single arbitrary constant, the above expression for y cannot be the complete solution of the given equation. To obtain the complete solution under these conditions, we write

$$(D - \lambda_1)^2 y = 0.$$

Hence $\quad (D - \lambda_1)z = 0, \quad \text{where } (D - \lambda_1)y = z,$

or $\quad z = A \exp \lambda_1 z.$

Substituting this value of z in $(D - \lambda_1)y = z$ we get

$$\frac{dy}{dx} - \lambda_1 y = A \exp \lambda_1 x.$$

The integrating factor is $\exp(-\lambda_1 x)$ so that

$$y \exp(-\lambda_1 x) = \int A\, dx = B + Ax.$$

$$\therefore\ y = (B + Ax) \exp \lambda_1 x.$$

If both roots are imaginary (imaginary roots occur in pairs), let $\lambda_1 = \theta + j\phi$ and $\lambda_2 = \theta - j\phi$, where $j = \sqrt{-1}$.

Then the solution to the differential equation we are discussing is

$$\begin{aligned} y &= A_1 \exp(\lambda_1 x) + A_2 \exp(\lambda_2 x) \\ &= \exp(\theta x)[(A_1 + A_2)\cos \phi x + j(A_1 - A_2)\sin \phi x] \\ &= \exp(\theta x)[F \cos \phi x + G \sin \phi x], \end{aligned}$$

where

$$F = A_1 + A_2, \quad G = -j(A_1 - A_2).$$

F and G are arbitrary constants and at first sight it looks as if G

must be imaginary, but this is not necessarily so. For example, if

$$A_1 = 3 + 2j \quad \text{and} \quad A_2 = 3 - 2j,$$

$$F = 6 \quad \text{and} \quad G = -4.$$

It is sometimes more convenient to write the solution in the form

$$y = C \exp \theta x \cos (\phi x + \alpha),$$

where C and α are arbitrary constants.

Examples.—(i) $$D^2y + 4Dy + 3y = 0.$$

The equation for λ is

$$\lambda^2 + 4\lambda + 3 = 0,$$

so that

$$\lambda = -1, \quad \text{or} \quad -3.$$

$\therefore\ y = A \exp(-x) + B \exp(-3x)$, where A and B are arbitrary constants.

(ii) $$(D^2 + \beta)y = 0.$$

Here the equation for λ is

$$\lambda^2 + \beta = 0, \qquad \text{which gives } \lambda = \pm j\beta^{\frac{1}{2}}.$$

Hence $y = A \cos (\beta^{\frac{1}{2}} x + \phi)$, where A and ϕ are constants.

(iii) $$\frac{d^2s}{dt^2} + \alpha \frac{ds}{dt} + \beta s = 0 \quad \text{or} \quad \ddot{s} + \alpha \dot{s} + \beta s = 0.$$

Then $$\lambda^2 + \alpha\lambda + \beta = 0, \quad \text{or} \quad \lambda = \frac{-\alpha \pm \sqrt{\alpha^2 - 4\beta}}{2}.$$

If $\alpha^2 > 4\beta$, $$s = \exp(-\tfrac{1}{2}\alpha t)[A \exp(\tfrac{1}{2}\sqrt{\alpha^2 - 4\beta}\,.\,t) + B \exp(-\tfrac{1}{2}\sqrt{\alpha^2 - 4\beta}\,.\,t)]$$

If $\alpha^2 < 4\beta$, $$\begin{aligned} s &= \exp(-\tfrac{1}{2}\alpha t)[A \exp(\tfrac{1}{2}j\sqrt{4\beta - \alpha^2}\,.\,t) + B \exp(-\tfrac{1}{2}j\sqrt{4\beta - \alpha^2}\,.\,t)] \\ &= C \exp(-\tfrac{1}{2}\alpha t) \cos(\tfrac{1}{2}\sqrt{4\beta^2 - \alpha^2}\,.\,t + \phi), \end{aligned}$$

where A, B, C and ϕ are constants.

Methods for finding the particular integral.—The equation to be solved is

$$(D^2 + \alpha D + \beta)y = f(x),$$

where $f(x)$ is a function of x only. Hence

$$y = \frac{f(x)}{D^2 + \alpha D + \beta}.$$

The following examples illustrate the methods to be adopted to evaluate the above when $f(x)$ is a polynomial in x or contains a factor which is an exponential function of x.

Example (i).— $(D^2 - 6D + 9)y = x^2.$

For the particular integral

$$y = \frac{x^2}{(3 - D)^2} = \frac{1}{9}\left[1 - \frac{D}{3}\right]^{-2} x^2$$
$$= \tfrac{1}{9}[1 + \tfrac{2}{3}D + \tfrac{1}{3}D^2 + \ldots]x^2$$
$$= \tfrac{1}{9}(x^2 + \tfrac{4}{3}x + \tfrac{2}{3}).$$

Since the complementary function is $(A + Bx) \exp 3x$, the primitive of the given equation is

$$y = (A + Bx) \exp 3x + \tfrac{1}{9}(x^2 + \tfrac{4}{3}x + \tfrac{2}{3}).$$

When $f(x)$ contains an exponential factor, let $f(x) = \{\exp(\kappa x)\}X$, where X is a function of x only. Then

$$y = \frac{1}{(D^2 + \alpha D + \beta)}\{\exp(\kappa x)\}X$$
$$= \exp \kappa x \,.\, \frac{1}{(D + \kappa)^2 + \alpha(D + \kappa) + \beta} \,.\, X \qquad \text{[cf. p. 32].}$$

Example (ii).— $(D^2 - 5D + 6)y = x \exp x.$

For the particular integral, we have

$$y = \frac{x \exp x}{D^2 - 5D + 6} = \exp x \,.\, \frac{1}{(D + 1)^2 - 5(D + 1) + 6} \,.\, x$$
$$= \exp x \,\{\tfrac{1}{2}(1 - \tfrac{3}{2}D + \tfrac{1}{2}D^2)^{-1}\}x$$
$$= \tfrac{1}{2} \exp x(1 + \tfrac{3}{2}D + \ldots)x = \tfrac{1}{2} \exp x(x + \tfrac{3}{2}).$$

Hence the primitive is

$$y = A_1 \exp 2x + A_2 \exp 3x + \tfrac{1}{2} \exp x(x + \tfrac{3}{2}).$$

When $f(x) = X \cos(mx + \kappa)$, where X is a function of x only, the particular integral is

$$y = \frac{1}{\psi(D)} X \cos(mx + \kappa), \quad \text{where } \psi(D) = (D^2 + \alpha D + \beta).$$

Hence y is the real part of $\dfrac{1}{\psi(D)} X \exp j(mx + \kappa)$

$$= \exp j(mx + \kappa) \,.\, \frac{1}{\psi(D + jm)} \,.\, X$$
$$= \{\exp j(mx + \kappa)\}\,(u + jv),$$

where u and v are functions of x.

$$\therefore\ y = u \cos(mx + \kappa) - v \sin(mx + \kappa).$$

Usually X is a constant, say C, then

$$\frac{1}{\psi(D + jm)} . C = \frac{1}{\psi(jm)} . C.$$

Example (iii).— $\dfrac{d^2s}{dt^2} + \alpha \dfrac{ds}{dt} + \beta s = \gamma \cos \omega t.$

To find the particular integral, we have

$$\frac{\gamma \exp j\omega t}{D^2 + \alpha D + \beta} = \exp j\omega t \, . \, \frac{1}{(D + j\omega)^2 + \alpha(D + j\omega) + \beta} \, . \, \gamma$$

$$= \exp j\omega t \frac{\gamma}{(\beta - \omega^2) + j\alpha\omega}$$

$$= \frac{\gamma(\cos \omega t + j \sin \omega t)(\beta - \omega^2 - j\alpha\omega)}{(\beta - \omega^2)^2 + \alpha^2\omega^2}.$$

The real part of the above expression gives

$$s = \frac{\gamma[(\beta - \omega^2) \cos \omega t + \alpha\omega \sin \omega t]}{(\beta - \omega^2)^2 + \alpha^2\omega^2}$$

$$= \frac{\gamma}{\sqrt{(\beta - \omega^2)^2 + \alpha^2\omega^2}} \cos (\omega t - \theta), \qquad \text{where } \tan \theta = \frac{\alpha\omega}{\beta - \omega^2}.$$

The complete solution is

$$s = \exp(-\tfrac{1}{2}\alpha t)[A \exp(\tfrac{1}{2}\sqrt{\alpha^2 - 4\beta} \, . \, t) + B \exp(-\tfrac{1}{2}\sqrt{\alpha^2 - 4\beta} \, . \, t)]$$

$$+ \frac{\gamma}{\sqrt{(\beta - \omega^2)^2 + \alpha^2\omega^2}} \, . \cos (\omega t - \theta).$$

[It is often more convenient to write $\beta = \omega_0^2$ when the particular integral assumes the more symmetrical form

$$s = \frac{\gamma}{\sqrt{(\omega_0^2 - \omega^2)^2 + \alpha^2\omega^2}} \cos (\omega t - \theta).]$$

Partial differential equations.—A partial differential equation is one involving partial differential coefficients. The only types here considered are those which may be solved by successive integration with respect to each of the variables, or by some elementary method. The following examples show how some of the partial differential equations of applied physics may be solved.

Examples.—(i) Solve $\dfrac{\partial^2 z}{\partial x \, \partial y} = 0.$

Integrating with respect to y we obtain

$$\frac{\partial z}{\partial x} = A,$$

where A may be any arbitrary function of x not containing y, for the partial derivative with respect to y of such a function is zero. We therefore have

$$\frac{\partial z}{\partial x} = \psi(x),$$

where $\psi(x)$ is an arbitrary function of x alone. Integrating with respect to x, we have

$$z = \int \psi(x)\, dx + \phi_2(y),$$

where $\phi_2(y)$ is a function of y only. This equation may be written

$$z = \phi_1(x) + \phi_2(y).$$

(ii) If $\left(\frac{\partial}{\partial t} - \lambda\right) z = 0$, where λ is independent of t, then

$$z = \mathrm{A} \exp \lambda t,$$

where A is independent of t.

Hence in a solution of the differential equation $\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) z = 0$, A can be made an arbitrary function of x, say $\phi(x)$.

$$\begin{aligned}
\therefore\ z &= \phi(x) \exp\left(ct\frac{\partial}{\partial x}\right) \\
&= \left[1 + \frac{ct\frac{\partial}{\partial x}}{1!} + \frac{c^2t^2\frac{\partial^2}{\partial x^2}}{2!} + \ldots\right]\phi(x) \\
&= \phi(x) + \frac{ct}{1!}\phi'(x) + \frac{c^2t^2}{2!}\phi''(x) + \ldots \\
&= \phi(x + ct),
\end{aligned}$$

by Taylor's theorem. This is the solution required.

(iii) **The equation of wave-motion.**—A partial differential equation of paramount importance in wave theory is

$$\frac{\partial^2\xi}{\partial t^2} - c^2\frac{\partial^2\xi}{\partial x^2} = 0,$$

where ξ is the displacement, t the time and x defines the position at which the displacement is ξ.

To solve this equation we write

$$v = (x - ct), \qquad \text{and} \qquad w = (x + ct).$$

Then
$$\frac{\partial\xi}{\partial v} = \frac{\partial\xi}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial\xi}{\partial t}\frac{\partial t}{\partial v} = \frac{\partial\xi}{\partial x} - \frac{1}{c}\frac{\partial\xi}{\partial t}.$$

$$\begin{aligned}
\therefore\ \frac{\partial}{\partial w}\left(\frac{\partial\xi}{\partial v}\right) &= \frac{\partial}{\partial x}\left[\frac{\partial\xi}{\partial x} - \frac{1}{c}\frac{\partial\xi}{\partial t}\right]\frac{\partial x}{\partial w} + \frac{\partial}{\partial t}\left[\frac{\partial\xi}{\partial x} - \frac{1}{c}\frac{\partial\xi}{\partial t}\right]\frac{\partial t}{\partial w} \\
&= \left[\frac{\partial^2\xi}{\partial x^2} - \frac{1}{c}\frac{\partial^2\xi}{\partial x\,\partial t}\right] + \left[\frac{\partial^2\xi}{\partial x\,\partial t} - \frac{1}{c}\frac{\partial^2\xi}{\partial t^2}\right]\frac{1}{c},
\end{aligned}$$

i.e.
$$\frac{\partial^2\xi}{\partial w\,\partial v} = \frac{\partial^2\xi}{\partial x^2} - \frac{1}{c^2}\cdot\frac{\partial^2\xi}{\partial t^2} = 0.$$

$$\therefore\ \xi = \phi_1(x - ct) + \phi_2(x + ct) \qquad \text{[cf. Ex. (i)]}.$$

An alternative method of deriving a solution to the equation of wave-motion is as follows. The equation may be written in the symbolic form

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\xi = 0.$$

It is known that the solution of

$$\left(\frac{\partial}{\partial t} - \lambda_1\right)\left(\frac{\partial}{\partial t} - \lambda_2\right)\xi = 0$$

is
$$\xi = \mathrm{A} \exp \lambda_1 t + \mathrm{B} \exp \lambda_2 t, \qquad \text{[cf. p. 34]}$$

where λ_1 and λ_2 are independent of t, and A and B are constants.

If λ_1 and λ_2 are functions of x, A and B may be functions of x, say $\phi_1(x)$ and $\phi_2(x)$. Thus the solution of

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\xi = 0$$

is
$$\begin{aligned}\xi &= \phi_1(x) \exp\left(-ct\frac{\partial}{\partial x}\right) + \phi_2(x) \exp\left(ct\frac{\partial}{\partial x}\right)\\ &= \phi_1(x - ct) + \phi_2(x + ct).\end{aligned}$$

The solution $\xi = \phi_1(x - ct)$ represents a wave travelling with velocity c in the positive direction along the x-axis. To prove this let ξ' be the displacement at a point x' at time t'. At time t'' ($>t'$) let the displacement be ξ' at x''. Then

$$\phi_1(x' - ct') = \xi' = \phi_1(x'' - ct''),$$

i.e.
$$x' - ct' = x'' - ct'',$$

or
$$c = \frac{x'' - x'}{t'' - t'}.$$

Thus, since $t'' > t'$, $x'' > x'$, i.e. the wave advances in the positive direction of the x-axis. Similarly, the solution $\xi = \phi_2(x + ct)$ is a wave travelling with velocity c in the negative direction along the x-axis.

(iv) The equation $a^2 \dfrac{\partial^2 z}{\partial x^2} = \dfrac{\partial z}{\partial t}$. This is a linear equation and it occurs in the theory of diffusion, the theory of heat conduction, etc. Now in the treatment of ordinary differential equations, solutions with exponentials occur frequently. This suggests $z = \exp(mx + nt)$ may be a solution in the present instant. Substituting, it is found that it satisfies the differential equation if $n = m^2a^2$. Thus

$$\exp(mx + m^2a^2t) \quad \text{and} \quad \exp(-mx + m^2a^2t)$$

are solutions.

To find a solution of the same equation which vanishes when $t \to \infty$, it is noticed that in the solutions just obtained t occurs in the term $\exp(m^2a^2t)$. This increases with t; to make it decrease we put $m = jp$, where $j = \sqrt{-1}$, so that $m^2a^2 = -p^2a^2$. Thus $\exp(jpx - p^2a^2t)$ and $\exp(jpx - p^2a^2t)$ are solutions. Since the equation is linear,

$$\exp(-p^2a^2t)[\mathrm{A} \exp(jpx) + \mathrm{B} \exp(-jpx)]$$

is also a solution, and this may be replaced by

$$\exp(-p^2a^2t)[\mathrm{P} \cos px + \mathrm{Q} \sin px],$$

where A, B, P and Q are constants.

The curve $y - b = \kappa(x - a)^n$.—Let Ox, Oy, Fig. 1·18, be the usual rectangular axes and let P_1 be a point on the curve $y - b = \kappa(x - a)^n$. Let OA $= a$, OB $= b$, and through A and B draw the straight lines AH and BK with slopes α and β respectively, the co-ordinates axes being considered as lines of zero slope. Let M be the projection of P_1 on Ox, and through M draw ML with slope θ to cut AH in L. Let N be the projection of L on Ox. Similarly, let R be the projection of P_1 on Oy; then draw RQ with slope ϕ with respect to Oy, β and ϕ being considered positive when the straight lines BK and RQ are as indicated. If S is the projection of Q on Oy, then NL and QS produced will meet in P_2, a point on the same curve as is P_1, provided certain conditions are fulfilled. To find these, if P_1 is (x_1, y_1) and P_2 is (x_2, y_2),

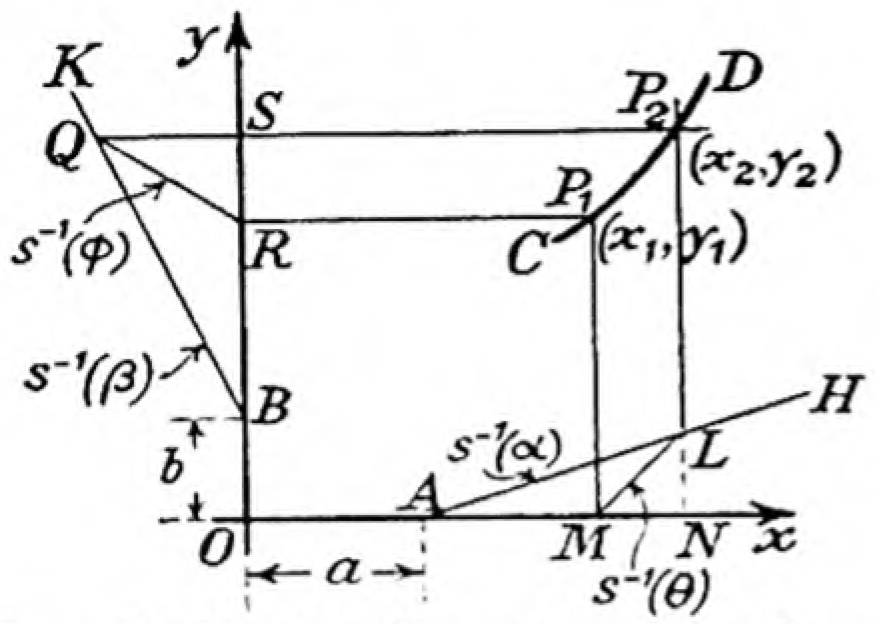


FIG. 1.18.—[$s^{-1}(\phi) \equiv$ the line whose slope is ϕ; etc.] The construction of the curve $y - b = \kappa(x - a)^n$, when one point on the curve is known.

$$x_2 = \mathrm{ON} = \mathrm{OM} + \mathrm{MN} = x_1 + \left(\frac{\mathrm{NL}}{\theta}\right) = x_1 + \left\{(x_2 - a)\frac{\alpha}{\theta}\right\},$$

and $$y_2 = \mathrm{OS} = y_1 + \mathrm{RS} = y_1 + \left(\frac{\mathrm{QS}}{\phi}\right) = y_1 + \left\{(y_2 - b)\frac{\beta}{\phi}\right\}.$$

These give

$$x_2 = \left[x_1 - a\left(\frac{\alpha}{\theta}\right)\right] \div \left[1 - \left(\frac{\alpha}{\theta}\right)\right]$$

and $$y_2 = \left[y_1 - b\left(\frac{\beta}{\phi}\right)\right] \div \left[1 - \left(\frac{\beta}{\phi}\right)\right],$$

so that $$x_2 - a = [x_1 - a] \div \left[1 - \left(\frac{\alpha}{\theta}\right)\right]$$

and $$y_2 - b = [y_1 - b] \div \left[1 - \left(\frac{\beta}{\phi}\right)\right].$$

Since $y_1 - b = \kappa(x_1 - a)^n$, we have

$$\frac{(y_2 - b)}{(x_2 - a)^n} = \left[\frac{(y_1 - b)}{1 - \left(\frac{\beta}{\phi}\right)}\right]\left[\frac{\left\{1 - \left(\frac{\alpha}{\theta}\right)\right\}^n}{(x_1 - a)^n}\right],$$

so that P_2 will lie on the curve† under discussion if

$$\left[1-\left(\frac{\alpha}{\theta}\right)^n\right]=\left[1-\left(\frac{\beta}{\phi}\right)\right].$$

In the same way, other points may be located and a portion CD of the complete curve obtained.

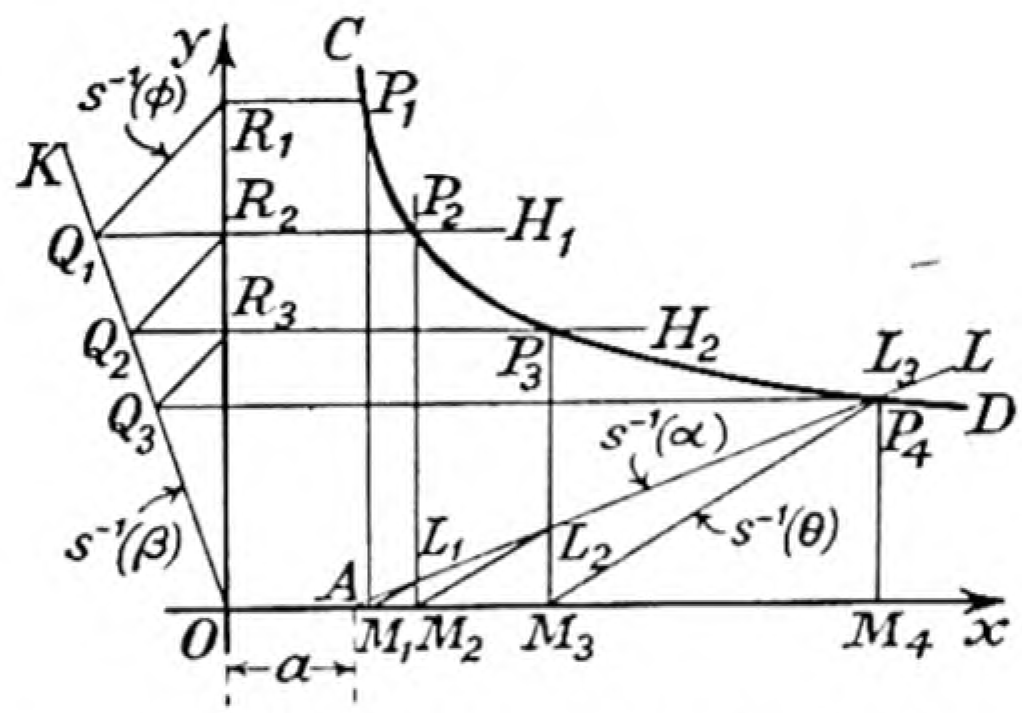


FIG. 1·19.—The curve $y = \kappa(x - a)^n$.

To determine whether or not a given curve obeys an equation of the type just discussed we have to proceed in a reverse manner, when it is soon found that it is only when a or b is zero that the method becomes a convenient one, i.e. the constants can only be determined when it is suspected that the equation to the curve is of the type

$$y = \kappa(x - a)^n, \quad \text{or} \quad y - b = \kappa x^n.$$

Thus, suppose CD, Fig. 1·19, is a smooth curve drawn through a set of 'points' determined experimentally. It is desired to test whether or not the curve may be represented, within the limits of experimental error, by the equation $y = \kappa(x - a)^n$. Since $b = 0$, a straight line OK is drawn through O, the slope of this line being chosen to be some convenient value β. Through P_1, a point on CD, draw P_1R_1 parallel to Ox, and draw R_1Q_1 to cut OK in Q_1, the actual angle OR_1Q_1 being 45°; thus ϕ, the slope of this line, is negative and is given by

$$\phi = \frac{\text{Quantity represented by } R_2Q_1\ (+ve)}{\text{Quantity represented by } R_1R_2\ (-ve)}$$

$$= -\frac{\text{Quantity represented by } |Q_1R_2|}{\text{Quantity represented by } |R_1R_2|}.$$

† Cf. Perry 'Elementary Practical Mathematics' and C. J. Smith *Phil. Mag.*, XXXVII, p. 505, 1946.

Then through Q_1 draw Q_1H_1 parallel to Ox to cut Oy in R_2 and CD in P_2. Through P_1 and P_2 draw P_1M_1 and P_2M_2 parallel to Oy and through M_1 draw M_1L_1 at an actual angle of say 30° (45° or 60° to suit the problem) to cut P_2M_2 in L_1. Then L_1 is a point on a straight line whose intercept on Ox is $OA = a$, and whose slope will be α, such that if θ is the slope of M_1L_1,

$$\left[1 - \left(\frac{\alpha}{\theta}\right)\right]^n = \left[1 - \left(\frac{\beta}{\phi}\right)\right].$$

So far, however, only one point, L_1, has been located, but by starting with P_2 (or any other point on the curve) we may locate P_3 and then L_2, by repeating the construction already described. Thus a succession of points $L_1, L_2, L_3, \ldots$ may be obtained; if they lie on a straight line then the equation $y = \kappa(x - a)^n$ fits the curve CD, and the constants a and n can be found from the known values of α, θ, β and ϕ. The constant κ may then be determined by plotting y against $(x - a)^n$, for both a and n are now known. To illustrate how this method may be used, let us consider the two following instances.

(*a*) The variation of the surface tension, γ, of a liquid with temperature θ, as measured on the centigrade nitrogen scale, is related to its surface tension γ_0 at 0° C. by the equation $\gamma = \gamma_0(1 - b\theta)^n$, where b and n are constants; cf. p. 516.

Calling $x = \dfrac{\gamma}{\gamma_0}$ and $y = \theta$, the above equation may be written

$$y = \left(\frac{1}{b}\right) - \left(\frac{1}{b}\right)x^{\frac{1}{n}}.$$

This is an equation of the type now under discussion so that the constants b and n may be found. The method is easier than that used by Ferguson† and, in addition, avoids the construction of tangents to a curve and the assumption that $b = \theta_c^{-1}$, where θ_c is the critical temperature for the liquid concerned.

(*b*) Slotte's equation, cf. p. 604, representing how the viscosity η of a liquid varies with the temperature θ (°C.) is $\eta = \eta_0(1 + b\theta)^n$, where η_0, b and n are constants. This equation may be written

$$\left[\frac{\eta}{\eta_0}\right] = b^n\left[\theta + \left(\frac{1}{b}\right)\right]^n,$$

so that if we write $y = \dfrac{\eta}{\eta_0}$ and $\theta = x$, the equation assumes the form which can be dealt with by the present graphical method.

† *Phil. Mag.* L. p. 971, 1925.

EXAMPLES I

1·01. Find the percentage error in weighing bodies of average density 10, 1 and 0·1 gm.cm.$^{-3}$ through neglecting the buoyancy of the air when this has a density 0·00125 gm.cm.$^{-3}$. Assume the density of the brass standard masses = 8·4 gm.cm.$^{-3}$. Give the essential theory. [−0·0024, 0·11, 1·24]

1·02. Define the sensitivity of a balance and obtain a formula which connects the sensitivity with other constants of the balance. Discuss how the sensitivity varies with the load.

In using a physical balance it is found that the sensitivity for zero load is 4·0 scale divisions per milligram. For a load of 100 gm. it is 3·5 divisions per milligram. Assuming the mass of each pan to be 20 gm. calculate a value for the sensitivity when the load is 50 gm. [3·7 div. per mgm.]

1·03. Explain the correction for the buoyancy of the air in an accurate determination of the mass of a body. Show that the true mass X is given by the expression

$$\mathrm{X} = \mathrm{M}\left[1 - \rho_a\left(\frac{1}{\rho_w} - \frac{1}{\rho_x}\right)\right],$$

where M is the apparent mass of the body, ρ_a, ρ_w, ρ_x are the densities of air, the standard masses and the material of the body respectively.

1·04. Prove that $[|\vec{A} \times \vec{B}|]^2 + (\vec{A} \cdot \vec{B})^2 = A^2B^2$.

1·05. If x_1, x_2 and x_3 are the scale readings for three consecutive points of rest of a damped galvanometer coil, show that the rest position of the coil is given by

$$x_0 = \frac{x_1x_3 - x_2^2}{x_1 + x_3 - 2x_2}.$$

If the damping is slight prove that

$$4x_0 = x_1 + 2x_2 + x_3,$$

and if δ is the decrement per cycle,

$$\delta(x_3 - x_2)^2 = (x_1 - x_3)(x_1 + x_3 - 2x_2).$$

1·06. Two forces F and $(1 + n)F$, where n is small, act on a particle. If θ is the angle between the lines of action of the two forces, show that their resultant makes with the larger force an angle ϕ, such that

$$\sin\phi \simeq (1 - \tfrac{1}{2}n)\sin\tfrac{1}{2}\theta.$$

1·07. Write down the first four terms in the binomial expansion of $(1 - x)^{-\frac{1}{2}}$.

By putting $x = 0{\cdot}020$, use the expression to find an approximate value of $\sqrt{2}$, and state to how many places of decimals it is accurate. [1·4142134, six.]

1·08. Determine the stationary values of the function

$$y = \exp(-ax)\sin bx,$$

where a and b are positive constants. Prove that these value form a geometrical progression whose common ratio is $-\exp\left(-\frac{\pi a}{b}\right)$ and illustrate your results by means of a diagram.

CHAPTER II

THE PRINCIPLES OF PARTICLE DYNAMICS

Rectilinear motion.—Consider the motion of a point or of a material particle along a straight line or axis Ox, Fig. 2·01, the point O being fixed. Let P_1 and P_2 be the positions of the particle at times t_1 and t_2 respectively. Then, in time $(t_2 - t_1)$ the displacement of the particle is $(x_2 - x_1)$, if $OP_1 = x_1$ and $OP_2 = x_2$. The average rate of displacement or ***average velocity*** during the time interval $(t_2 - t_1)$ is $\dfrac{(x_2 - x_1)}{(t_2 - t_1)}$. If the value u of this fraction is the same for all time intervals, the velocity of the particle is said to be uniform.

If time is measured from the instant when the particle is at O,

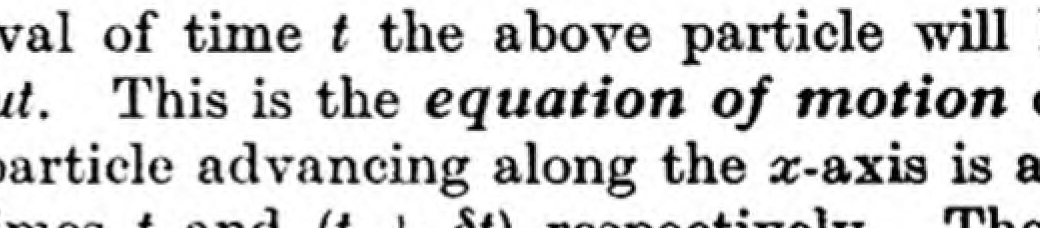


FIG. 2·01.—Rectilinear motion of a material particle.

after an interval of time t the above particle will have moved a distance $x = ut$. This is the ***equation of motion*** of the particle.

Suppose a particle advancing along the x-axis is at points x and $(x + \delta x)$ at times t and $(t + \delta t)$ respectively. Then the average velocity during the interval δt is $\dfrac{\delta x}{\delta t}$. As $\delta t \rightarrow 0$, the above fraction will tend to a limiting value, v, which is the velocity of the particle at time t. Hence

$$v = \lim_{\delta t \to 0} \frac{\delta x}{\delta t} = \frac{dx}{dt} = \dot{x}.$$

Acceleration.—This is defined as the rate of change of velocity. If v_1 and v_2 are the velocities of a moving point at times t_1 and t_2 respectively, then the average rate of change of velocity during this time interval is $\dfrac{(v_2 - v_1)}{(t_2 - t_1)}$, and the acceleration is said to be uniform when the above fraction is independent of the actual time interval selected. If v and $(v + \delta v)$ are the velocities of a moving particle at

times t and $(t + \delta t)$ respectively, the fraction $\frac{\delta v}{\delta t}$ is the average value of the acceleration during the given time interval, and the limiting value of the above fraction as $\delta t \to 0$ is the acceleration at the time t. If this is a, then

$$a = \lim_{\delta t \to 0} \frac{\delta v}{\delta t} = \frac{dv}{dt} = \dot{v} = \ddot{x}.$$

Velocity-time curves.—If a curve in which abscissae denote time and ordinates represent velocity is drawn, that curve is termed a ***velocity-time*** curve. Such a curve is shown in Fig. 2.02. Let us consider a small element PQNM of the area under the curve. Let

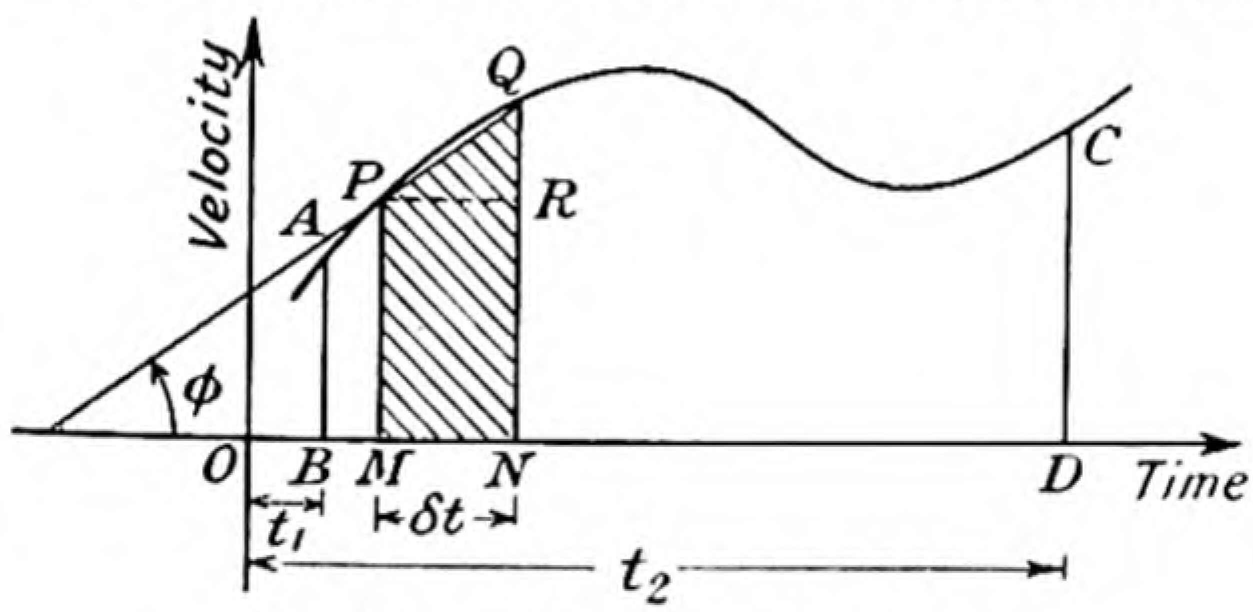


FIG. 2·02.—A velocity-time curve.

MN = δt be so small that during this interval the velocity of the moving point may be considered constant and equal to PM. If quantities of a higher order of smallness than the first are neglected, the area of this element is PM.δt which is the distance traversed in time δt. Hence, since each such element represents a distance traversed, it follows that the area included between the curve, the ordinates AB and CD, and the x-axis represents the distance traversed in time BD, i.e. in time $(t_2 - t_1)$. The above argument may be put somewhat more formally by writing

$$\text{Area} = \int_{t_1}^{t_2} v\, dt = \int_{t_1}^{t_2} \frac{dx}{dt}\, dt = \Big[x \Big]_{x_1}^{x_2}.$$

Also, the gradient of the curve or the slope of the tangent to it at a given point is $\frac{dv}{dt}$, and thus gives the acceleration at that instant.

Uniformly accelerated motion.—In uniformly accelerated motion the acceleration is constant and equal to a (say). Thus the motion is expressed either by $\dot{v} = a$ or by $\ddot{x} = a$. Integrating the former with respect to time we get

$$v = at + C,$$

where C is a constant. If the velocity is u (or $\dot{x}_0$) when $t = 0$, $C = u$, and

$$\left.\begin{aligned} v &= u + at \\ \dot{x} &= \dot{x}_0 + \ddot{x}t \end{aligned}\right\} \quad . \quad . \quad . \quad . \qquad \text{(i)}$$

or

Again, on integrating the equation $\ddot{x} = a$, if A and B are constants, we have

$$\dot{x} = at + A = at + u,$$

since at $t = 0$, $\dot{x} = u$ so that $A = u$.

Integrating again we get $x = \frac{1}{2}at^2 + ut + B$. But since $x = 0$ when $t = 0$, $B = 0$, and therefore

$$\left.\begin{aligned} x &= ut + \tfrac{1}{2}at^2 \\ &= \dot{x}_0 t + \tfrac{1}{2}\ddot{x}t^2 \end{aligned}\right\} \quad . \quad . \quad . \quad . \qquad \text{(ii)}$$

Eliminating t from (i) and (ii), we get

$$\left.\begin{aligned} v^2 &= u^2 + 2ax \\ \dot{x}^2 &= \dot{x}_0{}^2 + 2\ddot{x}x \end{aligned}\right\} \quad . \quad . \quad . \quad . \qquad \text{(iii)}$$

or

Newtonian mechanics.—The three laws of motion formulated by Newton in the seventeenth century are the foundation upon which the science of dynamics is based. The validity of this science in so far as it concerns the motion of common bodies, the so-called 'Newtonian Mechanics,' finds ample confirmation in many experiments. It is only when this mechanics is applied to the motions of the constituents of atoms or to the motions of celestial objects that small discrepancies occur; the smallness of these is realized when it is remembered that the Newtonian mechanics enabled astronomers to predict the occurrences of eclipses with an accuracy which banishes at once any suggestion that Newton's laws of motion are radically in error. The errors which do arise when the principles of Newtonian mechanics are applied to the motions of things microscopic or things macroscopic, will, in general, be neglected in the treatment which follows.

Force and the law of inertia.—Changes in the velocity (either in magnitude or direction or both) of a body are assumed to have their cause in external agencies and we say that ***forces*** are exerted by these agencies on the body whose motion is under investigation. ***Force is therefore defined as that which changes or tends to change the motion of a body.*** Newton's first law of motion (or the law of inertia) states that every body perseveres in its state of rest or of uniform motion in a straight line unless it is compelled to change that state by impressed forces.

Mass.—Matter is a primary concept and it is difficult to define it. All matter, however, is characterized by the fact that an effort is required to produce any finite change in the motion of a given piece of matter. In virtue of this property matter is said to possess ***inertia***, and the ***mass*** of a body when moving in a straight line is said to be a ***measure of its inertia***. One of the chief characteristics of any object near to the earth's surface is its ***weight***.

The measurement of force.—***Newton's second law of motion states that the rate of change of linear momentum is directly proportional to the impressed force and takes place along the direction in which the force is applied***, the momentum of a material particle being defined as the product of its mass and its velocity.

If m is the mass of a particle moving at any instant with a velocity v, its momentum at that instant is mv. If a is the acceleration of the particle and F the force producing it, then, according to the above law,

$$\mathrm{F} \propto \frac{d}{dt}(mv).$$

If the mass is constant

$$\mathrm{F} \propto m\dot{v}, \quad \text{or} \quad \mathrm{F} \propto ma.$$

If the unit of force is that force which produces unit acceleration when acting upon a body of unit mass, then

$$\mathrm{F} = ma = m\ddot{x}.$$

In the c.g.s. system this unit force is the dyne, while the poundal is the corresponding unit in the f.p.s. system.

Impulse.—***The impulse of a force is defined as the change of linear momentum produced by it.*** If the force F is constant and acts for a time t, the acceleration being $\ddot{x}$ and the velocity changing from $\dot{x}_1$ to $\dot{x}_2$, then I, the impulse of the force, is given by

$$\mathrm{I} = m(\dot{x}_2 - \dot{x}_1) = m\ddot{x}t = \mathrm{F}t.$$

If the force is variable,

$$\delta \mathrm{I} = m[(\dot{x} + \delta\dot{x}) - \dot{x}] = m\,\delta\dot{x} = m\ddot{x}\,\delta t = \mathrm{F}\,\delta t.$$

$$\therefore\ \mathrm{I} = \int_{t_1}^{t_2} \mathrm{F}\,dt,$$

the limits t_1 and t_2 being appropriate to the given problem.

Vertical motion under gravity.—The equation of motion in this instance is

$$\ddot{x} = -g,$$

if the positive direction of x is upwards and g is gravity.

$$\therefore\ \dot{x} = -gt + \mathrm{A},$$

and
$$x = -\tfrac{1}{2}gt^2 + \mathrm{A}t + \mathrm{B} \quad . \quad . \quad . \quad \text{(iv)}$$

where A and B are constants.

If, initially, $\dot{x} = u$ and $x = 0$, $\mathrm{A} = u$ and $\mathrm{B} = 0$.

$$\therefore\ \dot{x} = u - gt \quad . \quad . \quad . \quad . \quad \text{(v)}$$

and
$$x = ut - \tfrac{1}{2}gt^2 \quad . \quad . \quad . \quad . \quad \text{(vi)}$$

Equation (v) shows that after a time $\dfrac{u}{g}$, the particle has reached its highest point, for $\dot{x}$ is then zero. Using this value for t in (vi) we get for h, the maximum height reached,

$$h = \frac{u^2}{2g}.$$

Projectiles.—Let a particle be projected from a point O with an initial velocity u_0 in a direction making an angle α with the horizontal Ox, Fig. 2·03. Let u be the velocity of this particle when it is at a point (x, y) and let it then be moving in a direction making an angle θ with Ox. The initial velocity may be resolved into two components

$$y_0 = u_0 \sin \alpha,$$

and
$$x_0 = u_0 \cos \alpha.$$

The subsequent motion of the particle is expressed by

$$\ddot{y} = -g \text{ and } \ddot{x} = 0.$$

Hence, under the given initial conditions,

$$x = (u_0 \cos \alpha)t \quad \text{and} \quad y = (u_0 \sin \alpha)t - \tfrac{1}{2}gt^2.$$

The highest point in the path of the projectile is attained when $\dot{y} = 0$, i.e. when $t = \dfrac{u_0 \sin \alpha}{g}$. In this time the distance travelled in the x direction is

$$u_0 \cos \alpha \left(\frac{u_0 \sin \alpha}{g}\right) = \frac{{u_0}^2 \sin 2\alpha}{2g}.$$

The range on a horizontal plane is therefore $\dfrac{{u_0}^2 \sin 2\alpha}{g}$.

Similarly, for the highest point reached, we have

$$y_{\text{max.}} = u_0 \sin \alpha . \left(\frac{u_0 \sin \alpha}{g}\right) - \tfrac{1}{2}g\left(\frac{u_0 \sin \alpha}{g}\right)^2 = \frac{u_0{}^2 \sin^2 \alpha}{2g} .$$

To find the equation to the path of the projectile, its so-called trajectory, consider the state of affairs after a time t when the particle is at a point P.

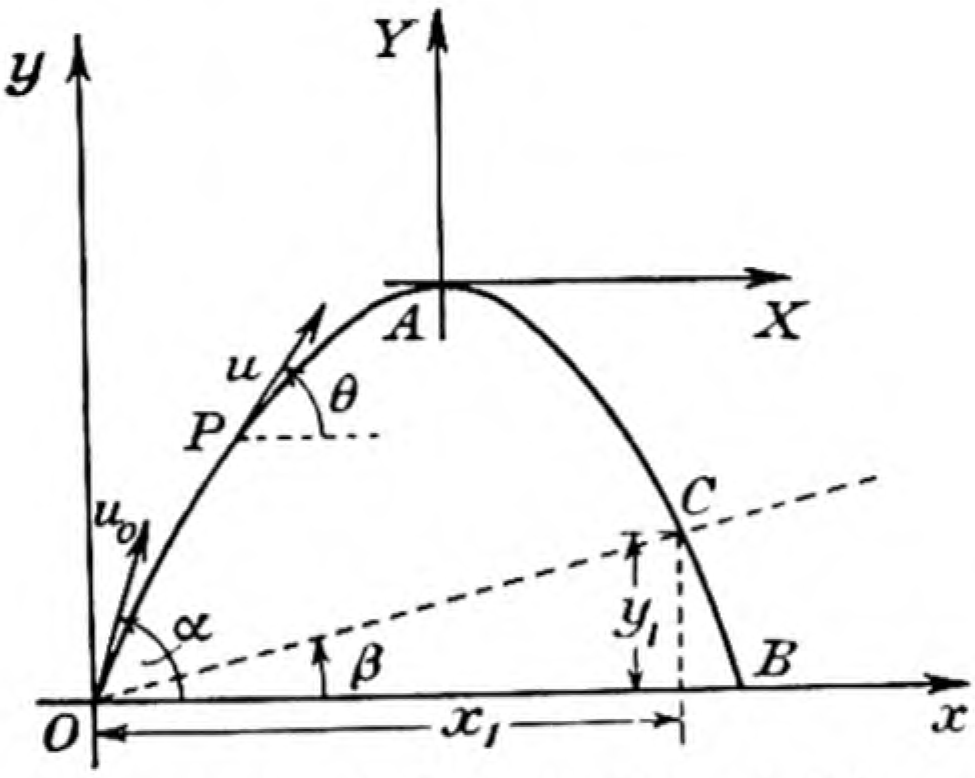


FIG. 2·03.—Trajectory of a projectile.

Then $\qquad \dot{x} = u \cos \theta = u_0 \cos \alpha,$

and $\qquad \dot{y} = u \sin \theta = u_0 \sin \alpha - gt.$

Hence $\qquad x = (u_0 \cos \alpha)t,$

and

$$\begin{aligned} y &= (u_0 \sin \alpha)t - \tfrac{1}{2}gt^2 \\ &= u_0 \sin \alpha \left(\frac{x}{u_0 \cos \alpha}\right) - \tfrac{1}{2}g \frac{x^2}{u_0{}^2 \cos^2 \alpha} \\ &= x \tan \alpha - \frac{1}{2} \frac{gx^2}{u_0{}^2 \cos^2 \alpha}, \end{aligned}$$

no integration constants being added since at $t = 0$, $x = 0$, $y = 0$.

Let us transfer the origin of coordinates from O to A, the highest point reached, and let (X, Y) be the coordinates of any point referred to a new system of axes having A as origin and directions parallel to the original axes. Then

$$\mathrm{Y} + \frac{u_0{}^2 \sin^2 \alpha}{2g} = \left(\mathrm{X} + \frac{u_0{}^2 \sin 2\alpha}{2g}\right) \tan \alpha - \tfrac{1}{2}g \frac{\left(\mathrm{X} + \dfrac{u_0{}^2 \sin 2\alpha}{2g}\right)^2}{u_0{}^2 \cos^2 \alpha} .$$

$$\therefore\ \mathrm{Y} = -\tfrac{1}{2}g \frac{\mathrm{X}^2}{u_0{}^2 \cos^2 \alpha} .$$

Hence the trajectory is a parabola lying wholly below the X-axis

of the new system of coordinates. If we write $U_0 = u_0 \cos \alpha$, the above equation becomes

$$2U_0^2 Y + gX^2 = 0.$$

Range on an inclined plane.—Let OC, Fig. 2·03, be the inclined plane making an angle β with Ox. Suppose that the projectile hits this plane at C. If (x_1, y_1) are the coordinates of this point these are determined by

$$y_1 = x_1 \tan \beta = x_1 \tan \alpha - \frac{1}{2} \frac{g x_1^2}{u_0^2 \cos^2 \alpha}.$$

Hence $\qquad x_1 = 0, \quad \text{or} \quad 2 \dfrac{(\tan \alpha - \tan \beta)}{g} u_0^2 \cos^2 \alpha.$

The range on the plane is given by

$$s = 0, \text{ or } s = x_1 \sec \beta = \frac{2u_0^2}{g}\left[\frac{\sin (\alpha - \beta) \cos \alpha}{\cos^2 \beta}\right].$$

Angular velocity.—Let S, Fig. 2·04, be a lamina or cross-section of a rigid body which always remains in the plane of the diagram.

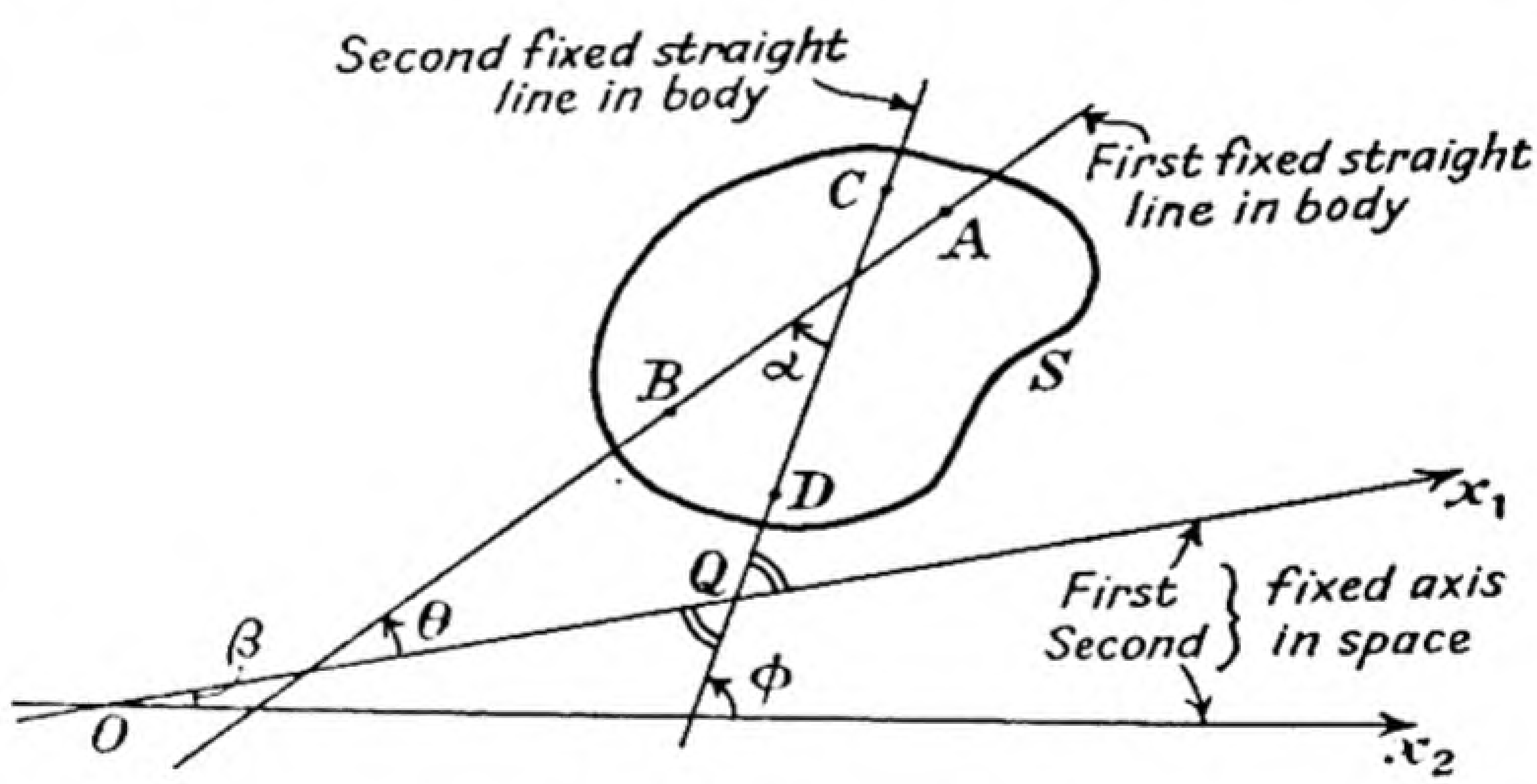


FIG. 2·04.—Angular velocity.

If θ is the angle between a straight line AB in S and fixed relative to it, and a fixed axis of reference Ox_1 in space, then

$$\dot{\theta} = \omega,$$

the ***angular velocity*** of the rotating object.

If ϕ is the angle between another straight line CD fixed in the body and a second axis Ox_2 fixed in space, then if CD and Ox_1 intersect at Q, we have

$$\theta + \alpha = \phi - \beta,$$

where α and β are the constant angles shown.

$$\therefore \ \dot{\theta} = \dot{\phi},$$

i.e. the angular velocity is independent of the two reference lines, one fixed in the body and the other fixed in space, which are selected.

The composition of angular velocities.—Let a body be subjected to two rotations simultaneously about axes OA and OB, Fig. 2·05, the angular velocities being ω_a and ω_b respectively. Consider any point P in the plane AOB. Let OA and OB be the oblique axes of coordinates to which the position of P is referred. Suppose that when viewed from O along the directions OA and

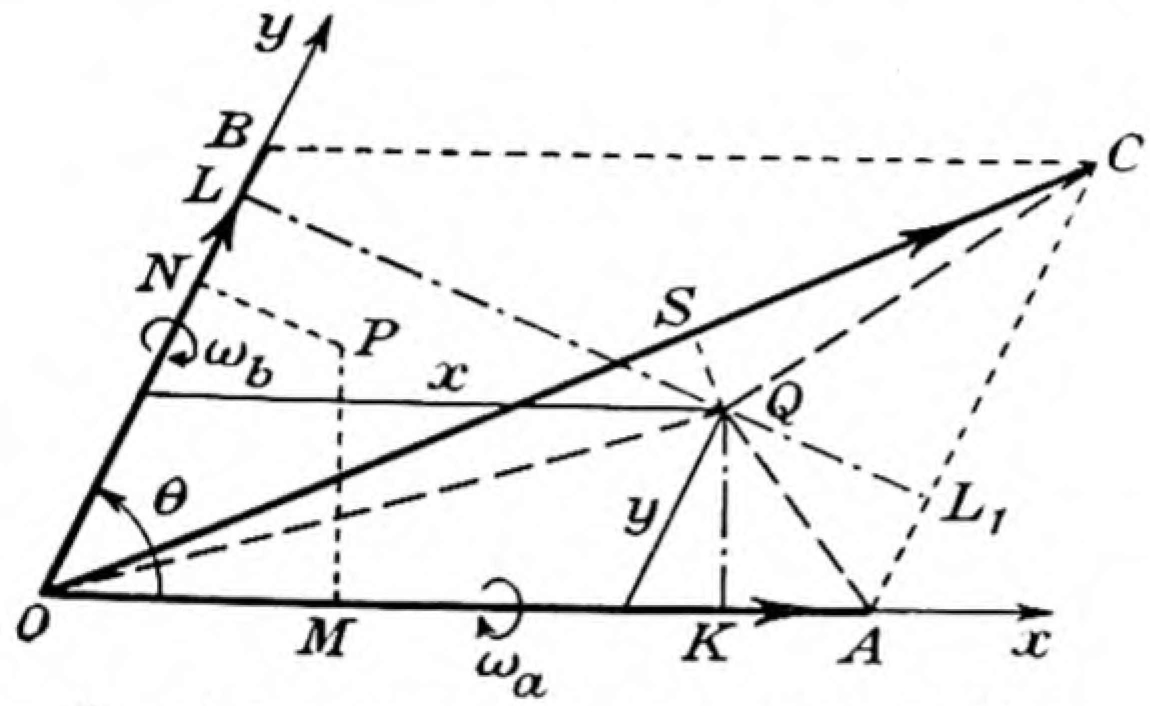


FIG. 2·05.—Composition of angular velocities.

OB the rotations are clockwise. Let OA and OB represent the magnitudes of the angular velocities. Then, due to the rotation about OA, the point P will, in time δt, be displaced upwards by an amount $\omega_a.\mathrm{PM}.\delta t$, while it will similarly suffer a displacement $\omega_b.\mathrm{PN}.\delta t$ downwards, where PM and PN are the perpendiculars from P on the axes of coordinates. Hence the total displacement upwards will be $\sin \widehat{\mathrm{AOB}}(y\omega_a - x\omega_b)$. This is zero if $y = \dfrac{x\omega_b}{\omega_a} = x\dfrac{\mathrm{OB}}{\mathrm{OA}}$; but this is the equation to the straight line OC, the diagonal of the parallelogram whose adjacent sides are OA and OB. Since any point on OC suffers no displacement OC is the axis about which the resultant angular velocity takes place. To show that OC represents the magnitude of this resultant angular velocity, let us consider the displacement of any point Q in the plane of the diagram. Through Q draw QK and QL normal to the axes, and also QS normal to OC and produce LQ to cut AC in $\mathrm{L_1}$. Then in time δt the displacement of Q upwards due to the rotations about OA and OB is $\omega_a.\mathrm{QK} - \omega_b.\mathrm{QL} = \kappa(\mathrm{OA}.\mathrm{QK} - \mathrm{OB}.\mathrm{QL})$, if κ is a constant giving the ratio of the magnitude of OA to the magnitude of ω_a, etc. Now

$$\mathrm{OA}.\mathrm{QK} - \mathrm{OB}.\mathrm{QL} = 2(\Delta\mathrm{OAQ} - \Delta\mathrm{OBC} + \Delta\mathrm{AQC}),$$

since $\mathrm{LQ} = \mathrm{LL_1} - \mathrm{QL_1}$. The above expression is equal to $-2\,\Delta\mathrm{OQC} = -\kappa\omega_c\mathrm{QS}$, which is the upward displacement of Q due

to an angular velocity ω_c about OC if OC represents the magnitude of the resultant angular velocity.

Tangential and normal accelerations.—Since the tangent and normal at a point on a curve are intrinsic to the curve and their directions do not involve any arbitrary system of coordinates, in certain problems it is often expedient to resolve the acceleration of a particle moving along the curve into components along the tangent and the normal to the curve at the point where the particle happens

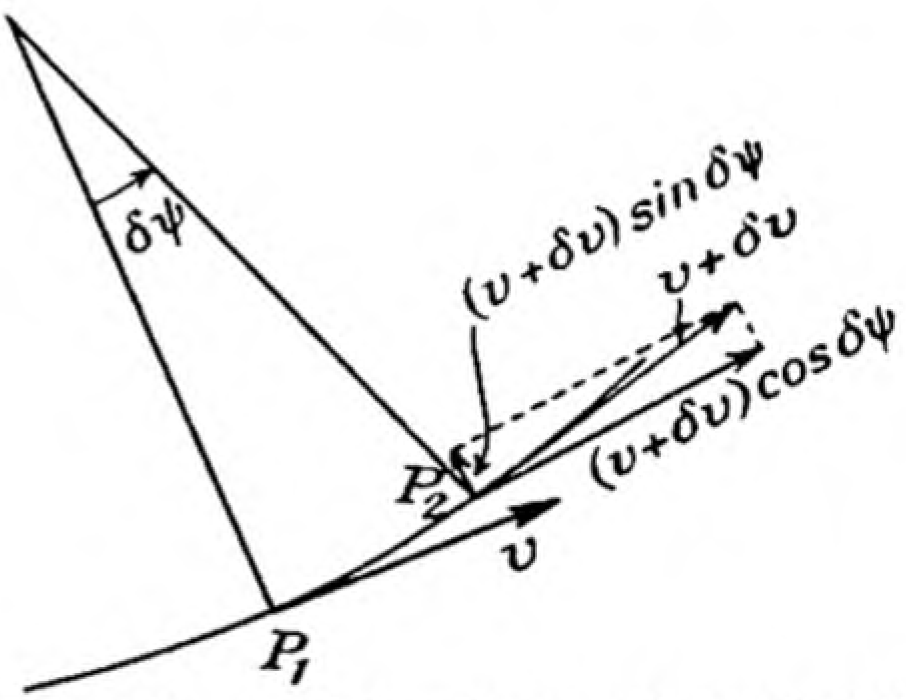


Fig. 2·06.—Tangential and normal accelerations.

to be at a given instant. Let P_1 and P_2, Fig. 2·06, be the positions of such a particle at times t and $(t + \delta t)$ and let the normals at these points meet at C, the angle between them being $\delta\psi$. Let v and $(v + \delta v)$ be the velocities (necessarily along the tangents at P_1 and P_2) at the above times. Then if δs is the distance P_1P_2 measured along the curve,

$$v = \frac{ds}{dt},$$

where $\frac{ds}{dt}$ is the limiting value of $\frac{\delta s}{\delta t}$ when $\delta t \to 0$. At P_2 the velocity $(v + \delta v)$ may be resolved into two components $(v + \delta v)\cos\delta\psi$ and $(v + \delta v)\sin\delta\psi$, parallel and perpendicular to the tangent at P_1. Since $\delta\psi$ is small, $\cos\delta\psi = 1 - \frac{\delta\psi^2}{2}$ and $\sin\delta\psi = \delta\psi$, so that if quantities of the second order of smallness are neglected, the above components are $(v + \delta v)$ and $v\delta\psi$ respectively. Hence, in a direction parallel to the tangent at P_1 the change in velocity is δv in time δt, so that the tangential acceleration at P_1 is

$$\lim_{\delta t\to 0} \frac{\delta v}{\delta t} = \frac{dv}{dt}.$$

Similarly, the normal component of the velocity changes by an

amount $v\,\delta\psi$ in time δt, so that $v.\dfrac{d\psi}{dt}$ is the normal acceleration at P_1. This may be written

$$v.\frac{d\psi}{ds}.\frac{ds}{dt}=\frac{v^2}{\rho}, \qquad \left[\because\ \frac{1}{\rho}=\frac{d\psi}{ds}\right]$$

where ρ is the radius of curvature at P_1. [N.B. This acceleration is directed towards C.]

Uniform circular motion.—For uniform circular motion v is constant, and the tangential acceleration is zero, while the normal acceleration is $\dfrac{v^2}{r}=\omega^2 r$, where r is the radius of the circle, and ω is the angular velocity. This expression may also be written $r\dot{\theta}^2$, where θ is the angular position of the moving particle with reference to an initial line passing through the centre of the circle.

Example.—To find the condition that a particle of mass m attached to a fixed point by a light string of length λ may describe with constant angular velocity ω a horizontal circle of radius a.

Let F be the tension in the string and θ the inclination of the string to the vertical. Then, resolving forces vertically and horizontally,

$$F\cos\theta = mg \quad \text{and} \quad F\sin\theta = m\omega^2 a.$$

Hence $$g\tan\theta = a\omega^2, \quad \text{where } \theta=\sin^{-1}\left(\frac{a}{\lambda}\right).$$

The balancing of rotating bodies.—Consider a material particle of mass m_1 at A, Fig. 24, rotating in a circle with constant angular velocity ω about a constant fixed axis through O the centre

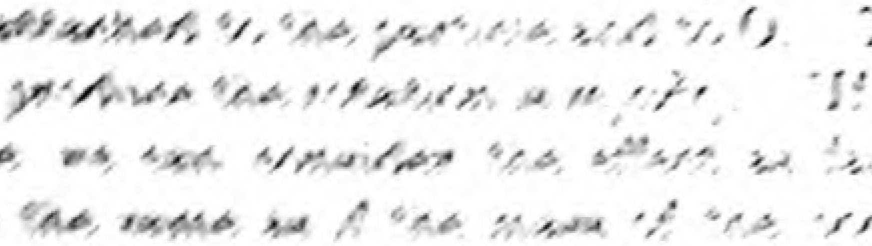


Fig. 24. The balancing of rotating bodies.

of the circle. Suppose that the constraint is provided by the tension in a light string attached to the particle and to O. The force F along AO necessary to produce the rotation is $m_1\omega^2 r_1$. If the body is not a material particle, we can consider the effect, so far as the force is concerned, to be the same as if the mass of the body were concentrated at a point at a distance r_1 from O. The corresponding force at O on the bearing (i.e. the reaction on the axle) is equal in magnitude to F, i.e. $m_1\omega^2 r_1$, and its direction is continually changing. This force, unless balanced, will tend to produce wear and may cause vibrations to be set up. Suppose, however, that a mass m_2 is placed at B, at a distance r_2 from O and on the side of O remote from A. If this mass is also rotating about the same axis through O with angular velocity ω, the total force on the axle due to the rotation of m_1 and m_2 is

$$m_1\omega^2 r_1 - m_2\omega^2 r_2.$$

and this will be zero if $m_1r_1 = m_2r_2$. The rotating masses are then said to be balanced.

Tension in a revolving hoop.—Suppose a hoop is revolving with constant angular velocity ω about an axis through its centre C and at right angles to its plane. Consider a small element δs of the hoop, Fig. 2·08, lying between A and B and subtending a small angle $\delta\theta$ at C. Let AC and BC be the normals to the hoop at these points and suppose the tangents at A and B meet in D; let AC = BC = r.

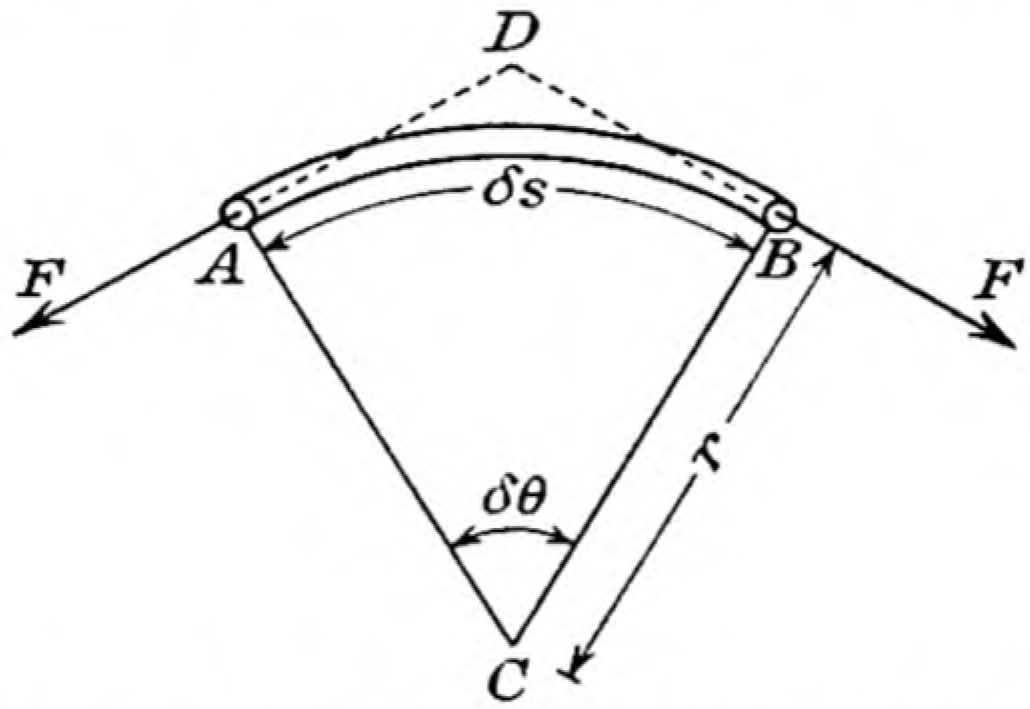


FIG. 2·08.—Tension in a revolving hoop.

Since the angular velocity of the hoop is constant the tension F in it is the same at all points; hence the resultant of the tensions at A and B is along the bisector of $\widehat{ADB}$ and equal to

$$2F \sin \tfrac{1}{2}\delta\theta = F\,\delta\theta.$$

If λ is the mass per unit length of the hoop, the ***centrepetal*** force is

$$(\lambda\,\delta s)r\dot{\theta}^2 = \lambda\,.\,r^2\dot{\theta}^2\,.\,\delta\theta,$$

and this is the resultant of the tensions at A and B. Thus

$$F = \lambda r^2\dot{\theta}^2 = \lambda v^2,$$

where v is the velocity of any point in the hoop. Thus the tension is independent of the radius of the hoop; hence when a flexible belt is running over smooth pulleys of different diameters, the tension in the belt is constant.

Slipping of a belt on a pulley.—Consider a belt ABCDEH, Fig. 2·09, passing over a circular pulley and being in contact with the pulley over an arc BE subtending an angle θ at O, the centre of the pulley. Suppose that the belt is just on the point of slipping in a clockwise direction. Let F_1 and F_2 be the tensions in the parts AB and EH of the belt. Then $F_2 > F_1$ on account of the friction between the belt and the pulley. Let μ be the coefficient of friction

and consider an infinitely small portion CD of the belt. Let $\widehat{COD} = \delta\theta$, and let F and F + δF be the tensions in the belt at C and D respectively.

$\therefore$ Force on pulley due to tensions at C and D is directed towards O and amounts to

$$F \sin\frac{\delta\theta}{2} + (F + \delta F)\sin\frac{\delta\theta}{2} = F\,\delta\theta.$$

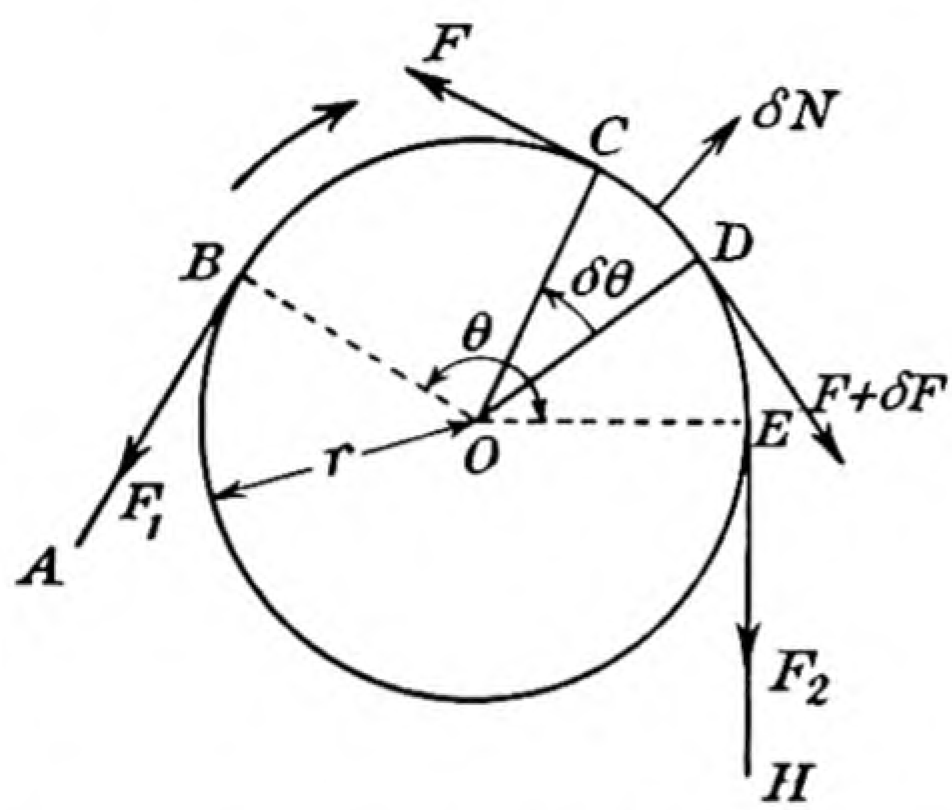


FIG. 2·09.—Slipping of a belt on a pulley.

Let δN be the normal component of the reaction of the pulley on CD, and λ the mass per unit length of the belt. Then

$$F\,\delta\theta - \delta N = (\lambda\,\delta s)r\dot{\theta}^2.$$

Also
$$\delta F = \mu\,\delta N.$$

If $v = r\dot{\theta}$, the above equations give, when one proceeds to the limit,

$$(F - \lambda v^2) = \frac{1}{\mu}.\frac{dF}{d\theta},$$

whence
$$\ln(F - \lambda v^2) = \mu\theta + \text{constant},$$

or
$$F - \lambda v^2 = A\exp(\mu\theta).$$

where A is a constant. If $F = F_1$ when $\theta = 0$, $A = F_1 - \lambda v^2$,

i.e.
$$F_2 - \lambda v^2 = (F_1 - \lambda v^2)\exp(\mu\theta).$$

Energy and work.—A force is said to do ***work*** when its point of application undergoes a displacement along the line of action of the force. Thus, if the point of application of a force F advances

through a distance s along the line of action of the force the work done by the force is Fs.

If the point of application of the force advances a distance s in a direction inclined at an angle θ to the line of action of the force, cf. Fig. 2·10, the work done may be found by resolving the force into components along the displacement s and at right angles to it. These components have magnitudes F cos θ and F sin θ respectively. The work done by the former is (F cos θ)s, while the latter component does no work. The total work done is

$$\mathrm{F}s \cos \theta = \{\vec{\mathrm{F}}\cdot\vec{s}\}.$$

The absolute units of work in the British and in the Metric systems of units are respectively the foot.poundal and the dyne.cm.

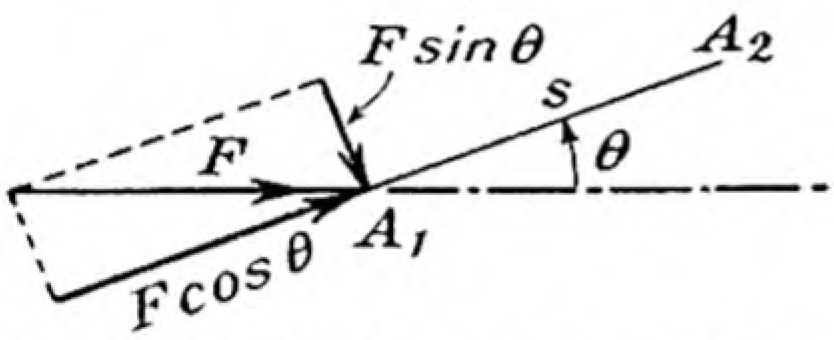


Fig. 2·10.—Work done by a force.

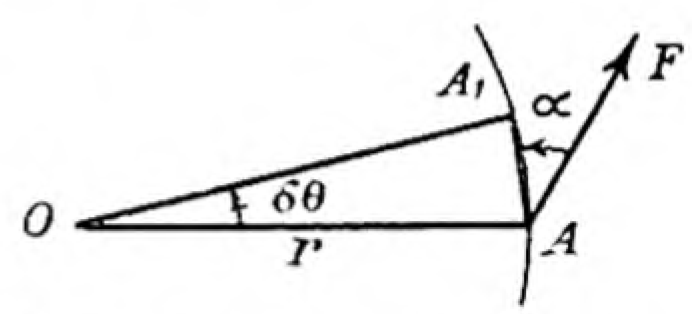


Fig. 2·11.—Work done by a couple.

or erg. Practical units are the ft.lb.-wt., the joule (which is 10^7 ergs), and the electron-volt, viz. $1{\cdot}602 \times 10^{-12}$ erg.

Energy is defined as the capacity which a body has for doing work. Energy due to motion is termed ***kinetic energy*** and the kinetic energy of a body is measured by the work that body could do in overcoming resistance before being brought to rest. The energy possessed by a body in virtue of its position is termed ***potential energy***. The potential energy of a body is measured by the work which would be done by the forces acting on the body if it moved from its position to some standard position where the potential energy is taken to be zero. Since the choice of a standard position is arbitrary it is usual to state that bodies on the earth's surface possess zero potential energy. Since we always have to deal with changes in potential energies rather than with absolute values no difficulties are presented by this arbitrary zero.

Work performed by a couple.—Consider a rigid body which is acted upon by a system of forces; let F, Fig. 2·11, be a typical force acting at one point in the body. Suppose that this system causes the body to rotate through a small angle $\delta\theta$ about an axis through O perpendicular to the plane of the diagram, and that A moves to A_1. If α is the angle between A_1A and F, the work done by the force F is

$$\begin{aligned}\mathrm{F}\cos\alpha(\mathrm{AA_1}) &= \text{component of F along } \mathrm{AA_1} \times r.\delta\theta\\ &= \delta\theta \times \text{moment of F about the axis of rotation.}\end{aligned}$$

If Γ is the moment about O of all the forces acting on the system, then the work done by the system is $\Gamma\delta\theta$. In particular, if the forces constitute a couple of constant magnitude the work done is equal to the product of the couple and the rotation.

Conservative systems of forces.—Let us consider (*a*) the work done in raising a mass m against gravity through a vertical distance h, and (*b*) the work done in pushing a body against a frictional force F through a distance s. Now in (*a*) the work performed against gravity is mgh; on permitting the body to descend to its original level the work becoming available is mgh, so that the total work performed on the mass is zero. In (*b*), however, the work done is Fs and an equal quantity of work must again be performed when the body is brought back to its original position, i.e. 2Fs is the total work performed under these conditions. The essential difference between the two kinds of work here typified is that in processes similar to (*a*) the work may be recovered from the system by making the system of bodies perform mechanical work as they return to their original positions, whereas in (*b*) no recovery of any work expended is possible.

Definition: When the forces acting on a system of bodies are of such a nature that the algebraic work done in performing any series of displacements whereby the original configuration of the system is regained is zero, then those forces constitute a conservative system of forces.

The forces associated with gravitational, electrostatic and magnetostatic fields are conservative, whereas forces due to friction or to the resistance offered by a medium in which a body may be moving are non-conservative.

Theorem: The work done by a system of conservative forces in moving a system of bodies from a configuration A to a second one B, is independent of the path from A to B.

Let the work done in passing from A to B along the paths (1) and (2), Fig. 2·12, be W_1 and W_2 respectively, while W_3 is the work done in passing from B to A via (3). Then, since the forces constitute a conservative system

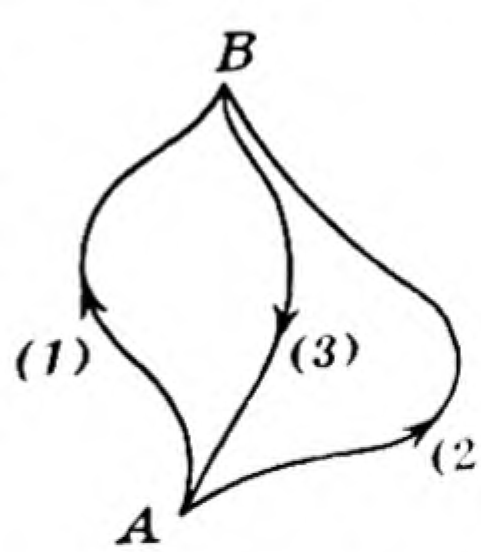


Fig. 2·12.—A conservative system of forces.

$$W_1 + W_3 = 0 \quad \text{and} \quad W_2 + W_3 = 0.$$

Hence $W_1 = W_2$, which establishes the theorem.

Definition: If any configuration A is taken as standard, the work done when a system of bodies, under the action of conservative forces, passes from a configuration A to a

configuration B, is termed the potential energy of the system in the B configuration.

Hence the work stored in the system when it has assumed the B configuration is measured by the potential energy of that configuration.

Theorem: The work done against conservative forces in moving a system of bodies from one configuration to another is $W_2 - W_1$, where W_1 and W_2 are the potential energies appropriate to the first and second configurations.

Now W_1 measures the work done when the system passes from a standard configuration to the first configuration and W_2 measures the work done when the system passes from the same standard configuration to the second configuration. But W_2 is the work from S to (i) plus the work from (i) to (ii). Hence the work from (i) to (ii) is $W_2 - W_1$.

The energy equation.—The general equation for a rectilinear motion of a mass m under the action of a force X is

$$m\ddot{x} = X,$$

where $\ddot{x}$ is the acceleration. Multiplying by $\dot{x}$, we have

$$m\ddot{x}\dot{x} = X\dot{x},$$

or

$$\frac{d}{dt}\left(\tfrac{1}{2}m\dot{x}^2\right) = X\frac{dx}{dt}.$$

Since $X\,\delta x$ is the work done by the force X in an infinitesimal displacement δx, $X\dfrac{dx}{dt}$ measures the rate at which work is being done on the body at time t. The above equation therefore shows that the rate at which the kinetic energy of a body increases is equal to the rate at which work is being done on the body. Integrating from time t_1 to t_2 we get

$$\tfrac{1}{2}m(\dot{x}_2^{\,2} - \dot{x}_1^{\,2}) = \int_{t_1}^{t_2} X\frac{dx}{dt}\,dt$$

$$= \int_{x_1}^{x_2} X\,dx,$$

where x_1 and x_2 are the initial and final positions, and $\dot{x}_1$ and $\dot{x}_2$ are the corresponding velocities.

If the system of forces is a conservative one, then $X = -\dfrac{\partial V}{\partial x}$, where V is the potential energy, and

$$\int_{x_1}^{x_2} X\,dx = -\int_{x_1}^{x_2}\frac{\partial V}{\partial x}\,dx = V_1 - V_2.$$

$$\therefore\ \tfrac{1}{2}m\dot{x}_2^{\,2} + V_2 = \tfrac{1}{2}m\dot{x}_1^{\,2} + V_1,$$

i.e. the sum of the kinetic and potential energies is constant. This sum is termed the ***total energy*** and measures the amount of work which the body can perform against external agencies in passing from its actual state as regards velocity and position to rest in a standard position.

Although the proof here given only applies to one variable, viz. x, the theorem is true in general.

Power.—Power is the rate of doing work, i.e. the work done per unit time. Suppose a particle, acted upon by a force F, Fig. 2·13, moves from a point A at time t_1 to a point B at time t_2. The work done by the force is

$$\int_A^B \vec{F}\cdot\hat{s}\, ds,$$

where $\hat{s}$ is the unit vector at an element MN, of length δs, in the path from A to B and the integral is to be evaluated along this path. Now

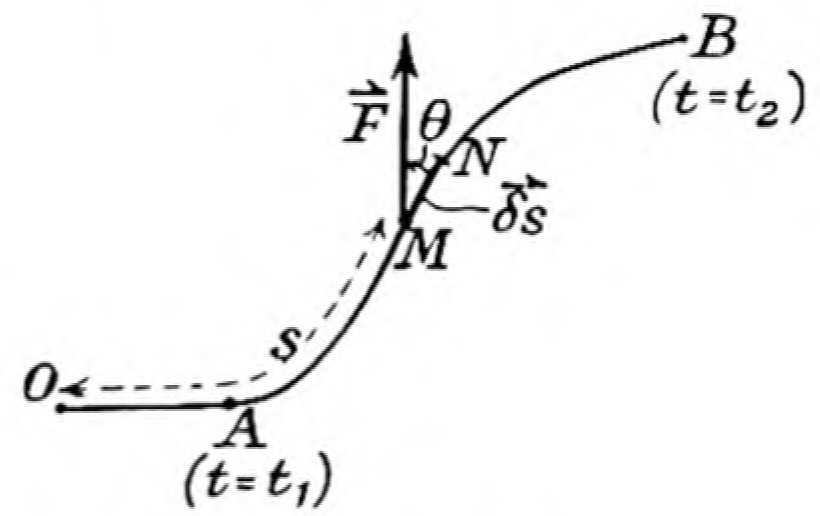


FIG. 2·13.

$$\hat{s}\,\delta s = \overrightarrow{\delta s} = \vec{v}\,dt,$$

where $\vec{v}$ is the velocity with which the particle moves along δs. Hence the work done is

$$\int_{t_1}^{t_2} \vec{F}\cdot\vec{v}\, dt,$$

i.e. the instantaneous value of the power P is given by

$$\begin{aligned} P &= \vec{F}\cdot\vec{v} \\ &= Fv\cos\theta, \end{aligned}$$

where $\theta = (\widehat{\vec{F}, \vec{v}})$. Thus power is a scalar quantity measured by the scalar product of force and velocity.

The theoretical unit of power in c.g.s. units is one erg per second, which is the rate of working when a force of one dyne causes a particle to move in one second a distance of one centimetre along the line of action of the force. This unit is inconveniently small for many purposes; a larger unit is the watt which is 10 erg.^{7}sec.$^{-1}$ or 1 joule.sec.$^{-1}$ Electrical engineers also use the kilowatt and megawatt.

Velocity and acceleration referred to polar coordinates.—Let P, Fig. 2·14, be the point (x, y) referred to rectangular axes Ox, Oy; similarly let (r, θ) be its polar coordinates referred to an origin O and an initial line Ox. Let v_1 and v_2 be the components of the velocity of P along OP and at right angles to it in the sense

of θ increasing. Since $x = r\cos\theta$ and $y = r\sin\theta$, by resolving velocities parallel to Ox, we get

$$v_1 \cos\theta - v_2 \sin\theta = \dot{x} = \frac{d}{dt}(r\cos\theta)$$

$$= \dot{r}\cos\theta - r\sin\theta.\dot{\theta}. \quad . \quad . \quad \text{(i)}$$

Similarly,

$$v_1 \sin\theta + v_2\cos\theta = \dot{y} = \dot{r}\sin\theta + r\cos\theta.\dot{\theta}. \quad . \quad \text{(ii)}$$

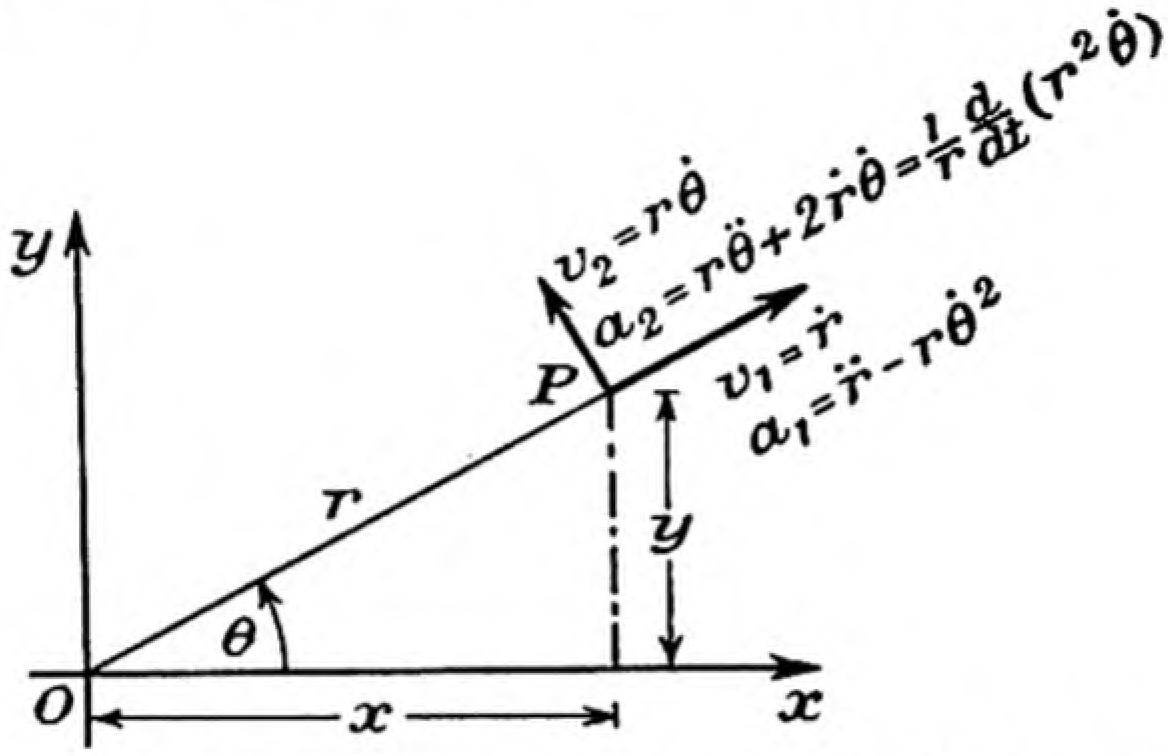


FIG. 2·14.—Transverse and radial components of velocity and acceleration.

Eliminating $\sin\theta$ and $\cos\theta$ from equations (i) and (ii) we get

$$(v_1 - \dot{r})^2 + (v_2 - r\dot{\theta})^2 = 0,$$

and since the sum of the squares of two real quantities can only be zero provided each quantity is zero, we have

$$v_1 = \dot{r} \quad \text{and} \quad v_2 = r\dot{\theta}.$$

If a_1 and a_2 are the accelerations in the above directions, we obtain in a similar manner,

$$a_1\cos\theta - a_2\sin\theta = \ddot{x} = \frac{d}{dt}(\dot{r}\cos\theta - r\sin\theta.\dot{\theta})$$

$$= \ddot{r}\cos\theta - (\dot{r}\sin\theta)\dot{\theta} - (\dot{r}\sin\theta)\dot{\theta}$$
$$- r(\cos\theta)\dot{\theta}^2 - r(\sin\theta)\ddot{\theta}$$
$$= (\ddot{r} - r\dot{\theta}^2)\cos\theta - (r\ddot{\theta} + 2\dot{r}\dot{\theta})\sin\theta. \quad \text{(iii)}$$

Similarly,

$$a_1\sin\theta + a_2\cos\theta = (\ddot{r} - r\dot{\theta}^2)\sin\theta + (r\ddot{\theta} + 2\dot{r}\theta)\cos\theta, \quad \text{(iv)}$$

Eliminating $\sin\theta$ and $\cos\theta$ from equations (iii) and (iv) we find, as above,

$$a_1 = \ddot{r} - r\dot{\theta}^2,$$

and

$$a_2 = r\ddot{\theta} + 2\dot{r}\dot{\theta} = \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}).$$

The components a_1 and a_2 are termed the **radial** and the **transverse components of the acceleration.**

Central orbits.—If a particle P, Fig. 2·15, is describing an orbit under the action of a force directed towards a fixed point O, then at any instant the particle is moving in a plane containing the radius vector OP and the tangent to the orbit at the point considered. Since the force acting on the particle lies in this plane, and there is no other force acting on the particle, it follows that the orbit must lie wholly in a plane.

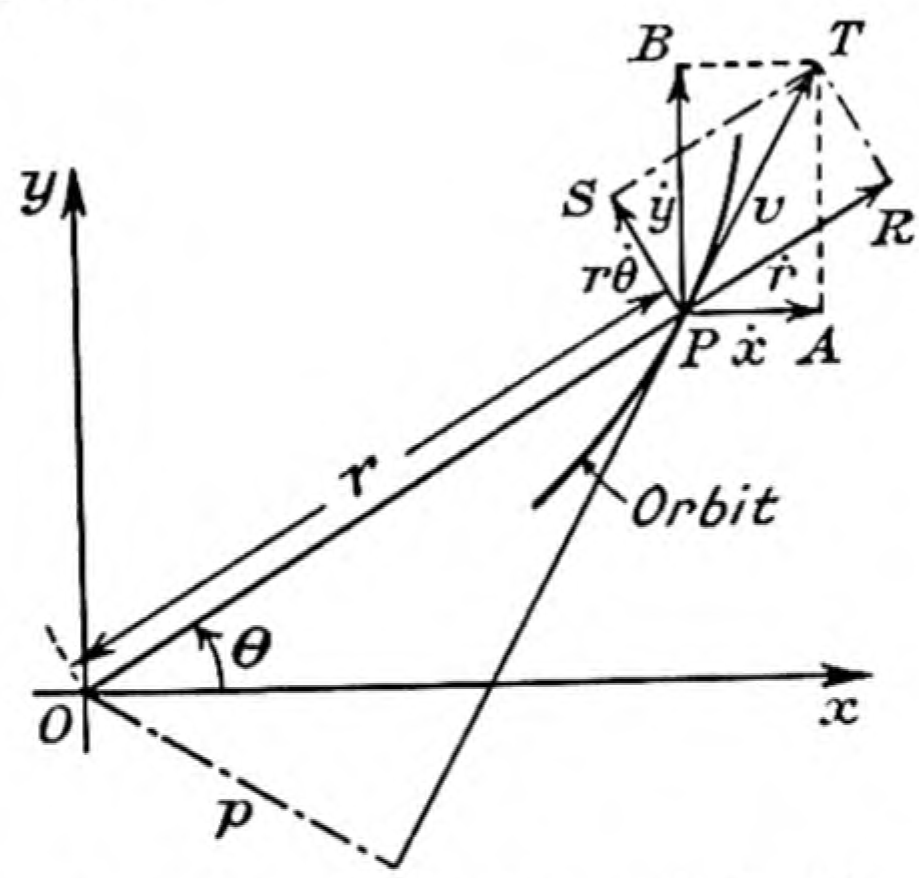


FIG. 2·15.—Motion in a central orbit.

Moreover, since there is no force at right angles to the radius vector, the transverse component of the acceleration is zero, i.e. if OP $= r$ and θ is the angle which OP makes with a fixed line Ox,

$$r^2\dot{\theta} = \text{constant} = h \text{ (say).}$$

Since the area of a sectorial element is $\frac{1}{2}r^2\,\delta\theta$, it follows from the above that the rate of description of area by the radius vector is constant. Moreover, since $r^2\dot{\theta}$ is the moment of the resultant velocity about O, for the moment of velocity $\dot{r}$ about O is zero, and since the moment of any vector about a given point is equal to the sum of the moments of any components into which that vector may be resolved, we have

$$h = r^2\dot{\theta} = x\dot{y} - y\dot{x} = pv,$$

where (x, y) are the rectangular coordinates, referred to O as origin, of a point on the orbit and p is the perpendicular distance from the origin on a straight line defining v the resultant velocity of the particle when it is at the point (x, y).

Orbit described under the action of an attraction inversely proportional to the square of the distance.—It has already been shown that when a particle moves in an orbit under the action of a

force directed to a fixed point, equal areas are described in equal times. If we couple with this the fact that the energy total of the particle is constant, the equation to the orbit may be obtained.

The velocity v of such a particle is expressed by

$$v^2 = (\dot{r})^2 + (r\dot{\theta})^2 = \left(\frac{dr}{d\theta}\frac{d\theta}{dt}\right)^2 + (r\dot{\theta})^2 = \dot{\theta}^2\left[\left(\frac{dr}{d\theta}\right)^2 + r^2\right]$$
$$= \frac{h^2}{r^4}\left[\left(\frac{dr}{d\theta}\right)^2 + r^2\right].$$

If $f(r)$ is the attractive force per unit mass when the particle is at distance r from O, the potential energy of the particle is $m\int_{\infty}^{r} f(r)\,dr$, since the work done per unit mass by an external agent and against the force acting on a small mass δm as it is brought up from infinity to the point considered is $\int_{\infty}^{r} - f(r)(-dr)$. In the present instance $f(r) = \kappa r^{-2}$, where κ is a constant, so that the potential energy is $-\kappa m r^{-1}$. If E is the total energy per unit mass, we have

$$\frac{1}{2}\frac{h^2}{r^4}\left[\left(\frac{dr}{d\theta}\right)^2 + r^2\right] - \frac{\kappa}{r} = \mathrm{E}.$$

This gives

$$d\theta = \frac{h\,dr}{r\sqrt{2\mathrm{E}r^2 + 2\kappa r - h^2}}$$

$$= \frac{-h\,dx}{x^2\left(\frac{1}{x}\right)\sqrt{\frac{2\mathrm{E}}{x^2} + \frac{2\kappa}{x} - h^2}} \qquad \left[\text{if } x = \frac{1}{r}\right]$$

$$= -\frac{dx}{\sqrt{\frac{2\mathrm{E}}{h^2} + \frac{\kappa^2}{h^4} - \left(x - \frac{\kappa}{h^2}\right)^2}}.$$

$$\therefore\ \theta = \sin^{-1}\left\{\left(\frac{\kappa}{h} - \frac{h}{r}\right) \div \sqrt{2\mathrm{E} + \frac{\kappa^2}{h^2}}\right\} + \beta,$$

where β is a constant. The above equation may be written

$$\sqrt{2\mathrm{E} + \frac{\kappa^2}{h^2}}.\sin(\theta - \beta) = \frac{\kappa}{h} - \frac{h}{r}.$$

$$\therefore\ r = \frac{h^2}{\kappa} \div \left\{1 + \sqrt{1 + \frac{2\mathrm{E}h^2}{\kappa^2}}.\cos\left(\theta - \beta + \frac{\pi}{2}\right)\right\}.$$

Comparing this with the equation $r = \dfrac{l}{1 + e\cos(\theta - \alpha)}$, we see that

the orbit is a conic section whose semi-latus rectum is $\frac{h^2}{\kappa}$, and whose eccentricity is $\sqrt{1 + \frac{2Eh^2}{\kappa^2}}$. Hence if $E < 0$, the orbit is an ellipse. Since $E = \frac{1}{2}v^2 - \frac{\kappa}{r}$, if $v^2 < \frac{2\kappa}{r}$, the orbit is an ellipse.

Alternative treatment: It has been established [cf. p. 60] that the radial and transverse components of the acceleration of a moving particle are

$$\ddot{r} - r\dot{\theta}^2, \quad \text{and} \quad \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})$$

respectively. If the attraction per unit mass is $f(r)$ and it is directed towards a fixed point, we have

$$\ddot{r} - r\dot{\theta}^2 = -f(r), \quad \text{and} \quad r^2\dot{\theta} = \text{constant} = h,$$

the latter equation being derived from the fact that the transverse acceleration is zero.

It is sometimes convenient to express these equations in terms of u, where $u = r^{-1}$. We have

$$\frac{d\theta}{dt} = \frac{h}{r^2} = hu^2,$$

and $$\frac{dr}{dt} = \frac{d}{dt}\left(\frac{1}{u}\right) = \frac{d}{d\theta}\left(\frac{1}{u}\right).\frac{d\theta}{dt} = -\frac{1}{u^2}.\frac{du}{d\theta}.\dot{\theta} = -h.\frac{du}{d\theta}.$$

Hence $$\frac{d^2r}{dt^2} = -h.\frac{d}{dt}\left(\frac{du}{d\theta}\right) = -h.\frac{d^2u}{d\theta^2}.\frac{d\theta}{dt} = -h^2u^2.\frac{d^2u}{d\theta^2}.$$

The differential equation giving the orbit is therefore

$$\frac{d^2u}{d\theta^2} + u = \frac{f\left(\frac{1}{u}\right)}{h^2u^2} = \frac{\kappa}{h^2},$$

if $f\left(\frac{1}{u}\right) = \kappa u^2$, i.e. if the attraction varies inversely as the square of the distance.

The above is a second order differential equation and the complete solution is obtained by finding (*a*) the particular integral, (*b*) the complementary function and adding the two together [cf. p. 33].

For the particular integral, we have, if $D = \frac{d}{d\theta}$,

$$u = \frac{1}{(D^2 + 1)}\left(\frac{\kappa}{h^2}\right) = \frac{\kappa}{h^2}.$$

The complementary function is $u = A\cos(\theta + \gamma)$, where A and γ are arbitrary constants; the complete solution is

$$u = [A\cos(\theta + \gamma)] + \frac{\kappa}{h^2}.$$

$$\therefore\ r = \frac{1}{\dfrac{\kappa}{h^2} + A\cos(\theta + \gamma)} = \frac{\dfrac{h^2}{\kappa}}{1 + \dfrac{Ah^2}{\kappa}\cos(\theta + \gamma)}.$$

The orbit is therefore a conic section with eccentricity $\dfrac{Ah^2}{\kappa}$ and a semi-latus rectum $\dfrac{h^2}{\kappa}$.

The constant A may be determined in terms of E, the total energy per unit mass of the particle, as follows. We have

$$E = \tfrac{1}{2}v^2 - \frac{\kappa}{r} = \frac{1}{2}\left[\left(\frac{dr}{d\theta}\right)^2 + r^2\right]\frac{h^2}{r^4} - \frac{\kappa}{r}.$$

Now
$$\frac{1}{r} = u = [A\cos(\theta + \gamma)] + \frac{\kappa}{h^2}.$$

$$\therefore\ -\frac{1}{r^2}\cdot\frac{dr}{d\theta} = -A\sin(\theta + \gamma).$$

$$\begin{aligned}\therefore\ E &= \frac{1}{2}\left[\{A^2h^2\sin^2(\theta + \gamma)\} + \frac{h^2}{r^2}\right] - \frac{\kappa}{r} \\ &= \frac{1}{2}A^2h^2\left[1 - \frac{1}{A^2}\left(\frac{1}{r} - \frac{\kappa}{h^2}\right)^2\right] + \frac{1}{2}\frac{h^2}{r^2} - \frac{\kappa}{r} \\ &= \frac{1}{2}\left[A^2h^2 - \frac{\kappa^2}{h^2}\right].\end{aligned}$$

$$\therefore\ 2E + \frac{\kappa^2}{h^2} = A^2h^2,$$

which is the expression obtained by equating the two values determined for the eccentricity of the orbit.

The periodic time when the orbit is an ellipse.—This is the time required for a radius vector to sweep out an area equal to that of the ellipse. If a and b are the semi-axes of the ellipse, its area is πab and the period T is therefore given by

$$T = \frac{\pi ab}{\frac{1}{2}h},$$

since an area $\frac{1}{2}h$ is swept out in unit time. The semi-latus rectum is $\frac{h^2}{\kappa}$ and in the present instance is also equal to $\frac{b^2}{a}$. Hence

$$\frac{b^2}{a} = \frac{h^2}{\kappa}, \quad \text{or} \quad b = h\sqrt{\frac{a}{\kappa}}.$$

$$\therefore \ \mathrm{T} = \frac{2\pi ab}{h} = 2\pi a^{\frac{3}{2}}\kappa^{-\frac{1}{2}}.$$

This expression is independent of b, the minor semi-axis of the ellipse, so that the periodic time is the same for all elliptic orbits having equal major semi-axes.

Kepler's laws.—The law of gravitation is an inverse square law so that the motions investigated above may be regarded as those of the planets around the sun. These orbits are ellipses of small eccentricity the sun being at one focus in each instance. Radar observations on meteors have shown that these bodies have elliptical orbits, often of quite short period, and are therefore part of the solar system. Comets are also accepted as members of the solar system, though the orbits are often highly eccentric, sometimes becoming parabolic through planetary perturbations. Thus our analysis applies to them but in all probability not to the orbits of meteorites whose origin may be different from that of meteors.

The three principal laws concerning the motions of the planets were discovered by Kepler during the period 1609–1619, although Newton first obtained them as the result of a mathematical investigation published in 1687.

Law I. Every planet describes an ellipse, the sun being at one of its foci.

Law II. In any given instance, equal areas are described by the radius vector drawn from the sun to the planet in equal times.

Law III. The squares of the periodic times are proportional to the cubes of the major semi-axes of the various orbits.

EXAMPLES II

2·01. Give statements of the principles of conservation of linear momentum and conservation of energy.

In Hicks' ballistic balance one of the platforms of mass 1·0 kgm. is pulled aside so that it is 10 cm. above the equilibrium position. After impact the two platforms move together. To what height will they rise if the second platform has mass 1·5 kgm. Calculate the loss of kinetic energy at impact.

2·02. A railway engine of mass 32 tons exerts a constant horse-power of 448 and the motion of the engine is opposed by a constant

force of 1·25 ton.-wt. Prove that the equation of motion of the engine is

$$v^2 \frac{dv}{dx} = \tfrac{5}{4}(88 - v),$$

where v is the speed in ft.sec.$^{-1}$, x the distance travelled in feet, and g may be taken as 32 ft.sec.$^{-2}$.

Find the maximum speed of the engine and prove that the engine, starting from rest, attains a speed of 30 ml.hr.$^{-1}$ in approximately 423 ft. [Assume $\ln 2 = 0{\cdot}6932$.]

2·03. Prove that the components of acceleration in terms of polar coordinates (r, θ) are $\ddot{r} - r\dot{\theta}^2$, $r\ddot{\theta} + 2\dot{r}\dot{\theta}$ along and perpendicular to the radius vector.

A small planet describes a circle round the sun. Suddenly the velocity is reduced by $\dfrac{1}{n}$ th, while its direction is unchanged. Find the change in the eccentricity and period.

2·04. The planet Neptune travels round the sun with a period of 165 years. Prove that the diameter of its orbit is about 30 times that of the earth's orbit, assuming both orbits to be circular.

Chapter III

THE PRINCIPLES OF RIGID DYNAMICS

Kinetic energy of a body rotating about a fixed axis.—Let X_1X_2, Fig. 3·01, be a fixed axis about which a body is rotating with angular velocity ω at a given instant. Consider a small element or particle of the body at P. Let m be the mass of this particle, and r its perpendicular distance from the axis X_1X_2. Since the velocity

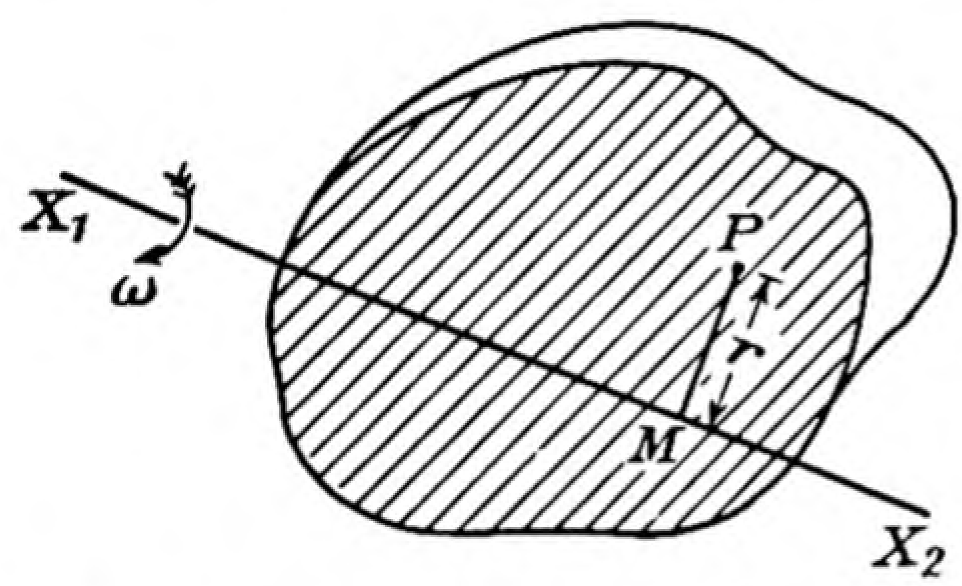


Fig. 3·01.—Kinetic energy of a rotating body.

of this element is $r\omega$, its kinetic energy is $\frac{1}{2}mr^2\omega^2$. The kinetic energy of the whole body, defined as the sum of the kinetic energies of the individual particles at the instant considered, is therefore

$$\Sigma\,(\tfrac{1}{2}mr^2\omega^2) = \tfrac{1}{2}\Sigma\,(mr^2)\omega^2,$$

where the summation is applied to all such elements of the body. The expression $\Sigma\, mr^2$ is called the ***moment of inertia*** of the body about the ***specified axis***, and is denoted by I.

If κ^2 is put equal to $\dfrac{\Sigma\,(mr^2)}{\Sigma\, m}$, the expression for the kinetic energy becomes $\frac{1}{2}\kappa^2(\Sigma\, m)\omega^2$, so that the energy is the same as if the whole mass of the body were concentrated at a point at a perpendicular distance κ from the axis of rotation. The quantity κ is called the ***radius of gyration*** of the body about the ***axis of rotation***. It is such that κ^2 measures the mean value of r^2 averaged over all the particles of the body. Some instances of practical importance are as follows.

Moment of inertia of a uniform thin rod about an axis at one end perpendicular to its length.—Let OA, Fig. 3·02, be the rod of length $2a$ and Y_1OY_2 the axis of rotation. Consider a small

element of the rod at P and of length δx where x is the distance of P from O. Let λ be the mass per unit length of the rod. Then $\lambda\,\delta x$

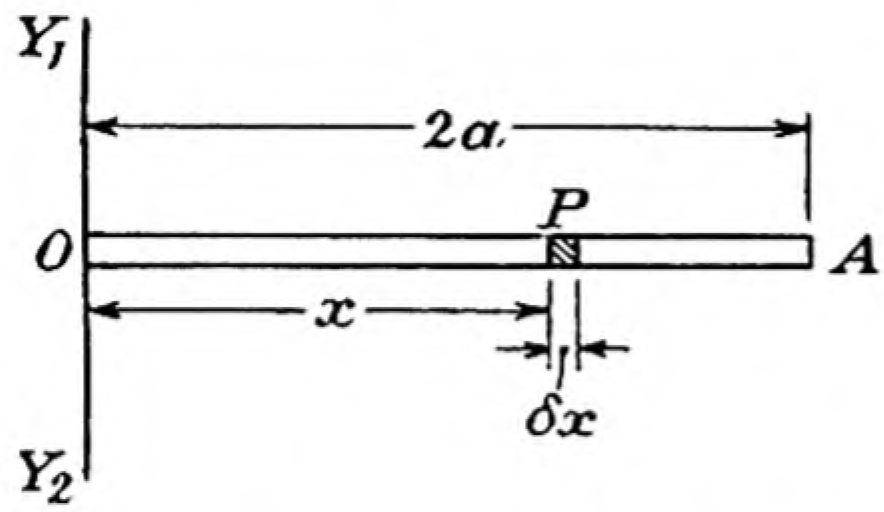


FIG. 3·02.—Moment of inertia of a uniform thin rod about an axis at one end perpendicular to its length.

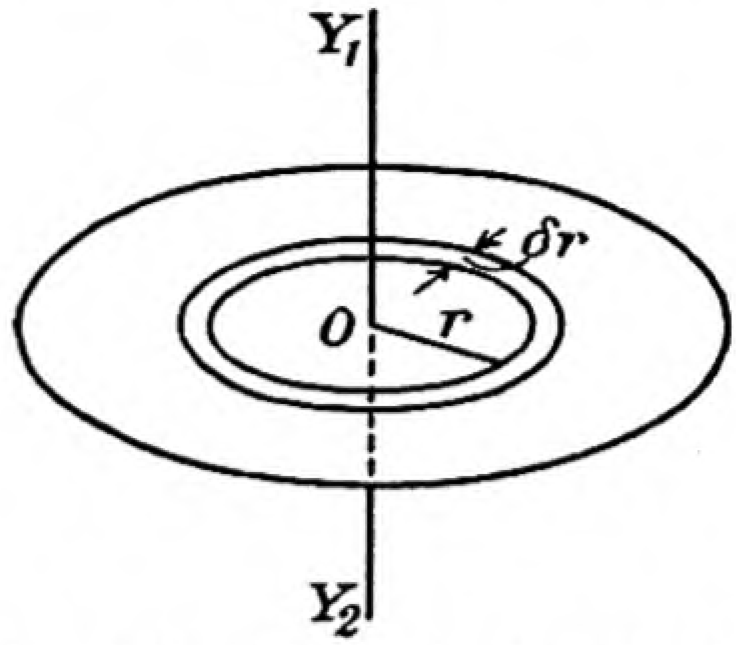


FIG. 3·03.—Moment of inertia of a uniform circular lamina about an axis through its centre and normal to its plane.

is the mass of the element at P, and since this is at a distance x from O its moment of inertia, δI, about Y_1OY_2 is $\lambda . \delta x . x^2$. Hence

$$I = \int_0^{2a} \lambda x^2\, dx = \lambda \left[\frac{x^3}{3}\right]_0^{2a} = \frac{8\lambda a^3}{3}.$$

But the mass of the rod is $2\lambda a$.

$$\therefore\ I = \frac{4ma^2}{3} = \tfrac{1}{3}m(2a)^2, \text{ i.e. } \quad \kappa = \frac{2a}{\sqrt{3}}.$$

Moment of inertia of a uniform thin rod about an axis normal to its length and passing through its centre.—Let $2a$ be the length of the rod, λ its mass per unit length and first of all consider one half of the rod. Then the problem is identical with that just discussed and we have, if $I_{\frac{1}{2}}$ is the moment of inertia of this portion of the rod about the axis considered,

$$I_{\frac{1}{2}} = \lambda \int_0^a x^2\, dx = \tfrac{1}{3}\lambda a^3.$$

Hence for the whole rod,

$$I = 2I_{\frac{1}{2}} = \frac{2\lambda a^3}{3} = \frac{ma^2}{3},$$

where m is the mass of the rod. Hence $\kappa = \dfrac{a}{\sqrt{3}}$.

Moment of inertia of a uniform circular lamina about an axis through its centre and normal to its plane.—Let a be the radius of the disc and σ its mass per unit area. Consider a ring element, Fig. 3·03, of radius r and width δr. Then the mass of this

ring is $2\pi r\sigma\,\delta r$ and since all parts of it are at a common distance r from the axis of rotation Y_1Y_2, the radius of gyration for the ring about the above axis is r. Hence the moment of inertia of this ring element about Y_1Y_2 is given by

$$\delta I = 2\pi r^3 \sigma\,\delta r.$$

$$\therefore\ I = \int_0^a 2\pi\sigma r^3\,dr = \tfrac{1}{2}\pi\sigma a^4 = \tfrac{1}{2}ma^2,$$

where m is the mass of the disc.

If the problem concerns a circular disc of thickness t, let ρ be the density of the material of the disc. Let the axis of rotation pass through the centre of the disc and be normal to its plane. Then, for a cylindrical element of radius r, width δr, and length t,

$$\delta I = (2\pi r t\rho\,\delta r)r^2.$$

$$\therefore\ I = 2\pi\rho t\int_0^a r^3\,dr = 2\pi\rho t\left(\frac{a^4}{4}\right) = \tfrac{1}{2}\pi\rho t a^4.$$

But $m = \pi a^2 t\rho$.

$$\therefore\ I = \tfrac{1}{2}ma^2, \quad \text{or} \quad \kappa^2 = \tfrac{1}{2}a^2.$$

Hence the radius of gyration of a circular disc about its own axis of revolution is independent of the thickness of the disc.

Moment of inertia of a uniform shell about a diameter.—Let us assume that the centre of the shell is the origin of a system of rectangular coordinates and Ox the axis about which the moment of inertia is required. Let P be a point in the shell and suppose that a particle of mass δm is situated at this point, whose coordinates are (x, y, z). Then if r is the radius of the shell, we have

$$\delta m\,.\,r^2 = \delta m(x^2 + y^2 + z^2),$$

and this is not a moment of inertia but we may call it, say δS. Hence for the whole sphere, of mass m, we have

$$S = \Sigma\,\delta m(x^2 + y^2 + z^2).$$

But, on account of symmetry,

$$\Sigma\,(\delta m\,.\,x^2) = \Sigma\,(\delta m\,.\,y^2) = \Sigma\,(\delta m\,.\,z^2) = \tfrac{1}{3}mr^2.$$

Hence I_x, the moment of inertia of the shell about the axis is given by

$$I_x = \Sigma\,\delta m(y^2 + z^2) = \tfrac{2}{3}mr^2.$$

Moment of inertia of a uniform sphere about a diameter.—The result just obtained may be applied at once to the present problem. For this purpose consider a uniform sphere of radius a

and let its material have a density ρ. Then if the sphere is divided up into thin concentric shells, let r and $r + \delta r$ be the radii which determine one of them. Its moment of inertia about Ox, is given by

$$\delta I_x = \tfrac{2}{3}(4\pi r^2 \rho\, \delta r) r^2,$$

so that, for the whole sphere,

$$I_x = \frac{2}{3}\int_0^a (4\pi r^4 \rho)\, dr$$
$$= \frac{8}{3}\pi\rho \cdot \frac{a^5}{5} = \frac{2}{5} ma^2,$$

where m is the mass of the sphere, viz. $\tfrac{4}{3}\pi\rho a^3$.

Theorems on moments of inertia.—Theorem I. ***If* I *is the moment of inertia of a body about any axis through its centre of mass and* I_a *is the moment of inertia about a parallel axis,***

$$I_a = I + ma^2,$$

where a is the perpendicular distance between the two axes, and m is the mass of the body.

Let P, Fig. 3·04(a), be the point (x, y, z) referred to rectangular

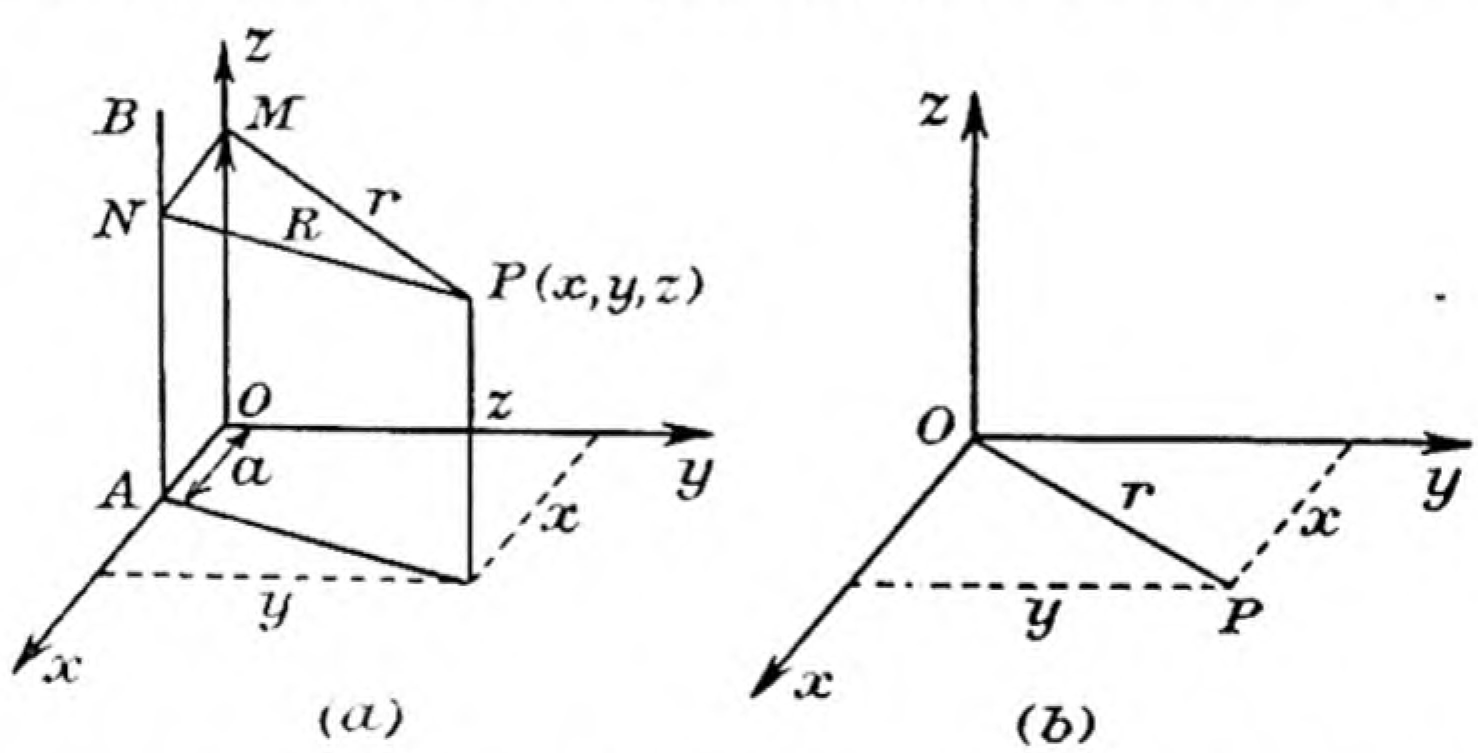


FIG. 3·04.—(a) Moments of inertia of a rigid body about parallel axes.
(b) Moment of inertia of a lamina about an axis normal to its plane.

axes Ox, Oy, Oz, where O is the centre of mass of the body, and let Oz be the axis of rotation about which the moment of inertia of the body is I. Let AB be a straight line parallel to Oz and at a distance a from it. [There is no loss of generality in this procedure for, when Oz is fixed, Ox and Oy can always be chosen so that AB lies in the plane xOz and is normal to Ox.] Consider an element of mass δm at P, and let r be the shortest distance of P from Oz. Then

$$I = \Sigma\, (r^2\, \delta m),$$

where the summation extends over the whole body. Let R be the shortest distance of P from AB, i.e. R = PN. Then

$$I_a = \Sigma\,(R^2 . \delta m).$$

But

$$R^2 = (x - a)^2 + y^2$$
$$= r^2 - 2ax + a^2.$$
$$\therefore\ I_a = \Sigma\,[(r^2 - 2ax + a^2)\,\delta m]$$
$$= I + a^2\,\Sigma\,\delta m - 2a\,\Sigma\,(x\,\delta m)$$
$$= I + ma^2,$$

for the x component of the centre of mass is $\dfrac{\Sigma\,(x\,\delta m)}{\Sigma\,\delta m}$ and in the present instance this, and therefore $\Sigma\,(x\,\delta m)$, is zero since the centre of mass is at the origin of coordinates.

Hence

$$\kappa_a^{\,2} = \kappa^2 + a^2,$$

where κ_a is the radius of gyration about the axis AB.

One method whereby this relation may be tested experimentally is described on p. 122.

Theorem II. ***If* Ox, Oy, *Fig. 3.04(b) are two axes mutually at right angles, in the plane of a lamina, and* Oz *is an axis normal to the plane* xOy, *then***

$$I_x + I_y = I_z,$$

***where* I_x, I_y *and* I_z *are respectively the moments of inertia of the lamina about the axes* Ox, Oy *and* Oz.**

Let P be a particle of mass δm in the ***lamina*** and let P be the point (x, y). Let OP = r. Then

$$I_x + I_y = \Sigma\,(\delta m . x^2) + \Sigma\,(\delta m . y^2)$$
$$= \Sigma\,[\delta m(x^2 + y^2)]$$
$$= I_z \qquad [\because\ r^2 = x^2 + y^2].$$

[It will be noticed that theorem I refers to any body, whereas theorem II is only true for a ***lamina***.]

Rectangular lamina.—Consider a rectangular lamina of sides $2a$, $2b$, and of mass σ per unit area. Let us find its radius of gyration about (i) a straight line through its centre and parallel to the side of length $2a$, (ii) an axis through its centre of mass and normal to its plane.

(i) Let O be the centre of mass of the lamina, and let axes Ox, Oy, Fig. 3·05, be chosen so that they are in the plane of the lamina and parallel to the sides of length $2a$ and $2b$ respectively. Let P_1P_2

be an element of the lamina parallel to Ox, at a distance y from it, and of width δy. Then the mass of this element is $2a\sigma . \delta y$ and its moment of inertia about Ox, viz. $\delta \mathrm{I}_x$, is given by

$$\delta \mathrm{I}_x = (2a\sigma . \delta y)y^2.$$

Hence
$$\mathrm{I}_x = 2\left[2a\sigma \int_0^b y^2\, dy\right] = \frac{4\sigma ab^3}{3}.$$

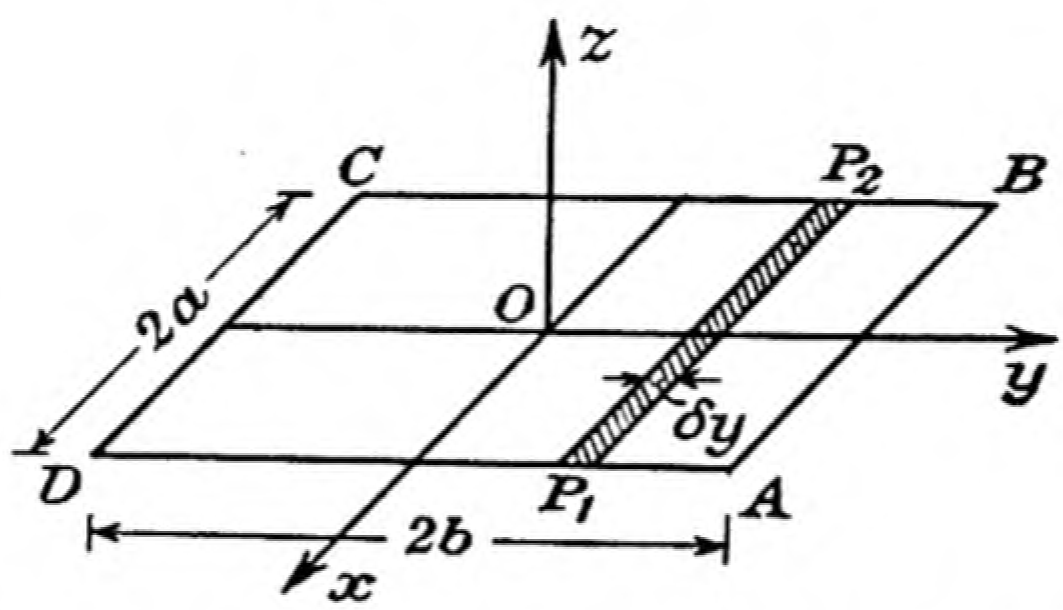


FIG. 3·05.—Moment of inertia of a uniform rectangular lamina about (a) Ox, (b) Oz.

But $m = 4ab\sigma$, so that

$$\mathrm{I}_x = \frac{mb^2}{3}.$$

Similarly,
$$\mathrm{I}_y = \frac{ma^2}{3}.$$

(ii) If Oz is drawn through O normal to the plane of the lamina,

$$\mathrm{I}_z = \mathrm{I}_x + \mathrm{I}_y = \tfrac{1}{3}m(a^2 + b^2).$$

Alternative method.—If $\delta x\, \delta y$ is a small element in the plane of the lamina, we have, since its distance from the axis of rotation is $(x^2 + y^2)^{\frac{1}{2}}$,

$$\mathrm{I}_z = 4\int_0^a \int_0^b (x^2 + y^2)\sigma\, dx\, dy,$$
$$= 4\sigma \int_0^a (x^2 b + \tfrac{1}{3}b^3)\, dx = \tfrac{4}{3}\sigma(a^3 b + b^3 a)$$
$$= m[\tfrac{1}{3}(a^2 + b^2)].$$

The first theorem on p. 70 enables us to write down the moment of inertia of the above lamina about AD. It is

$$m\left(\frac{a^2}{3} + a^2\right) = \tfrac{4}{3}\, ma^2.$$

Circular lamina.—It has already been shown that the moment of inertia of a circular lamina about an axis through its centre and normal to its plane is $\frac{1}{2}ma^2$. If this axis is Oz, and if Ox and Oy are two axes at right angles to one another and to Oz,

$$I_x = I_y = \tfrac{1}{2}I_z = \tfrac{1}{4}ma^2$$

Hence the radius of gyration about any diameter is $\dfrac{a}{2}$.

Uniform elliptical lamina.—Consider now a thin lamina of mass m in the form of an ellipse with semi-axes a and b parallel to the coordinate axes Ox, Oy, as shown in Fig. 3·06. Let I_x be the moment of inertia of the lamina about Ox. Then for the element PN, of height y and width δx, we have, if σ is the surface density of the lamina,

$$\delta I_x = \tfrac{1}{3}\sigma(y\,\delta x)y^2 = \tfrac{1}{3}\sigma y^3\,\delta x.$$

$$\therefore\ I_x = 4\left[\frac{\sigma}{3}\int_0^a y^3\,dx\right].$$

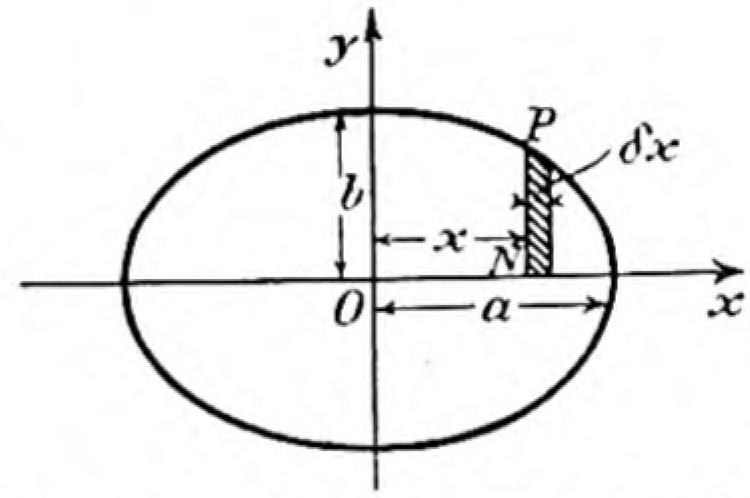


FIG. 3·06.—Moment of inertia of a uniform elliptical lamina.

To carry out this integration we may make use of the eccentric angle ϕ, and write $x = a\cos\phi$, $y = b\sin\phi$. Then $\delta x = -a\sin\phi\,\delta\phi$, so that

$$I_x = \frac{4}{3}\sigma\int_{\frac{\pi}{2}}^{0} b^3\sin^3\phi(-a\sin\phi)\,d\phi$$

$$= -\frac{4}{3}\sigma ab^3\left[-\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2}\right]$$

$$= \tfrac{1}{4}ma^2, \qquad [\because\ m = \pi ab\sigma].$$

Similarly,

$$I_y = \tfrac{1}{4}mb^2.$$

$$\therefore\ I_z = I_x + I_y = \tfrac{1}{4}m(a^2 + b^2).$$

where Oz is normal to Ox and Oy.

Moment of inertia of a uniform cylinder about an axis through its centre and at right angles to its length.—Let $2l$ be the length of the cylinder and a its radius of cross-section. Let O, Fig. 3·07, be the centre of mass of the cylinder, Ox its axis of revolution, and Oy, an axis at right angles to Ox, the axis about which the moment of inertia is to be calculated. Let ρ be the density of the material of the cylinder. Consider a circular element

AB, defined by the planes x and $x + \delta x$; then its centre O_1 is at a distance x from O. Then the moment of inertia of this disc about Ox is

$$(\pi a^2 . \delta x)\rho \frac{a^2}{2} = \tfrac{1}{2}\pi \rho a^4 . \delta x.$$

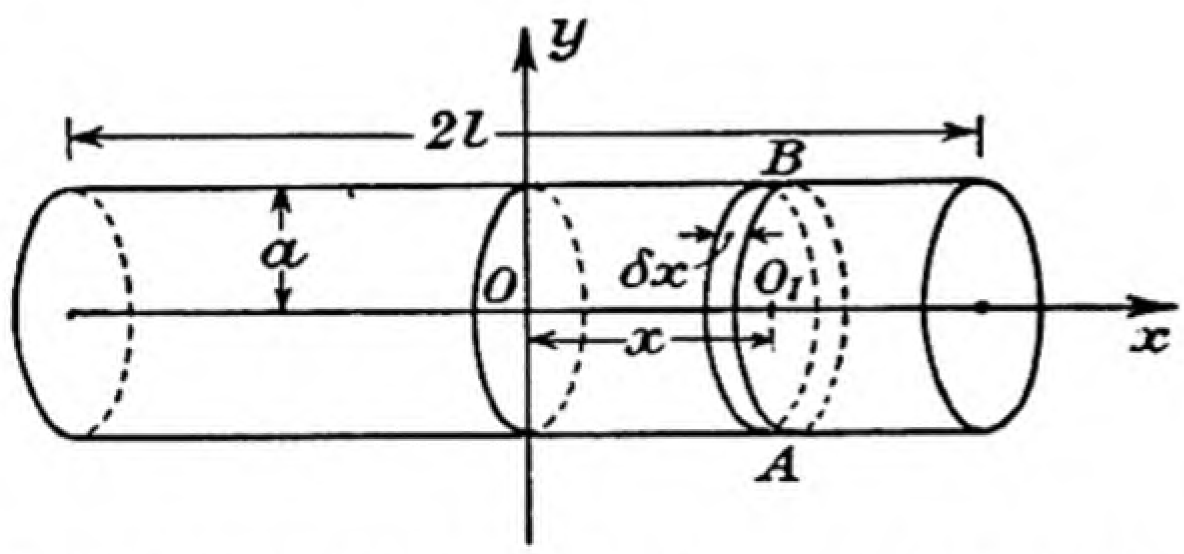


FIG. 3·07.—Moment of inertia of a uniform circular cylinder about Oy.

Its moment of inertia about a diameter is $\frac{1}{4}\pi\rho a^4 . \delta x$. Hence its moment of inertia about Oy is

$$(\pi \rho a^2 \, \delta x)x^2 + \tfrac{1}{4}\pi\rho a^4 \, \delta x = \delta \mathrm{I} \text{ (say)}.$$

$$\therefore \ \mathrm{I} = 2\left[\pi\rho a^2 \int_0^l \left(x^2 + \frac{a^2}{4}\right) dx\right] = 2\pi\rho a^2 \left[\frac{l^3}{3} + \frac{a^2 l}{4}\right]$$

$$= m\left(\frac{l^2}{3} + \frac{a^2}{4}\right),$$

where $m = 2\pi a^2 l \rho$, the mass of the cylinder.

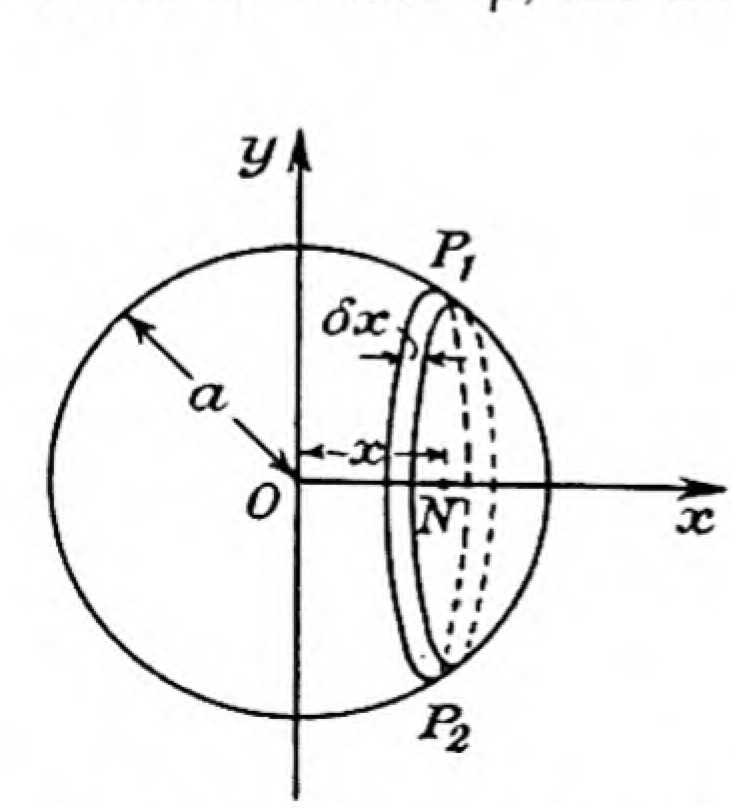


FIG. 3·08.—Moment of inertia of a uniform sphere about a diameter.

Moment of inertia of a uniform sphere about a diameter.—Let O, Fig. 3·08, be the centre of the sphere, radius a, whose material has a density ρ; divide the sphere into sections by planes normal to Ox, the axis about which the moment of inertia is required. Consider the element formed by two such planes at a distance δx apart. If N is the centre of this element of radius y, let ON $= x$. The moment of inertia of this element about Ox is $m . \frac{y^2}{2}$, where m, its mass, is $\pi y^2 \rho . \delta x$.

Hence, I, the moment of inertia of the whole sphere about Ox, is given by

$$\begin{aligned} I &= 2\int_0^a \pi y^2 \rho \left(\frac{y^2}{2}\right) dx \\ &= \pi\rho \int_0^a (a^2 - x^2)^2\, dx \qquad [\because\ a^2 = x^2 + y^2] \\ &= \pi\rho\left[a^5 - 2a^2 . \frac{a^3}{3} + \frac{a^5}{5}\right] \\ &= \tfrac{8}{15}\,\pi\rho a^5. \end{aligned}$$

But the mass of the sphere $= \frac{4}{3}\pi a^3 \rho = m$ (say).

$$\therefore\ I = \tfrac{2}{5}\, ma^2, \quad \text{i.e.} \quad \kappa^2 = \tfrac{2}{5}\, a^2.$$

Uniform bar of rectangular cross-section.—Let O, Fig. 3·09, be the centre of mass of the rectangular bar whose material has

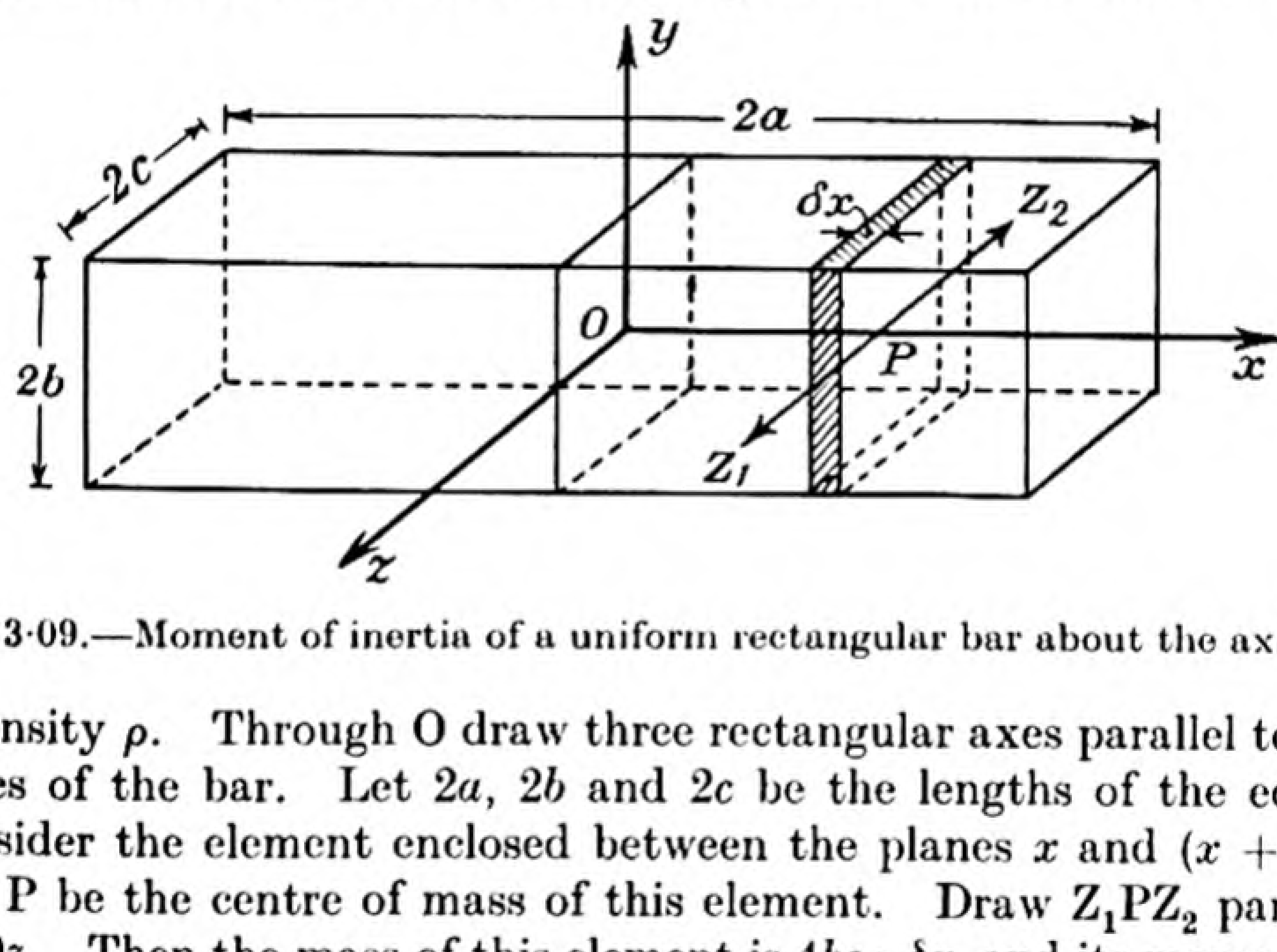


FIG. 3·09.—Moment of inertia of a uniform rectangular bar about the axis Oz.

a density ρ. Through O draw three rectangular axes parallel to the edges of the bar. Let $2a$, $2b$ and $2c$ be the lengths of the edges. Consider the element enclosed between the planes x and $(x + \delta x)$. Let P be the centre of mass of this element. Draw Z_1PZ_2 parallel to Oz. Then the mass of this element is $4bc\rho\,\delta x$, and its moment of inertia about Z_1Z_2 is

$$[4bc\rho\,\delta x] . \tfrac{1}{3}b^2.$$

Hence its moment of inertia about Oz is

$$4bc\rho[\tfrac{1}{3}b^2 + x^2]\,\delta x,$$

and I, the moment of inertia of the whole bar about Oz, is given by

$$\begin{aligned} I &= 2\left[4bc\rho \int_0^a \{\tfrac{1}{3}b^2 + x^2\}\, dx\right] \\ &= 8\rho bc[\tfrac{1}{3}ab^2 + \tfrac{1}{3}a^3]. \end{aligned}$$

But the mass of the bar $= 8abc\rho = m$ (say).

$$\therefore\ \mathrm{I} = \tfrac{1}{3}m(a^2 + b^2),$$

i.e.
$$\kappa^2 = \tfrac{1}{3}(a^2 + b^2).$$

A uniform triangular lamina and some of its moments of inertia.—Let ABC, Fig. 3·10(a), be a uniform triangular lamina of surface density σ. Let Ax and Ay be two mutually perpendicular axes through A, the axis Ax being parallel to the base BC of the

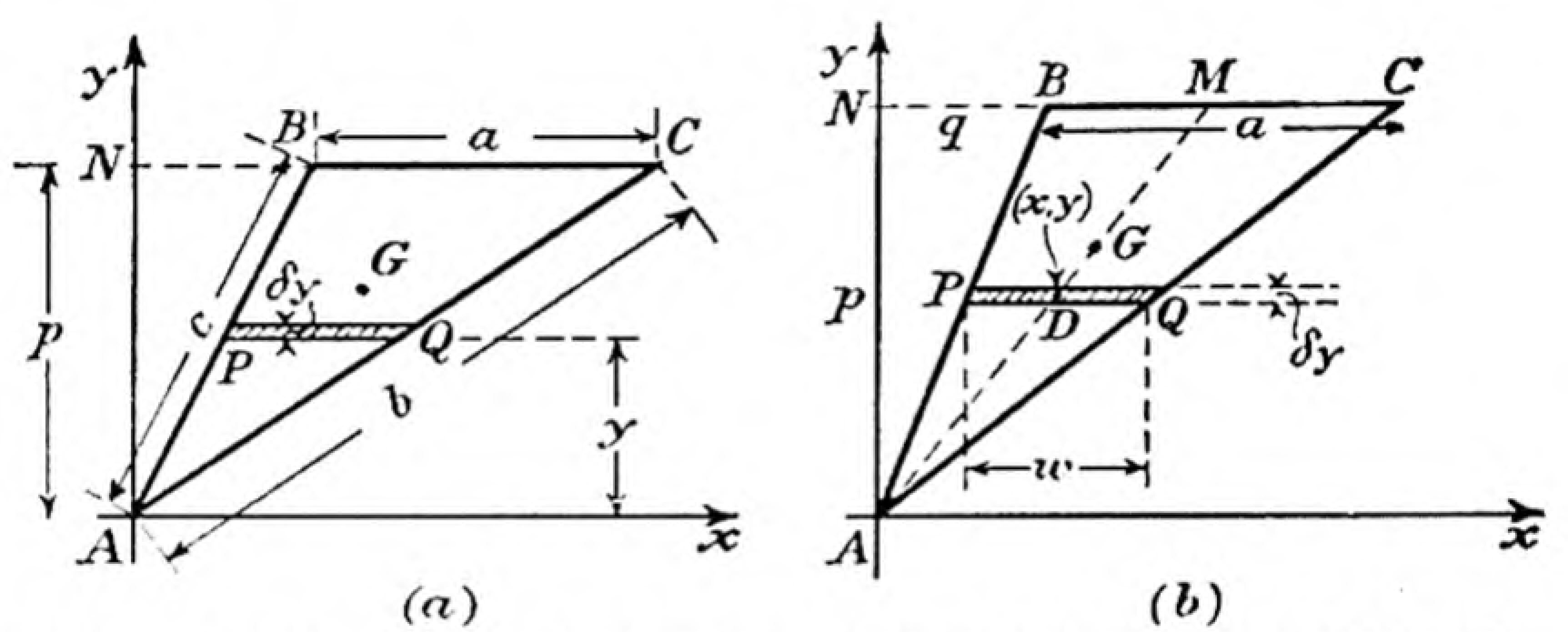


FIG. 3·10.—A uniform triangular lamina and some of its moments of inertia.

triangle. Let p be AN, the projection of AB on Ay. Consider the strip PQ of ordinate y and width δy and its moment of inertia ($\delta \mathrm{I}_x$) about Ax. The length of the strip is $\dfrac{a}{p}.y$.

$$\therefore\ \mathrm{I}_x = \frac{a\sigma}{p}\int_0^p y.y^2\,dy = \frac{a\sigma}{4}p^3$$
$$= \tfrac{1}{2}mp^2,$$

where $m = \frac{1}{2}ap\sigma$, the mass of the triangle.

Using the theorem of parallel axes, we find, if I_G is the moment of inertia of the triangle about a parallel axis through its centre of gravity G,

$$\mathrm{I}_\mathrm{G} + m(\tfrac{4}{9}p^2) = \mathrm{I}_x,$$

i.e.
$$\mathrm{I}_\mathrm{G} = \tfrac{1}{18}mp^2.$$

Similarly,
$$\mathrm{I}_{\mathrm{BC}} = \tfrac{1}{6}mp^2.$$

To find the moment of inertia of the triangle about Ay, consider the element PQ, Fig. 3·10(b), of length w, say, and width δy. If D

is the centre of this element, and (x, y) its coordinates referred to A, applying the theorem of parallel axes, we have

$$\delta I_y = (w\sigma\,\delta y)\left[\frac{1}{3}\left(\frac{w}{2}\right)^2 + x^2\right].$$

Now
$$\frac{y}{x} = \frac{p}{q + \frac{1}{2}a}, \quad \text{and} \quad w = \frac{a}{p}\,y.$$

$$\therefore\ I_y = \int_0^p \sigma\frac{a}{p}\,y\left[\frac{1}{3}\left(\frac{a}{p}\frac{y}{2}\right)^2 + \left(\frac{q + \frac{1}{2}a}{p}.y\right)^2\right]dy$$

$$= \frac{\sigma a}{p}\left[\frac{1}{48}\,a^2p^2 + (q + \tfrac{1}{2}a)^2\frac{p^2}{4}\right]$$

$$= 2m\left[\frac{a^2}{12} + \frac{q^2 + aq}{4}\right]$$

$$= m\left[\frac{a^2}{6} + \frac{aq + q^2}{2}\right].$$

Hence, if I_a, is the moment of inertia of the triangle about an axis through A, perpendicular to the plane of the diagram, we have

$$I_a = I_x + I_y = m\left[\frac{p^2}{2} + \frac{a^2}{6} + \frac{aq + q^2}{2}\right]$$

$$= m\left[\frac{b^2 + c^2}{4} - \frac{a^2}{12}\right],$$

using $b^2 = p^2 + (a + q)^2$, and $c^2 = p^2 + q^2$.

This important result may be obtained more quickly as shown in the next paragraph.

To find the radius of gyration of a uniform triangular lamina, ABC, about an axis through A normal to its plane.—At a point P, Fig. 3·11, in the plane of the lamina let δS be an element

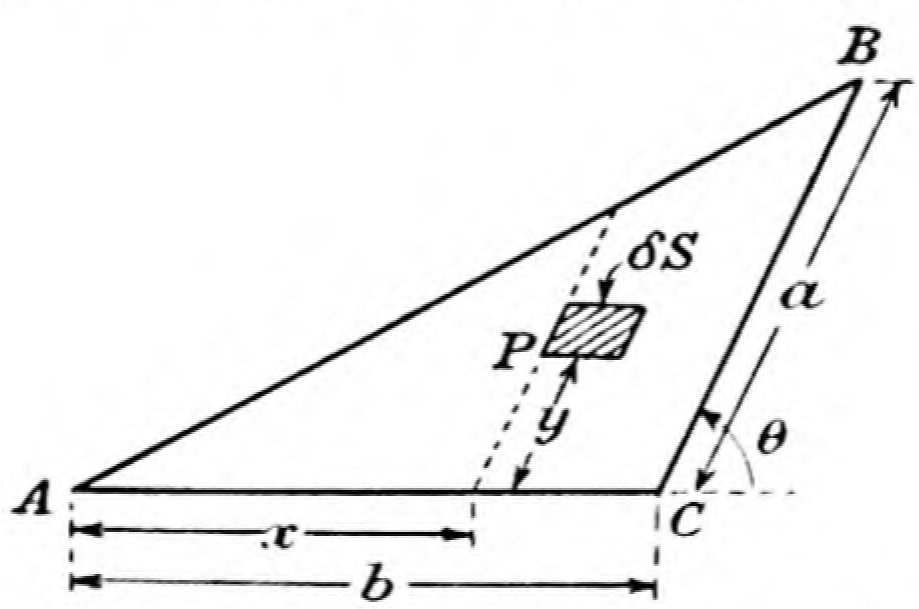


FIG. 3·11.—Moment of inertia of a uniform triangular lamina about an axis through an apex and normal to the plane of the lamina.

of area, its sides being parallel to AC and CB respectively. Let (x, y) be the coordinates of P referred to oblique axes through A, θ or $(\pi - C)$ being their mutual inclination. If σ is the mass per unit area of the triangle and κ the required radius of gyration

$$\tfrac{1}{2}(\sigma ab \sin C)\kappa^2 = \int_0^b \int_{y=0}^{y=\frac{a}{b}x} (x^2 + y^2 + 2xy\cos\theta)\sigma \sin\theta \, dx\, dy$$

$$= \int_0^b \left[x^2 y + \tfrac{1}{3}y^3 + xy^2 \cos\theta \right]_{y=0}^{y=\frac{a}{b}x} \sigma \sin\theta \, dx$$

$$= \left(\frac{a}{b} + \frac{1}{3}\frac{a^3}{b^3} + \frac{a^2}{b^2}\cos\theta \right) \sigma \sin\theta \int_0^b x^3 \, dx.$$

Since $\sin C = \sin\theta$, and $c^2 = a^2 + b^2 + 2ab\cos\theta$, from the above we obtain, after some reduction,

$$\kappa^2 = \tfrac{1}{12}[3(b^2 + c^2) - a^2].$$

Routh's rule.—The values of several radii of gyration are easily remembered by the following convenient rule first given by Routh. It applies to linear, plane, and solid homogeneous bodies which are

(α) rectangular (rod, lamina)
(β) circular or elliptical (disc)
(γ) spherical, spheroidal, or ellipsoidal,

and states that ***the radius of gyration about an axis of symmetry passing through the mass-centre of the body is given by***

$$\kappa^2 = \frac{\textit{sum of squares of perpendicular semi-axes}}{3,\ 4 \text{ or } 5.},$$

where the denominator is 3, 4 or 5, according as the body falls into the (α), (β) ***or*** (γ) ***division of the above classification.***

Lees' rule.—If m is the mass of a homogeneous solid of revolution, a and b its semi-axes normal to that about which the moment of inertia, I, is required, C_a and C_b the type of curvature of the surface of revolution in which each semi-axis a and b ends (Single, $C = 1$; double $C = 2$), then

$$I = m\left[\frac{a^2}{3 + C_a} + \frac{b^2}{3 + C_b}\right].$$

Thus, for a sphere, its moment of inertia about a diameter of length $2a$ is

$$I = m\left[\frac{a^2}{3+2} + \frac{a^2}{3+2}\right] = \frac{2}{5}ma^2.$$

Example.—A thin plate is symmetrical about perpendicular axes Ox and Oy. An axis OA is drawn in the plane of Ox and Oy to make an angle θ with Ox. If the moments of inertia of the plate about Ox and Oy are I_x and I_y respectively, show that the moment of inertia of the plate about OA is

$$\mathrm{I}_x \cos^2 \theta + \mathrm{I}_y \sin^2 \theta.$$

The equation to OA is $y = ax$, where $a = \tan \theta$. Consider an element of the plate at P (x_1, y_1) and let its mass be δm. If p is the shortest distance of P from OA then, with the usual notation,

$$\delta\mathrm{I} = \delta m(p^2).$$

To find p it is observed that a straight line through P parallel to OA has for its equation

$$y - y_1 = a(x - x_1),$$

so that this line makes an intercept $(y_1 - ax_1)$ on the y-axis.

$$\therefore\ p = (y_1 - ax_1)\cos\theta = y_1 \cos\theta - x_1 \sin\theta.$$

$$\therefore\ \mathrm{I} = \Sigma\,[\delta m(y_1 \cos\theta - x_1 \sin\theta)^2]$$

$$= \mathrm{I}_x \cos^2\theta + \mathrm{I}_y \sin^2\theta,$$

since $\Sigma\ \delta m(y_1)^2 = \mathrm{I}_x$, etc. and, on account of symmetry,

$$\Sigma\,[\delta m(x_1 y_1 \sin\theta \cos\theta)] = 0.$$

Example.—A thin rod can rotate about a horizontal axis passing through one end of the rod. The rod is initially vertical and in unstable equilibrium. If it is given a small displacement and falls from rest in this position, determine from energy considerations the angular velocity with which it passes through (a) the horizontal position and (b) the position of stable equilibrium.

Let AB, Fig. 3·12, be the initial position of the rod. When the rod has fallen through an angle θ, its centre of gravity will have fallen through a vertical distance $a(1 - \cos\theta)$, if $2a$ is the length of the rod. The loss in potential energy is therefore $mga(1 - \cos\theta)$, if m is the mass of the rod. Now the kinetic energy of the rod is $\frac{1}{2}\mathrm{I}\dot{\theta}^2$, where I is the moment of inertia of the rod about the axis through A which is normal to the plane of the diagram. The energy equation is therefore

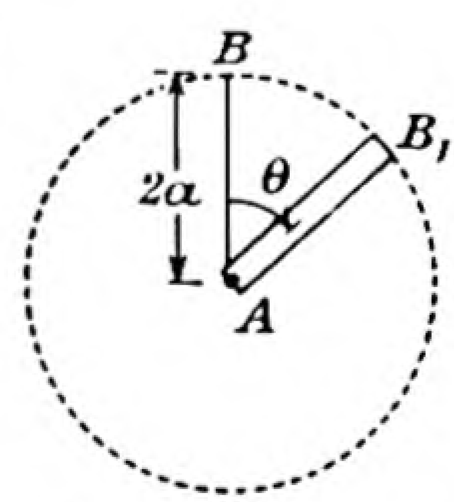


FIG. 3·12.—Motion of a thin rod about a horizontal axis.

$$mga(1 - \cos\theta) = \tfrac{1}{2}m[a^2 + \tfrac{1}{3}a^2]\dot{\theta}^2 = \tfrac{2}{3}ma^2\dot{\theta}^2.$$

In the horizontal position $\theta = \dfrac{\pi}{2}$. If ω_1 is the angular velocity in this position

$$\tfrac{2}{3}ma^2\omega_1{}^2 = mga.$$

$$\therefore\ \omega_1 = \sqrt{\frac{3g}{2a}}.$$

In the position of stable equilibrium, $\theta = \pi$. If ω_2 is the angular velocity in this instance

$$\tfrac{2}{3}ma^2\omega_2^{\,2} = 2mga.$$

$$\therefore\ \omega_2 = \sqrt{\frac{3g}{a}}.$$

Example.—A uniform rod OA of length $2a$ and mass m can turn freely about a horizontal axis through O. Initially the rod is at rest in a horizontal plane. After being set free the end A picks up a particle of mass M; if this occurs when the rod is passing through the vertical position, determine the vertical distance the particle moves before the system first becomes momentarily at rest.

In Fig. 3·13, the straight line OA_0 represents the position from which

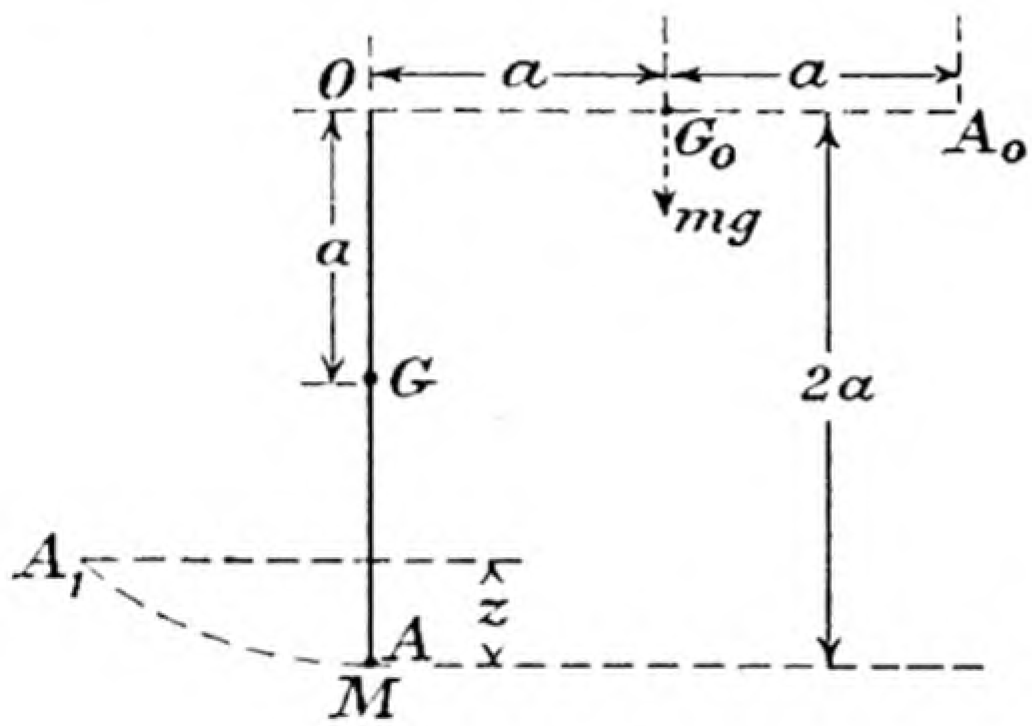


FIG. 3·13.

the rod is released, while OA is its position when the particle M is picked up. In falling to this position the rod has lost potential energy amounting to mga and this is equal to the kinetic energy of the rod as it passes through OA. Hence, if ω_0 is the angular velocity of the rod in this position and I its moment of inertia about the axis of rotation,

$$\begin{aligned} mga &= \tfrac{1}{2}\mathrm{I}\omega_0^{\,2} \\ &= \tfrac{1}{2}\,.\,\tfrac{1}{3}\,.\,m\,.\,4a^2\,.\,\omega_0^{\,2} = \tfrac{2}{3}ma^2\omega_0^{\,2}. \end{aligned}$$

$$\therefore\ \omega_0^{\,2} = \frac{3}{2}\,\frac{g}{a}.$$

If ω is the angular velocity with which the system is moving just after picking up the particle and z the vertical distance through which A rises to A_1, when the system is momentarily at rest, the energy principle gives us

$$\tfrac{1}{2}\{\tfrac{4}{3}ma^2 + \mathrm{M}(2a)^2\}\omega^2 = \mathrm{M}gz + mg(\tfrac{1}{2}z) \qquad . \quad . \quad \text{(i)}$$

The angular momentum is conserved when the particle is picked up so that

$$\tfrac{4}{3}ma^2\,.\,\omega_0 = \{\mathrm{M}(2a)^2 + \tfrac{4}{3}ma^2\}\omega.$$

$$\therefore\ \omega^2 = \frac{m^2}{(3\mathrm{M}+m)^2}\,.\,\frac{3}{2}\,.\,\frac{g}{a} \qquad . \quad . \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

Hence, eliminating ω^2 from (i) and (ii) we find, after some reduction,

$$z = \frac{2m^2a}{(3M + m)(2M + m)}.$$

Theorems of Pappus.—(*a*) ***If an arc of a plane curve revolve about an axis in its plane, but not intersecting it, the area of the surface generated is equal to the length of the arc multiplied by the length of the path of its centroid.***

Let Ox be the axis of rotation and y the ordinate of a point on the arc. The area of the surface generated in a complete revolution is $2\pi\int y\,ds$, where δs is an element of the arc and the integration extends over it. But if $\bar{y}$ refers to the centroid of the arc

$$\bar{y} = \left\{\int y\,ds\right\} \div \int ds$$

$$\therefore\ 2\pi \int y\,ds = 2\pi\bar{y} \int ds,$$

which establishes the theorem.

(*b*) ***If a plane area revolve about an axis in its plane, but not intersecting it, the volume generated is equal to the area multiplied by the path of its centroid.***

Let Ox be the axis of rotation and y the perpendicular distance of an element of area δS from Ox. If δS describes a complete revolution about Ox the volume generated is $2\pi\int y\,d\mathrm{S}$.

But
$$\bar{y} = \left\{\int y\,d\mathrm{S}\right\} \div \int d\mathrm{S},$$

so that $2\pi\int y\,d\mathrm{S} = 2\pi\bar{y}\int d\mathrm{S}$, which establishes the theorem.

Moment of momentum.—Consider a particle of mass m at a point (x, y, z) referred to three rectangular axes. If X, Y and Z be the components of the resultant external forces acting on the particle, then the equations of motion are

$$\left.\begin{aligned} m\ddot{x} &= \mathrm{X} \\ m\ddot{y} &= \mathrm{Y} \\ m\ddot{z} &= \mathrm{Z} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

The moment about the x-axis of the forces acting on the particle is

$$y\mathrm{Z} - z\mathrm{Y} = m(y\ddot{z} - z\ddot{y}) \quad . \quad . \quad . \quad \text{(ii)}$$

the moment about the x-axis being considered positive when the force tends to rotate the body in the direction indicated in Fig. 3·14.

Now the velocity of the particle is $(\dot{x}, \dot{y}, \dot{z})$, so that the moment of the velocity about the x-axis is

$$y\dot{z} - z\dot{y}.$$

The moment of momentum about the same axis is m times the moment of the velocity, viz.

$$m(y\dot{z} - z\dot{y}).$$

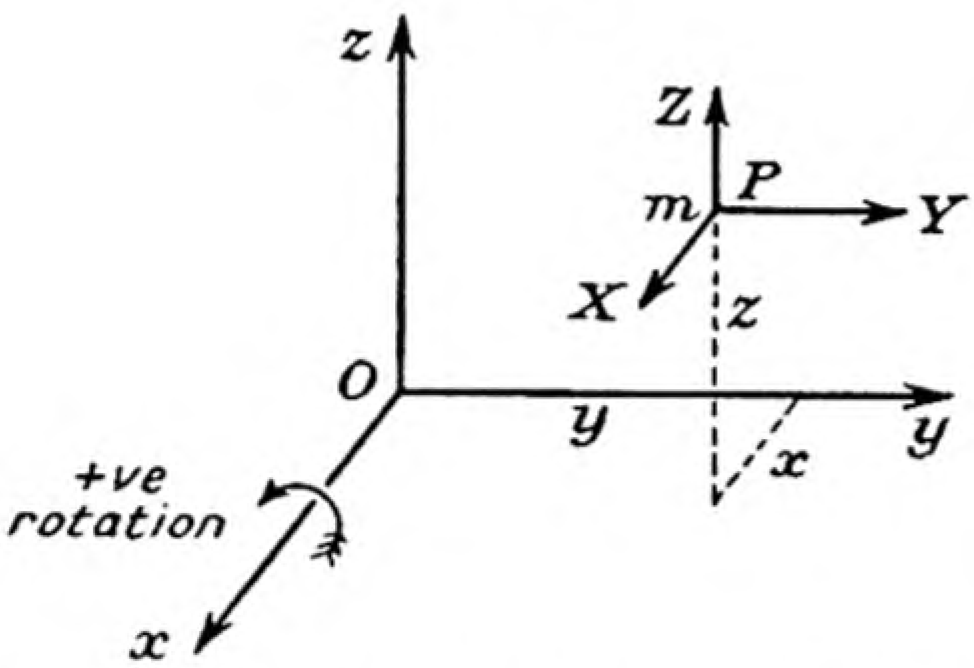


FIG. 3·14.—Moment of momentum; angular momentum

To find the rate at which the moment of momentum about this axis is changing, we differentiate the above expression with respect to the time, t, and obtain

$$m(\dot{y}\dot{z} + y\ddot{z} - \dot{z}\dot{y} - z\ddot{y}) = m(y\ddot{z} - z\ddot{y}) = y\mathrm{Z} - z\mathrm{Y} \qquad \text{[by (i)]}.$$

Hence the rate of change of the moment of momentum of a particle about any fixed axis is equal to the moment, about the same axis, of the external forces acting on the particle.

Motion of a rigid body; rotation about a fixed axis. d'Alembert's principle.—It has already been shown that if a particle of mass m is at a point (x, y, z) at time t, then its motion is found by equating $m\ddot{x}$ to the x-component of the force, together with similar equations for the other two axes. The quantity $m\ddot{x}$ is called the effective force acting on the particle; it is a vector parallel to the x-axis. By Newton's second law of motion the effective force acting on a particle must be equal in every respect to the resultant in the x-direction of all the forces acting on it. To deal with the motion of a rigid body a new principle is needed; this principle was given by d'ALEMBERT in 1743 and to appreciate its meaning we may proceed as follows.

In the case of a particle forming part of a material assemblage, the forces which act upon it may be divided into two classes:—

(a) The 'external (or applied) forces' acting from without the assemblage.

(b) The 'internal forces' or reactions due to the remaining particles.

Considering the whole assemblage it may be said that the system of localized vectors which represent the effective forces is statically equivalent to the combined effect of the external and internal forces. The assumption made by d'Alembert is that the internal forces constitute a system which is in equilibrium. If this assumption is granted it follows at once that the system of effective forces is, as a whole, statically equivalent to the system of external forces. To express this result analytically, let (x, y, z) denote the position, relative to fixed rectangular axes, of a particle P, Fig. 3·15. Then if m is the mass of this particle and (X, Y, Z) the external force acting upon it

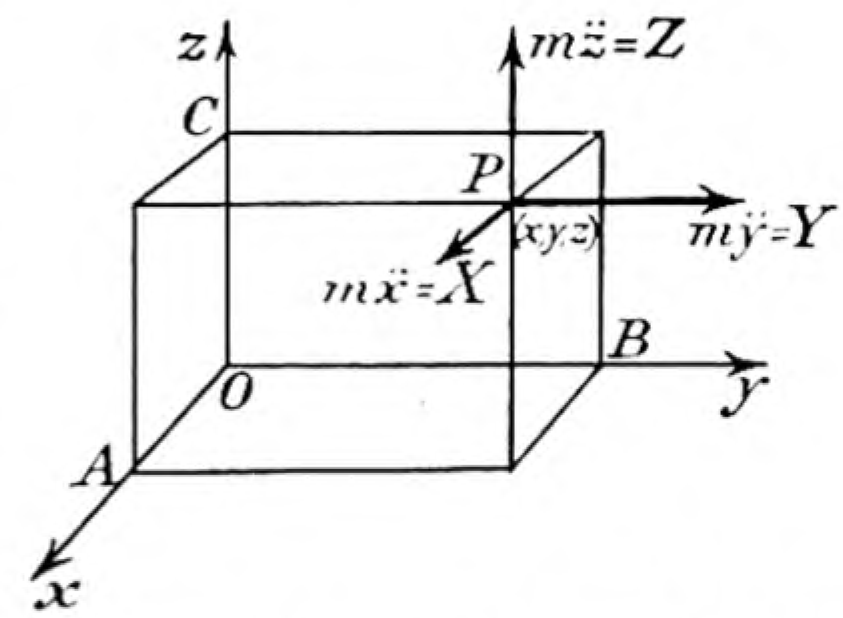


FIG. 3·15.—Motion of a rigid body; notation about a fixed axis.

$$m\ddot{x} = \mathrm{X}, \qquad m\ddot{y} = \mathrm{Y}, \qquad m\ddot{z} = \mathrm{Z}.$$

If we now regard P as a particle in a rigid body we have

$$\Sigma\, m\ddot{x} = \Sigma\, \mathrm{X}, \quad \Sigma\, m\ddot{y} = \Sigma\, \mathrm{Y}, \qquad \Sigma\, m\ddot{z} = \Sigma\, \mathrm{Z},$$

and taking moments of forces about the axis Oz, obtain

$$\Sigma\,(x.m\ddot{y} - y.m\ddot{x}) = \Sigma\,(x\mathrm{Y} - y\mathrm{X}),$$

where the summations include all the particles in the rigid body. Since $\dfrac{d}{dt}\Sigma\,(m\dot{z}) = \Sigma\,(m\ddot{z}) = \Sigma\,\mathrm{Z}$ etc., we may write

$$\frac{d}{dt}\Sigma\,(x.m\dot{y} - y.m\dot{x}) = \Sigma\,(x\mathrm{Y} - y\mathrm{X}),$$

together with two other pairs of equations of the same type. Since the z-axis may have any fixed position in space, the pair of equations written down for that axis really express the following laws.

(a) ***The rate of increase of linear momentum in any given direction is equal to the resultant external force in that direction.***

(b) ***The rate of increase of the moment of momentum of a rigid body about any fixed axis is equal to the total moment of all the external forces about that axis.***

If the moment, about any fixed axis, of the external forces, acting on a rigid body, is zero, the moment of momentum of the body about that axis is constant.

Atwood's machine.—To illustrate the theory of rigid bodies rotating about a fixed axis, let us consider a simple form of Atwood's machine for determining the intensity of gravity, g. Fig. 3·16(a) shows a pulley wheel of diameter $2a$; a light string passes over the groove in the pulley and masses m_1 and m_2 are attached to its free ends. It will be assumed that the free portions of the string are vertical and there is no slipping between the string and the disc. Suppose $m_1 > m_2$ and that the system moves from rest.

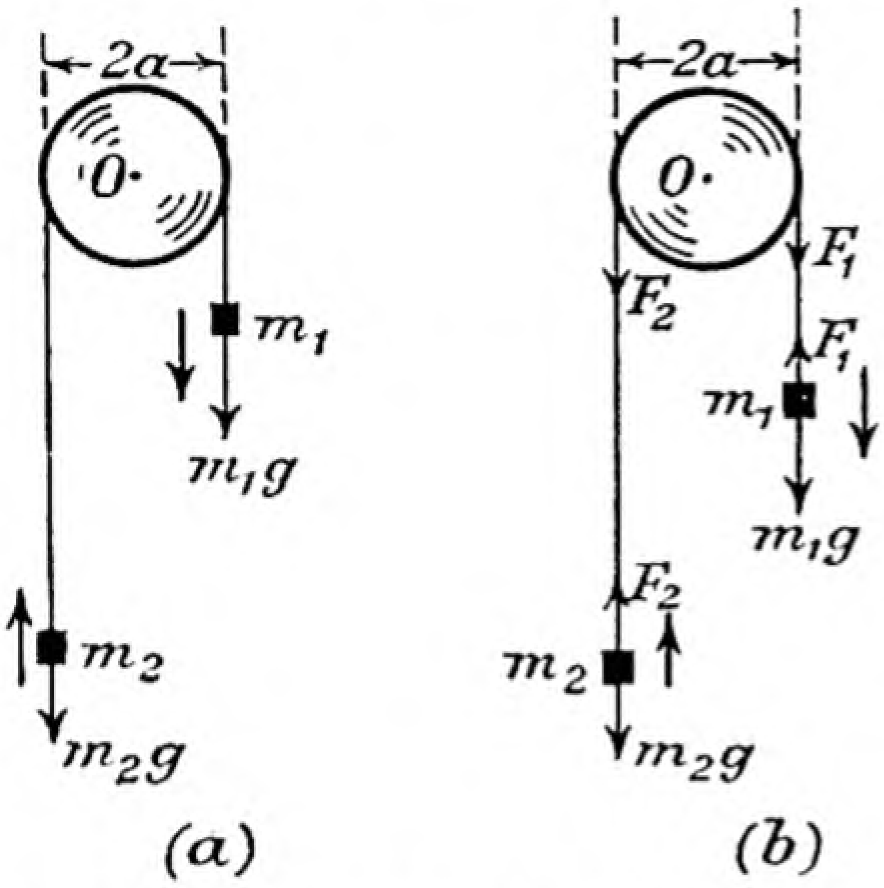


Fig. 3·16.—The motion of the two masses and pulley in an Atwood's machine.

(a) First consider the system as a whole and let z be the distance the masses move from rest. Then the couple acting on the system is $(m_1 - m_2)ga$ and this is equal to the increase in angular momentum of the system in so far as it is correct to regard this as a rigid body. If I is the moment of inertia of the disc about its axis of revolution, the angular momentum of the system is

$$[\quad + (m_1 + m_2)a^2]\dot{\theta},$$

where $\dot{\theta}$ is the angular velocity of the disc,

i.e.
$$(m_1 - m_2)ga = [\mathrm{I} + (m_1 + m_2)a^2]\,\frac{\ddot{z}}{a}.$$

$$[\because\ z = a\theta, \text{ since } \dot{z} = a\dot{\theta}, \text{ etc.}]$$

$$\therefore\ \ddot{z} = \frac{(m_1 - m_2)g}{(m_1 + m_2) + \dfrac{\mathrm{I}}{a^2}}.$$

(b) To verify this solution, let us apply the principle of the conservation of energy. The loss in potential energy of the system is

$$(m_1 - m_2)\,gz,$$

which is equal to its gain in kinetic energy, viz.

$$\tfrac{1}{2}(m_1 + m_2)\dot{z}^2 + \tfrac{1}{2}\mathrm{I}\dot{\theta}^2,$$

or
$$\tfrac{1}{2}(m_1 + m_2)\dot{z}^2 + \tfrac{1}{2}\,\frac{\mathrm{I}}{a^2}.\dot{z}^2.$$

Equating these two expressions and then differentiating with respect to time, we get

$$(m_1 - m_2)g\dot{z} = (m_1 + m_2)\dot{z}\ddot{z} + \frac{I}{a^2}\dot{z}\ddot{z},$$

whence, as before

$$\ddot{z} = \frac{(m_1 - m_2)g}{(m_1 + m_2) + \dfrac{I}{a^2}}.$$

(*c*) Now let F_1 and F_2 be the tensions in the string when the acceleration of each mass, (irrespective of sign), is $\ddot{z}$. Then, cf. Fig. 3·16(*b*),

$$m_1\ddot{z} = m_1 g - F_1, \quad \text{and} \quad m_2\ddot{z} = F_2 - m_2 g.$$

The couple acting on the wheel and giving it an angular acceleration $\ddot{\theta}$, where $a\ddot{\theta} = \ddot{z}$, is $(F_1 - F_2)a$, so that

$$(F_1 - F_2)a = I\ddot{\theta} = I\frac{\ddot{z}}{a}.$$

Eliminating $(F_1 - F_2)$, we find

$$\begin{aligned} m_1\ddot{z} + m_2\ddot{z} &= (m_1 - m_2)g - (F_1 - F_2) \\ &= (m_1 - m_2)g - \frac{I}{a^2}.\ddot{z}. \end{aligned}$$

$$\therefore\ \ddot{z} = \frac{(m_1 - m_2)g}{m_1 + m_2 + \dfrac{I}{a^2}}.$$

The agreement between the results thus obtained by three different methods shows that the underlying assumptions are correct and this is particularly satisfactory since it shows that the system may be regarded as a rigid body; but this is only because the distances of m_1 and m_2 from the vertical through O remain constant. If the masses begin to oscillate as simple pendulums, the system is no longer a rigid one; cf. the oscillations of a physical balance, p. 140.

***Experiment*.—To determine the moment of inertia of a fly-wheel about its axis of rotation.**—The moment of inertia, about its axis of rotation, of a wheel with a long axle may be determined as follows. The wheel is mounted in a horizontal position, the axle being supported on ball bearings to diminish friction as much as possible—cf. Fig. 3.17. A known mass is attached to the axle, at a point where there is either a hole or a pin, by a cord wrapped several times round the axle. The potential energy lost by the mass in its descent is, in the absence of

frictional forces, communicated to the wheel and to the mass itself as kinetic energy, i.e.

$$\begin{bmatrix}\text{Potential energy lost}\\ \text{by falling mass}\end{bmatrix} = \begin{bmatrix}\text{Kinetic energy gained}\\ \text{by fly-wheel}\end{bmatrix} + \begin{bmatrix}\text{Kinetic energy gained}\\ \text{by the mass itself}\end{bmatrix}.$$

For this experiment the length of cord is adjusted so that when the mass reaches its lowest position (the floor) the other end of the cord just leaves the axle.

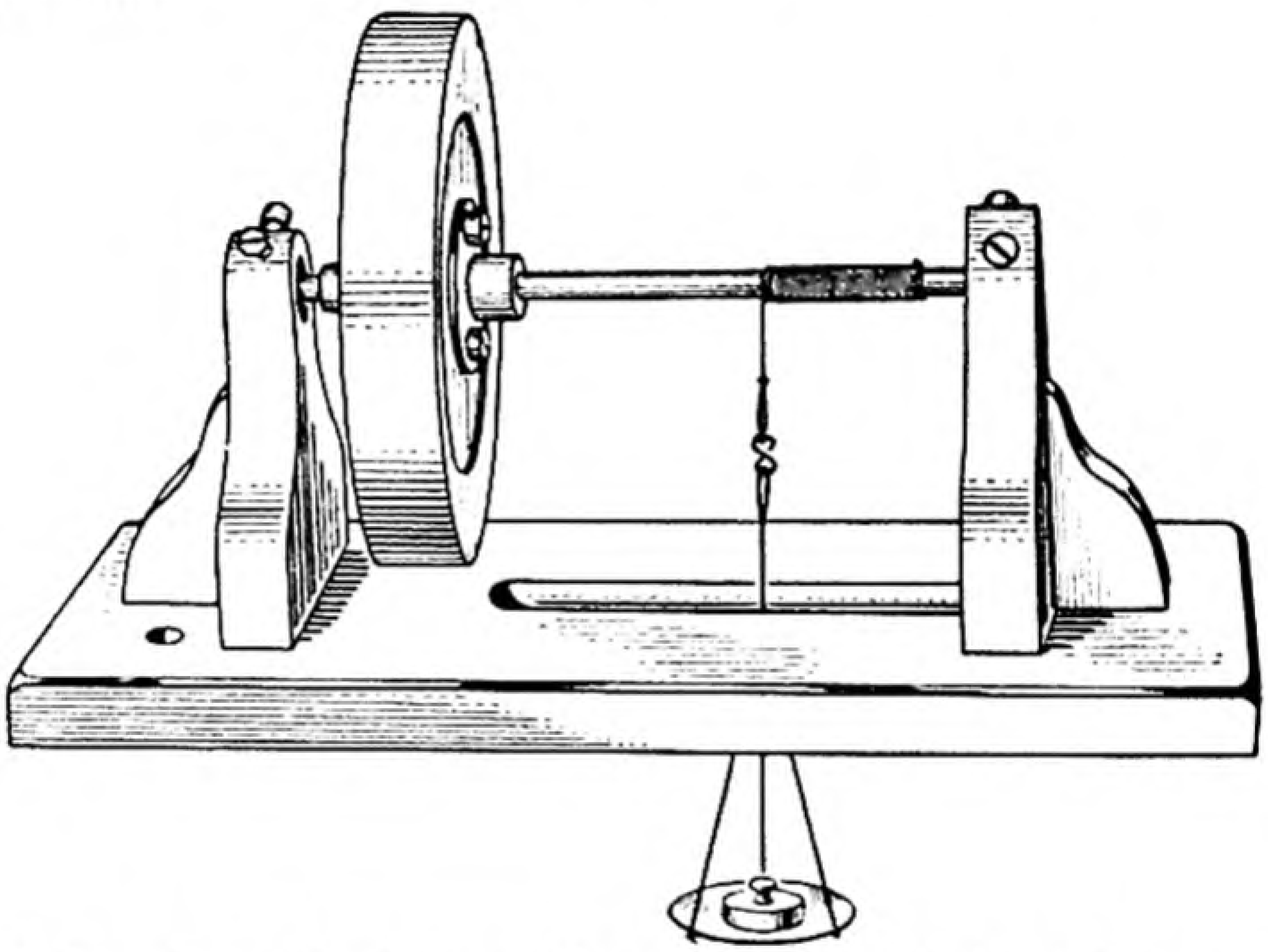


FIG. 3·17.—A fly-wheel—experimental determination of its moment of inertia about its axis of rotation.

In any actual experiment it is essential to make a correction for the work done against friction; to do this we proceed as follows.

Let m be the mass (that of the pan being included) and h the distance it falls before being arrested; the wheel make N_1 revolutions from rest before the falling mass m is removed. If X is the work done against friction in one revolution, and this is assumed constant,

$$mgh = \tfrac{1}{2}I\omega^2 + N_1X + \tfrac{1}{2}mv^2 = \tfrac{1}{2}(I + ma^2)\omega^2 + N_1X,$$

where I is the moment of inertia of the wheel about its axis of rotation, ω the angular velocity of the wheel when the cord falls away from the axle, and a is the radius of the axle.

If the wheel makes N_2 revolutions before coming to rest

$$\tfrac{1}{2}I\omega^2 = N_2X.$$

$$\therefore mgh = \tfrac{1}{2}\left(I + \frac{N_1}{N_2}\right)\omega^2 + \tfrac{1}{2}ma^2\omega^2$$

$$= \left[\tfrac{1}{2}\left(I + \frac{N_1}{N_2}\right) + \tfrac{1}{2}ma^2\right]\frac{4h^2}{a^2t^2} \quad \left[\because \omega = \frac{2h}{at}\right]$$

This equation gives

$$I = ma^2\left(\frac{N_2}{N_1 + N_2}\right)\left(\frac{gt^2}{4\pi aN_1} - 1\right).$$

The experiment should be repeated two or three times with the same mass and a mean value of $\frac{N_1}{t}$ used in deducing ω. The value for I thus obtained should be checked by using other masses.

To determine N_1 and N_2 it is convenient to place a chalk mark on the circumference of the wheel so that it is visible when the falling mass is arrested.

The diameter of the cord used should be small compared with that of the axle, otherwise the value of a in the above equations is the sum of the radii of the axle and of the cord.

A less accurate method of ascertaining the velocity of the mass when it touches the floor (and hence ω) is to determine the time of fall of the mass. Let this be t_1. Since the mass has fallen with uniform acceleration, the final velocity will be twice the average velocity, the initial velocity being zero, i.e.

$$v = 2\left(\frac{h}{t_1}\right).$$

Unfortunately t_1 is usually small and cannot be measured accurately.

Motion of the mass-centre of a rigid body and motion relative to that centre.—Referred to rectangular coordinates let $(\bar{x}, \bar{y}, \bar{z})$ be the mass-centre of a rigid body of mass M. Then

$$M\bar{x} = \Sigma\,(mx),\quad M\bar{y} = \Sigma\,(my)\quad \text{and}\quad M\bar{z} = \Sigma\,(mz)$$

throughout the motion. Thus, as on p. 83,

$$M\ddot{\bar{x}} = \Sigma\,X,\quad M\ddot{\bar{y}} = \Sigma\,Y\quad \text{and}\quad M\ddot{\bar{z}} = \Sigma\,Z.$$

These, however, are the equations of motion of a particle of mass M, placed at the mass-centre of the body and acted upon by forces parallel to, and equal to, the external forces acting on the body.

Hence ***the mass-centre of a rigid body moves as if all the mass of the body were collected at it and all external forces were acting on it in directions parallel to those in which they act.***

Next let (ξ, η, ζ) be the coordinates, relative to G, of a particle of the body whose coordinates referred to the original rectangular axes were (x, y, z); corresponding axes are parallel to each other. Then, throughout the motion,

$$x = \bar{x} + \xi,\qquad y = \bar{y} + \eta,\qquad z = \bar{z} + \zeta.$$

$$\therefore\ \ddot{x} = \ddot{\bar{x}} + \ddot{\xi},\qquad \ddot{y} = \ddot{\bar{y}} + \ddot{\eta},\qquad \ddot{z} = \ddot{\bar{z}} + \ddot{\zeta}.$$

Hence the equation

$$\Sigma\, m(x\ddot{y} - y\ddot{x}) = \Sigma\,(xY - yX),$$

given on p. 83, becomes

$$\Sigma\, m(\bar{x}\ddot{\bar{y}} - \bar{y}\ddot{\bar{x}}) + \Sigma\, m(\xi\ddot{\eta} - \eta\ddot{\xi}) + \Sigma\, m(\bar{x}\ddot{\eta} + \xi\ddot{\bar{y}} - \bar{y}\ddot{\eta} - \eta\ddot{\bar{x}})$$
$$= \Sigma\,(\bar{x} + \xi)Y - (\bar{y} + \eta)X.$$

Now $\dfrac{\Sigma\,(m\xi)}{\Sigma\, m}$ = the ξ-coordinate of the mass-centre referred to itself as origin

$= 0.$

$$\therefore\ \Sigma\,(m\xi) = 0, \quad \text{and} \quad \Sigma\,(m\xi) = 0.$$

Similarly,

$$\Sigma\,(m\eta) = 0, \quad \text{and} \quad \Sigma\,(m\ddot{\eta}) = 0.$$

$$\therefore\ \mathrm{M}(\bar{x}\ddot{\bar{y}} - \bar{y}\ddot{\bar{x}}) + \Sigma\, m(\xi\ddot{\eta} - \eta\ddot{\xi}) = \Sigma\,(\bar{x}\mathrm{Y} - \bar{y}\mathrm{X} + \xi\mathrm{Y} - \eta\mathrm{X}).$$

But the first two terms on the l.h.s. of this equation equal the first two terms on the r.h.s.

$$\therefore\ \Sigma\, m(\xi\ddot{\eta} - \eta\ddot{\xi}) = \Sigma\,(\xi\mathrm{Y} - \eta\mathrm{X}),$$

an equation which implies that the motion of a body about its mass-centre is the same as it would be if the mass-centre were fixed and the same forces acted on the body.

The independence of translation of the centre of mass of a rigid body and rotation about an axis passing through it.—In the two previous paragraphs we have established the complete independence of the translation of the centre of mass of a rigid body moving in any manner and its rotation about an axis through its mass-centre.

As an example we may consider a shell which is in motion in a vacuum and in a uniform gravitational field. Suppose an explosion occurs and the shell bursts into fragments. The internal forces exerted by the explosion balance one another and do not in any way influence the motion of the mass-centre of the shell; this centre will therefore continue to move in the same parabolic path in which it was moving before the explosion occurred.

Final form of the two-dimensional equations of rotation.— Now while the equations

$$\mathrm{M}\ddot{\bar{x}} = \Sigma\,\mathrm{X}, \quad \mathrm{M}\ddot{\bar{y}} = \Sigma\,\mathrm{Y} \quad \text{and} \quad \mathrm{M}\ddot{\bar{z}} = \Sigma\,\mathrm{Z}$$

are in a form suitable for direct application to a practical problem, yet the equations, of which

$$\Sigma\, m(\xi\ddot{\eta} - \eta\ddot{\xi}) = \Sigma\,(\xi\mathrm{Y} - \eta\mathrm{X})$$

is typical, need a simple transformation before they become really useful. Consider the two dimensional case, as expressed by the above equation. The l.h.s. may be written

$$\frac{d}{dt}\,\Sigma\, m(\xi\dot{\eta} - \eta\dot{\xi}).$$

Let (r, θ) be the polar coordinates, referred to G as origin, of the particle whose mass is m. Then $\xi = r\cos\theta$ and $\eta = r\sin\theta$, where r is independent of time.

$$\therefore\ \xi\dot{\eta} - \eta\dot{\xi} = r\cos\theta(r\cos\theta\,.\,\dot{\theta}) - r\sin\theta(-r\sin\theta\,.\,\dot{\theta})$$
$$= r^2\dot{\theta}.$$

Now $\dot{\theta}$ is the angular velocity of the lamina, cf. p. 50, say ω, so that

$$\begin{aligned}\Sigma\, m(\xi\ddot{\eta} - \eta\ddot{\xi}) &= \frac{d}{dt}\,\Sigma\, m(\xi\dot{\eta} - \eta\dot{\xi}) \\ &= \frac{d}{dt}\,\Sigma\, mr^2\dot{\theta} \\ &= \ddot{\theta}\,\Sigma\, mr^2 \\ &= \mathrm{I}\ddot{\theta}, \quad \text{or} \quad \mathrm{I}\dot{\omega},\end{aligned}$$

where I is the moment of inertia of the lamina about an axis through G normal to the plane of the lamina.

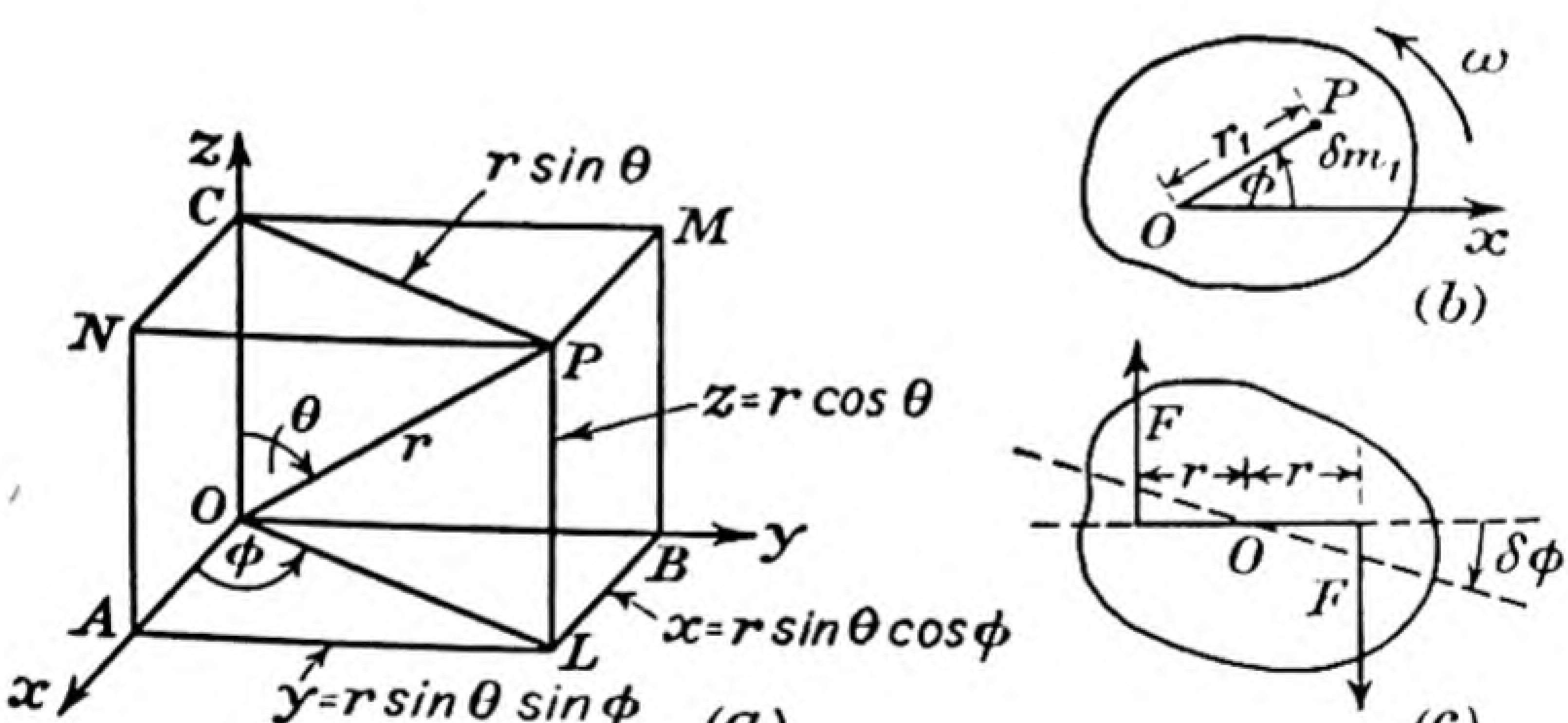


FIG. 3·18.—Cartesian and spherical polar coordinates; angular momentum.

To extend this argument to a rigid body rotating about a fixed axis, let us consider the moment of momentum about the z-axis of a particle of mass m at (x, y, z). If P, Fig. 3·18(a), is the position of the particle at a given instant and (r, θ, ϕ) the spherical polar coordinates defining this point, then

$$\left.\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned}\right\}.$$

Since the particle is rotating about the z-axis, θ is constant, r is constant and $\dot{\phi} = \omega$, the angular velocity. Hence

$$m(x\dot{y} - y\dot{x}) = m[r \sin \theta \cos \phi . r \sin \theta \cos \phi . \dot{\phi} + r \sin \theta \sin \phi . r \sin \theta \sin \phi . \dot{\phi}] = m[r^2 \sin^2 \theta . \dot{\phi}].$$

Hence, the moment of momentum of a rigid body rotating about the z-axis is $\Sigma\,[mr^2(\sin^2 \theta)\dot{\phi}]$, where the summation extends over all the particles in the body. Since $r \sin \theta$ is the radius of gyration for the particle and axis we have specified, the above expression may be written $I\dot{\phi}$ where I is the moment of inertia of the whole body about the axis considered.

Moment of momentum and angular momentum.—To clarify the conception of the term moment of momentum, let us consider a rigid body rotating with angular velocity ω about a fixed axis through a point O, Fig. 3·18(b), and normal to the plane of the diagram. Let δm_1 be the mass of a small particle at P a point in the body.

If OP $= r_1$, the linear velocity of P will be $r_1\dot{\phi}$ and the momentum $(\delta m_1)r_1\dot{\phi}$. The moment of this momentum about the axis of rotation will be $(\delta m_1)r_1^2\dot{\phi}$, and hence the moment of momentum of the whole body about the axis of rotation will be

$$\Sigma\,[(\delta m)r^2]\dot{\phi} = I\dot{\phi}, \quad \text{or} \quad I\omega,$$

where I is the moment of inertia of the body about the axis of rotation.

Similarly, the relation between angular acceleration and the applied couple may be obtained as follows. Let a couple be applied to a rigid body, as shown in Fig. 3·18(c), and the body rotate through a small angle $\delta\phi$, the angular velocity increasing from ω to $\omega + \delta\omega$. Since the work done by the couple is equal to the increase in kinetic energy of the rotating body, we have

$$F(2r\,\delta\phi) = \tfrac{1}{2}I[(\omega + \delta\omega)^2 - \omega^2].$$

$$\therefore\ \Gamma\,\delta\phi = I\omega\,\delta\omega. \qquad [\text{If } \Gamma = 2Fr \text{ and } \delta\omega^2 \rightarrow 0.]$$

i.e. $$\Gamma\frac{\delta\phi}{\delta t} = I\omega\frac{\delta\omega}{\delta t}.$$

$$\therefore\ \Gamma = I\ddot{\phi}.$$

The laws of rotation.—It has just been established that Γ, the moment of the external forces about a fixed axis, is given by

$$\Gamma = \frac{d}{dt}(I\dot{\phi}),$$

where I is the moment of inertia of the body about the given axis

and ϕ is the angular velocity about that axis. If I is constant, the above expression becomes $I\ddot{\phi}$, so that we may then write

$$\Gamma = I\ddot{\phi}.$$

Hence we have the following laws of rotation.

(*a*) ***If the moment of the external forces acting on a rigid body about a fixed axis is zero, then the angular velocity of that body about the fixed axis is constant.***

(*b*) ***The angular acceleration produced in a rigid body rotating about a fixed axis is directly proportional to the moment of the external forces about that axis.***

The kinetic energy of a rotating body; König's formula.—Let us now consider a lamina S, Fig. 3·19, moving in a plane xOy; let (x, y) be the coordinates of a typical particle in the lamina at P and m its mass. If the coordinates of this particle relative to G, the centroid of the lamina, are (ξ, η), the kinetic energy of the lamina, defined as the sum of the kinetic energies of all its particles, is given by

$$\begin{aligned} W &= \tfrac{1}{2}\Sigma\, m(\dot{x}^2 + \dot{y}^2) \\ &= \tfrac{1}{2}\Sigma\, m[(\dot{\xi} + \dot{\alpha})^2 \\ &\quad + (\dot{\eta} + \dot{\beta})^2], \end{aligned}$$

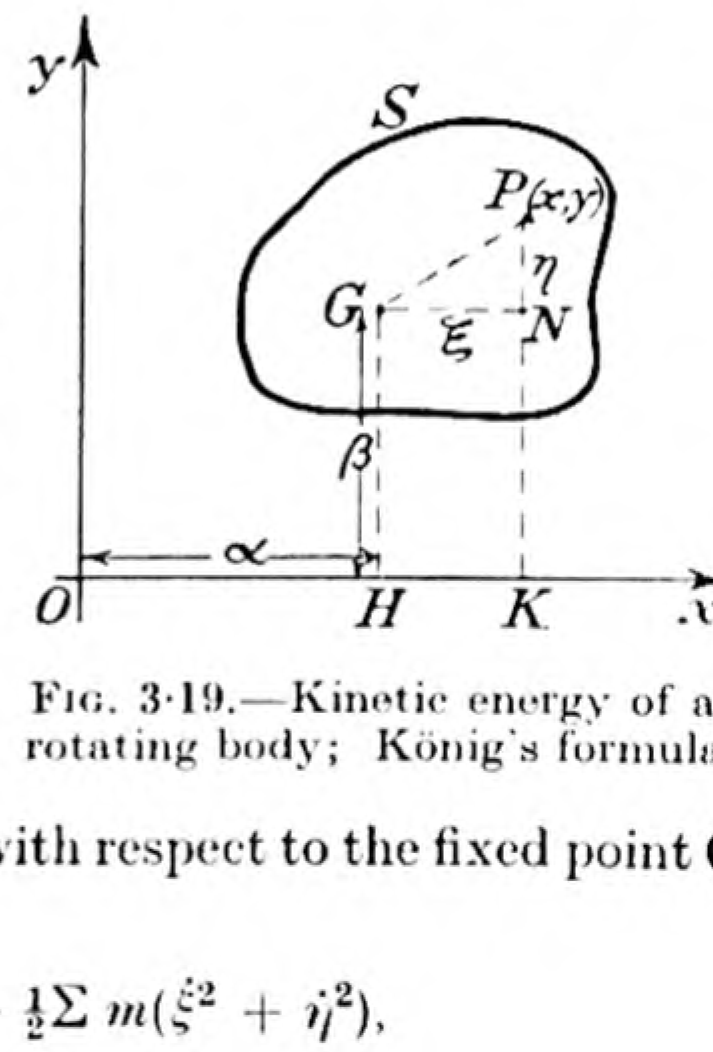


FIG. 3·19.—Kinetic energy of a rotating body; König's formula

where (α, β) are the coordinates of G with respect to the fixed point O. Hence, if $M = \Sigma\, m$,

$$W = \tfrac{1}{2}M(\dot{\alpha}^2 + \dot{\beta}^2) + \tfrac{1}{2}\Sigma\, m(\dot{\xi}^2 + \dot{\eta}^2),$$

for $$\Sigma\, m\xi = 0 \quad \text{and} \quad \Sigma\, m\eta = 0,$$

so that $$\Sigma\, m\dot{\xi} = 0 = \Sigma\, m\dot{\eta}.$$

Since the lamina is rigid $\dot{\xi}^2 + \dot{\eta}^2 = r^2\omega^2$, where $r^2 = \xi^2 + \eta^2$ and $\omega = \dot{\theta}$, θ being the angle which a fixed line in the lamina makes with a fixed line in space. [Actually $\dot{\xi} = -\eta\omega$ and $\dot{\eta} = \xi\omega$.]

$$\begin{aligned} \therefore\ W &= \tfrac{1}{2}M(\dot{\alpha}^2 + \dot{\beta}^2) + \tfrac{1}{2}I_G\omega^2, \\ &= \tfrac{1}{2}Mu^2 + \tfrac{1}{2}I_G\omega^2, \end{aligned}$$

where u is the velocity of G, and I_G the moment of inertia of the lamina about an axis through G and normal to its plane. The above is known as ***König's formula*** and it expresses the kinetic energy of a lamina as the sum of two terms; the first is the kinetic energy as if the whole mass of the lamina were concentrated at G and moving with the same velocity as G, while the second gives the energy for a

rotation with angular velocity ω about an axis at G fixed normally to the plane of the lamina.

It may be shown that the above formula can be extended to the motion of any rigid body.

Energy equation for a rigid body rotating about a fixed axis.—Let Γ be the moment of the forces which is causing the rotation of the rigid body about a fixed axis. Then

$$\Gamma = \mathrm{I}\ddot{\theta} = \mathrm{I}\dot{\omega},$$

where θ is the angle turned through from rest, and ω is the angular velocity of the body at the instant considered.

Now $$\frac{d\omega}{dt} = \frac{d\omega}{d\theta}.\frac{d\theta}{dt} = \omega.\frac{d\omega}{d\theta}.$$

Hence $$\Gamma = \mathrm{I}\omega.\frac{d\omega}{d\theta} = \tfrac{1}{2}\mathrm{I}\frac{d}{d\theta}(\omega^2).$$

Hence $$\int \Gamma\, d\theta = \tfrac{1}{2}\mathrm{I}\omega^2 + \lambda,$$

where λ is a constant. If the initial angular velocity is ω_0, i.e. $\omega = \omega_0$ when $\theta = 0$, we have

$$0 = \tfrac{1}{2}\mathrm{I}\omega_0^{\,2} + \lambda,$$

whence $$\int \Gamma\, d\theta = \tfrac{1}{2}\mathrm{I}\omega^2 - \tfrac{1}{2}\mathrm{I}\omega_0^{\,2},$$

i.e. ***the work done by the couple is equal to the change in the rotational energy of the body.***

Impulsive torques.—In linear dynamics, the equation

$$\mathrm{F} = m\ddot{x},$$

leads, by integration with respect to t, to the ***impulse equation*** viz.

$$\int_0^t \mathrm{F}\, dt = m[\dot{x} - (\dot{x})_0] = m(v - u),$$

i.e., the impulse is equal to the change of linear momentum.

In a similar way, from the equation for a rotating body,

$$\Gamma = \mathrm{I}\ddot{\theta} = \mathrm{I}\dot{\omega},$$

we have

$$\int_0^t \Gamma\, dt = \mathrm{I}\int_{\omega_0}^{\omega} d\omega$$

$$= \mathrm{I}(\omega - \omega_0), \qquad [\text{if } (\omega)_{t=0} = \omega_0]$$

or $$\mathrm{I}\omega, \text{ if } \omega_0 \rightarrow 0.$$

To discover the physical significance of the expression $\int_0^t \Gamma\,dt$, let Γ be due to two forces F at distance $2r$ apart. Then if this couple is applied for a time δt, the impulse is $2F\delta t$, and the moment of the impulse is

$$(2F\,\delta t)r = \Gamma\,\delta t = I\,\delta\omega.$$

If the impulse is finite, of duration t,

$$\text{Moment of impulse} = I\int_{\omega_0}^{\omega} d\omega = I(\omega - \omega_0),$$

i.e. the moment of the impulse about the axis of rotation is equal to the change in the angular momentum of the rotating body.

The moment of the impulse about the axis is called the ***impulsive torque.***

Equations of motion of a rigid body rotating about an axis not fixed in space, but moving parallel to itself.—Let G, Fig. 3·20, be the centre of mass of a body and suppose it moves with

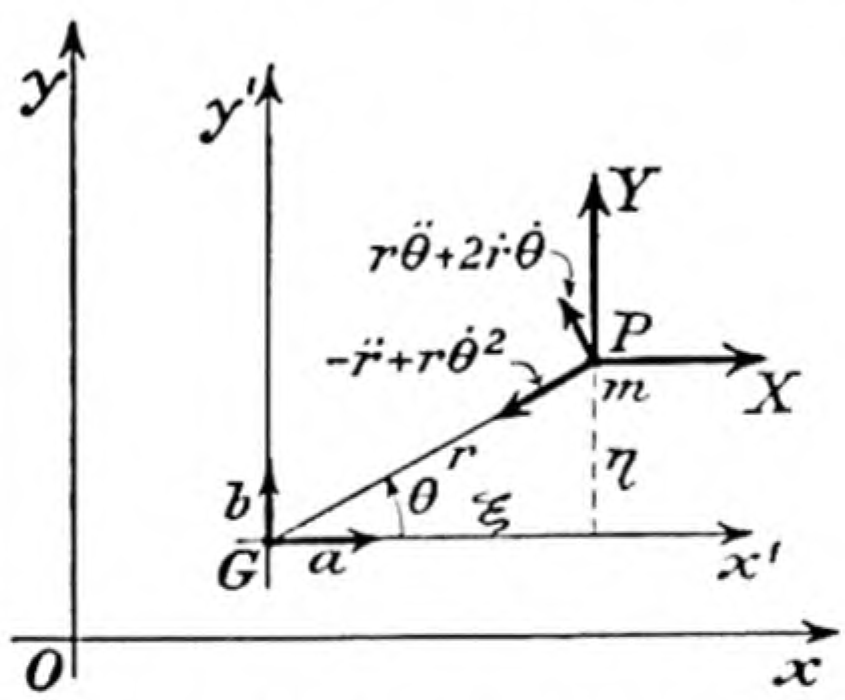


FIG. 3·20.—Motion of a rigid body, not fixed in space but moving parallel to itself.

accelerations a and b respectively in directions parallel to axes Ox and Oy, fixed in space. Let the forces acting on a small particle, m, at P be X and Y. If ξ and η are the coordinates of P relative to G, its accelerations relative to G are $\ddot{\xi}$ and $\ddot{\eta}$, so that the equations of motion, relative to the fixed axes Ox and Oy, are

$$X = m(a + \ddot{\xi}) \quad \text{and} \quad Y = m(b + \ddot{\eta}).$$

Taking moments of forces about G we have

$$(Y\xi - X\eta) = m[(b + \ddot{\eta})\xi - (a + \ddot{\xi})\eta].$$

Assuming that the internal reactions cancel one another when the forces acting on every particle are considered, there remains only the

sum of the moments of the external impressed forces about G, say Γ.

$$\therefore\ \Gamma = \Sigma\, m[(b + \ddot{\eta})\xi - (a + \ddot{\xi})\eta]$$
$$= b\, \Sigma\, m\xi - a\, \Sigma\, m\eta + \Sigma\, m(\xi\ddot{\eta} - \eta\ddot{\xi}).$$

Since the coordinates of G with respect to itself are (0, 0) the first two terms in the above equation vanish and we have

$$\Gamma = \Sigma\, m(\xi\ddot{\eta} - \eta\ddot{\xi}).$$

Let the polar coordinates of P relative to G be (r, θ); then since r is constant the acceleration of P at right angles to GP is $r\ddot{\theta}$ and along PG it is $r\dot{\theta}^2$. Resolving these parallel to Gx' and Gy', we have

$$\ddot{\xi} = -r\dot{\theta}^2 \cos\theta - r\ddot{\theta}\sin\theta,$$

and
$$\ddot{\eta} = r\ddot{\theta}\cos\theta - r\dot{\theta}^2 \sin\theta.$$
$$\therefore\ \xi\ddot{\eta} - \eta\ddot{\xi} = r^2\ddot{\theta}.$$
$$\therefore\ \Gamma = \ddot{\theta}\, \Sigma\, mr^2 = M\kappa^2\ddot{\theta},$$

where $M\kappa^2$ is the moment of inertia of the body about an axis through G and normal to the plane of the diagram.

This equation is the same as the one obtained when the body rotated about an axis through G which was fixed in space. The equations of motion for a body rotating about an axis moving parallel to itself are therefore obtained as follows.

(*a*) ***By resolving the forces parallel to two axes at right angles and supposing them to act on the body as if all its mass were collected at the centre of mass, and***

(*b*) ***By taking moments of the external forces about an axis through the centre of mass as if that axis were fixed in space.***

Moments about an instantaneous axis of rotation.—One of the equations of motion for a rigid body rotating about a fixed axis is

$I\dot{\omega}$ = sum of the moments about the fixed axis of rotation of all the external forces acting on the body.

This equation does not apply in general when the moments are evaluated with reference to an instantaneous axis of rotation. With the notation already used we have

$$W = \tfrac{1}{2}Mu^2 + \tfrac{1}{2}I_G\omega^2.$$

Let Γ be the sum of all the moments of all the external forces about the instantaneous axis of rotation and let $\delta\phi$ be the change in

position in time δt of the 'arm of the equivalent torque'. Then

$$\Gamma\,\delta\phi = \delta\mathrm{W}.$$

$$\therefore\ \Gamma\frac{d\phi}{dt} = \frac{d\mathrm{W}}{dt}.$$

$$\therefore\ \Gamma = \frac{1}{\omega}\frac{d}{dt}[\tfrac{1}{2}\mathrm{M}(r\omega)^2 + \tfrac{1}{2}\mathrm{I_G}\omega^2],$$

where r is the distance of G from the instantaneous axis of rotation; in general it will vary with time.

$$\therefore\ \Gamma = \frac{1}{\omega}[\tfrac{1}{2}\mathrm{M}(2r\omega)(r\dot{\omega} + \dot{r}\omega) + \tfrac{1}{2}\mathrm{I_G}.2\omega.\dot{\omega}]$$

$$= \mathrm{M}(r^2 + \kappa^2)\dot{\omega} + \mathrm{M}r\dot{r}\omega.$$

The additional term $\mathrm{M}r\dot{r}\omega$ vanishes when the distance of G from the instantaneous axis of rotation is constant.

Example.—A light thread is wound round a circular reel which is then allowed to fall, the end of the thread being fixed. Determine the acceleration of the reel.

Let m be the mass of the reel, and a its radius. Suppose that initially the point P, Fig. 3.21, is in contact with the string at O. Let Q be the point on the string which is just about to break contact with the reel. Let G be the mass-centre of the reel, and let $\theta = \widehat{\mathrm{PGQ}}$.

Resolving forces vertically,

$$m\ddot{x} = mg - \mathrm{F},$$

where F is the tension in the string, and g the intensity of gravity. Taking moments of forces about G,

$\mathrm{F}a$ = moment of external forces about the axis of spin,

= rate of change of angular momentum about that axis,

$= \mathrm{I}\ddot{\theta} = \tfrac{1}{2}ma^2\ddot{\theta}.$

But $x = a\theta$. Hence $\ddot{x} = a\ddot{\theta}$.

$$\therefore\ m\ddot{x} = mg - \tfrac{1}{2}ma\left(\frac{\ddot{x}}{a}\right),$$

i.e.

$$\ddot{x} = \tfrac{2}{3}g.$$

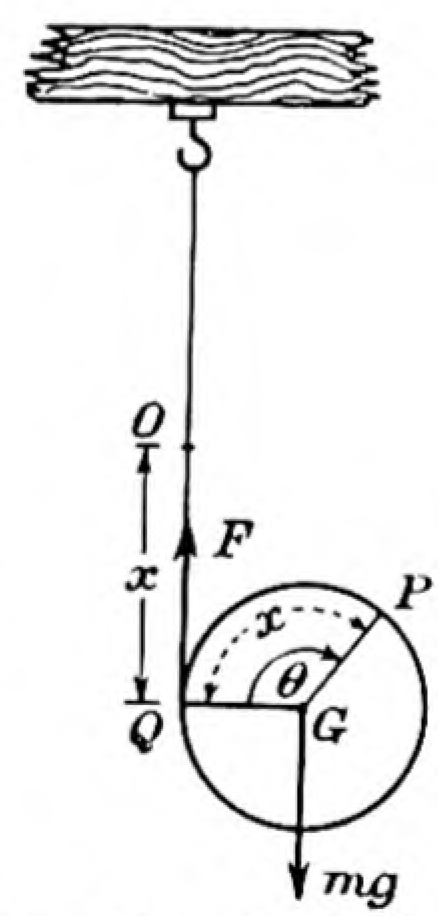


FIG. 3·21.—Motion of a reel suspended by a thread attached to a fixed support (the thread is unwound as the wheel falls).

[Alternatively, since the instantaneous axis of rotation passes through Q and is normal to the plane of the diagram,

$$mga = m(a^2 + \kappa^2)\ddot{\theta},$$

which, with $\ddot{x} = a\ddot{\theta}$ and $\kappa^2 = \tfrac{1}{2}a^2$, gives

$$\ddot{x} = \tfrac{2}{3}g.]$$

Motion under gravity on an inclined plane.*—Let us assume that a solid of revolution, of mass m, rolls down a plane under the influence of gravity, the axis of rotation being normal to the line of greatest slope in the plane. Let R and F, Fig. 3·22, be the normal and tangential components of the reaction of the plane on the body [we assume that this reaction may be reduced to a single force, i.e.

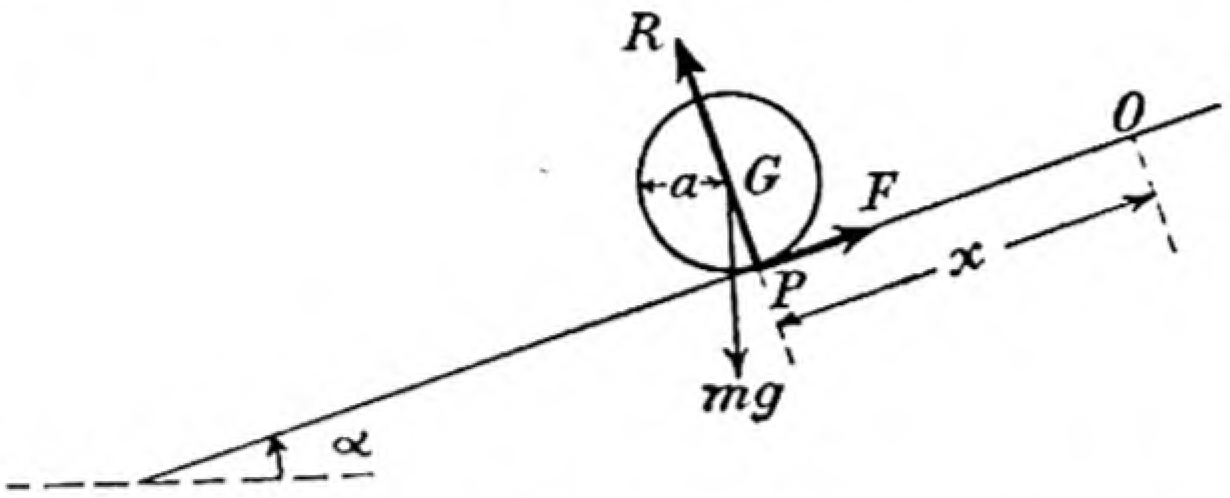


FIG. 3·22.—Motion under gravity on an inclined plane.

the frictional couple is zero]. If a is the radius of that circular section of the body which rolls in contact with the plane and x the distance which G, the mass-centre of the body, has moved from rest, so that $\dot{x}$ is the velocity of the mass-centre, then

$$\dot{x} = a\dot{\theta} \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

where $\dot{\theta}$ is the angular velocity of the body.

Resolving forces parallel and perpendicular to the plane, we have, since all these forces may, for this purpose, be considered to act at G,

$$\left.\begin{aligned} m\ddot{x} &= mg \sin \alpha - \mathrm{F} \\ 0 &= mg \cos \alpha - \mathrm{R} \end{aligned}\right\} \quad . \quad . \quad . \quad \text{(ii)}$$

where α is the slope of the plane and g is the intensity of gravity.

Now the moment of the external forces about an axis through G normal to the plane of the diagram (this is the so-called ***axis of spin***) is Fa, and this is equal to the rate of change of the angular momentum of the body (about the above axis) viz. $m\kappa^2.\ddot{\theta}$, where κ is the radius of gyration of the solid about that same axis. Hence

$$\mathrm{F}a = m\kappa^2\ddot{\theta} \quad . \quad . \quad . \quad . \quad \text{(iii)}$$

Eliminating F and $\ddot{\theta}$ from these equations, we have

$$m\ddot{x} = mg \sin \alpha - \frac{m\kappa^2\ddot{x}}{a^2},$$

or

$$\ddot{x} = \left(\frac{a^2}{a^2 + \kappa^2}\right) g \sin \alpha \quad . \quad . \quad . \quad \text{(iv)}$$

* In all such problems it is assumed that the bodies in contact are perfectly rigid, i.e. there is no frictional couple; also that the motion is one of pure rolling so that no work is done against friction.

Again, $R = mg \cos \alpha$ and $F = \left(\dfrac{\kappa^2}{a^2 + \kappa^2}\right) mg \sin \alpha.$

[By taking the moment of the external forces about P, the instantaneous centre of rest, we have

$$mga \sin \alpha = m(a^2 + \kappa^2)\ddot{\theta},$$

which leads at once to equation (iv)].

If we assume the law of sliding friction that F cannot be greater than μR, where μ is the coefficient of friction, the condition for no slipping is

$$\mu mg \cos \alpha > \left(\frac{\kappa^2}{a^2 + \kappa^2}\right) mg \sin \alpha, \qquad \text{i.e. } \mu\left(\frac{a^2 + \kappa^2}{\kappa^2}\right) > \tan \alpha.$$

If the plane is perfectly smooth, $F = 0$, and the motion of the solid is determined by

$$m\kappa^2\ddot{\theta} = 0, \quad \text{and} \quad m\ddot{x} = mg \sin \alpha,$$

i.e.

$$\ddot{x} = g \sin \alpha.$$

Hence, when friction is present, the acceleration of the mass-centre G is less than in the case of frictionless sliding in the ratio

$$\left(\frac{a^2}{a^2 + \kappa^2}\right) : 1.$$

Alternative treatment.—Let the solid roll from rest a distance x down the plane. Then its potential energy is diminished by $mgx \sin \alpha$. The increase in the kinetic energy of the solid is $\frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\kappa^2\dot{\theta}^2$, where the symbols have their usual meanings. This is $\frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\kappa^2\left(\dfrac{\dot{x}^2}{a^2}\right)$. Since the sum of the potential energy and kinetic energy of a body is constant, we have,

$$\tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}m\kappa^2\left(\frac{\dot{x}^2}{a^2}\right) = mgx \sin \alpha.$$

$$\therefore \; \dot{x}^2 = 2xg \sin \alpha\left(\frac{a^2}{a^2 + \kappa^2}\right).$$

Whence, by differentiation,

$$2\dot{x}\ddot{x} = 2\dot{x}g \sin \alpha\left(\frac{a^2}{a^2 + \kappa^2}\right),$$

or

$$\ddot{x} = g \sin \alpha\left(\frac{a^2}{a^2 + \kappa^2}\right),$$

as before.

Experiment.—When the solid of revolution discussed above is a uniform sphere ($\kappa^2 = \frac{2}{5}a^2$), cylinder or disc ($\kappa^2 = \frac{1}{2}a^2$), or a circular hoop ($\kappa^2 = a^2$); by carrying out the following experiment a value for the intensity of gravity may be obtained.

The inclined plane, arranged at a small angle to the horizontal, is adjusted by means of screws attached to the plane so that a metal sphere (4 cm. in diameter) rolls down the plane without passing over its edges. To measure the angle of inclination of the plane, place a long piece of wood on the plane and raise it by means of two rectangular pieces of brass of known thicknesses until a long spirit level placed on the wood indicates that the latter is horizontal. Be sure that the wood is parallel to the line of greatest slope of the plane and then measure the distance between the points of contact of the wood with the brass pieces. The required angle is easily deduced.

Observe the time, t, required for the sphere to roll from the top to the bottom of the plane. [A piece of wood fixed near the upper end of the plane and at right angles to its length serves as a fiducial mark from which the sphere starts to roll.] Let x be the above distance, i.e. the actual distance from the above fiduial mark to the lower end of the plane less the radius of the sphere. Then, assuming the acceleration to be uniform,

$$\ddot{x} = \frac{2x}{t^2}.$$

Take the mean of several such readings for each inclination of the plane and obtain a series of results for different inclinations, altering the inclination by means of a single screw at one end of the plane.

Since
$$\ddot{x} = \frac{a^2}{a^2 + \kappa^2}\cdot g \sin \alpha,$$
it follows that

$$x = \frac{1}{2}\cdot\frac{a^2}{a^2 + \kappa^2}\cdot g \sin \alpha . t^2, \quad \text{or} \quad x \operatorname{cosec} \alpha = \frac{1}{2} g\cdot\frac{a^2}{a^2 + \kappa^2}\cdot t^2.$$

Hence, if cosec α is plotted against t^2, a straight line whose slope is $\frac{1}{2} g\cdot\frac{a^2}{a^2 + \kappa^2}$ should be obtained. Measure this slope and deduce a value for g, the intensity of gravity.

Reliable results will only be obtained if the surfaces of the plane and the sphere are free from grit and grease. These may be removed by means of a detergent; after cleaning, the sphere should only be touched with a piece of clean silk.

Investigation of the motion of a wheel and axle down an inclined plane.—Let us suppose that a wheel and axle, of mass m, starts to roll from rest down an inclined plane making an angle α with the horizontal. We shall assume that the axle is always normal to the line of greatest slope in the plane. In this particular instance, the plane consists of a long wooden box, the edges of the sides being accurately planed. The axle is in contact with the plane at two points, one on each edge of the box. Let R and F, Fig. 3·23, be the components of the reaction at each point of contact in

directions normal and parallel to the plane. Then the forces acting on the wheel are its weight mg, and the components of the reaction, viz. 2R and 2F. If there is no frictional couple and the wheel rolls without slipping, the equations which determine the motion are

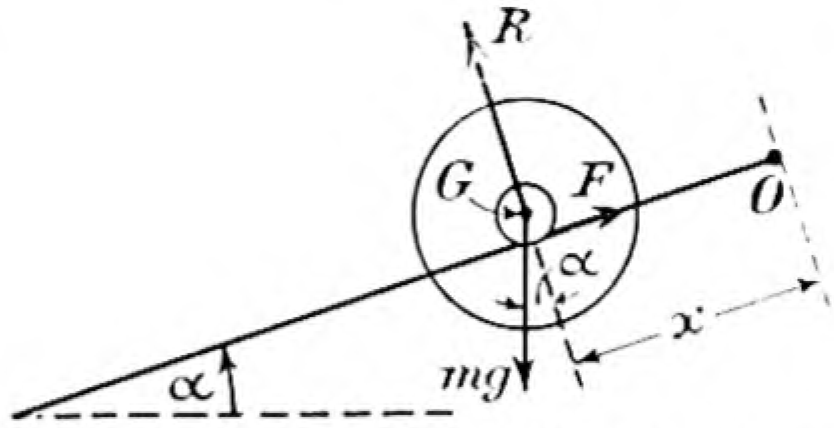


FIG. 3·23.—A wheel and axle on an inclined plane.

$$mg \sin \alpha - 2\mathrm{F} = m\ddot{x} \quad . \quad (\mathrm{i})$$

$$2\mathrm{R} - mg \cos \alpha = 0 \quad . \quad (\mathrm{ii})$$

and, taking moments of forces about the mass-centre G, we have, if a is the radius of the axle

$$2\mathrm{F}a = \mathrm{I}\ddot{\theta}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{iii})$$

$$= \text{rate of change of angular momentum about the axis of spin,}$$

where x is the distance G has moved from rest and θ is the angle through which any radius of the wheel has turned from rest. Then

$$\mathrm{I} = \frac{2\mathrm{F}a}{\ddot{\theta}} = \frac{(mg \sin \alpha - m\ddot{x})}{\ddot{\theta}} a.$$

But $\dot{x} = a\dot{\theta}$, since the points of contact of the wheel with the plane are each instantaneous centres of rest. Hence, $\ddot{x} = a\ddot{\theta}$, so that

$$\mathrm{I} = \frac{(mg \sin \alpha - m\ddot{x})}{\ddot{x}} a^2,$$

or

$$\ddot{x} = \frac{mga^2 \sin \alpha}{(\mathrm{I} + ma^2)} = \frac{a^2}{a^2 + \kappa^2} \cdot g \sin \alpha, \quad . \quad . \quad (\mathrm{iv})$$

where, as usual, $\mathrm{I} = m\kappa^2$.

Experiment.—The above equation shows that the acceleration down the plane is uniform. To establish this fact experimentally observe the times required by the wheel to roll various distances down the plane—the distance is measured along the line of greatest slope of the plane and as usual three observations are made for the time of rolling each particular distance selected. Plot the distance-time curve. Since the graph is not linear it follows that the motion is not one of uniform velocity. Draw tangents at several points to the above curve and determine their slopes, i.e. the values of $\dot{x}$ corresponding to different positions of the wheel on the plane. If the velocity-time curve which may then be constructed is linear, the motion is uniformly accelerated.

Having thus established that the acceleration is uniform, it follows that

$$x = \tfrac{1}{2}\ddot{x}t^2 \quad . \quad . \quad . \quad (\mathrm{v})$$

The required moment of inertia may be determined graphically as follows. Using equations (iv) and (v) we have

$$\operatorname{cosec} \alpha = \frac{1}{2} \cdot \frac{mga^2}{x(\mathrm{I} + ma^2)} \cdot t^2,$$

so that if, for a fixed value of x, cosec α is plotted against t^2, the slope of the line gives

$$\frac{1}{2} \cdot \frac{mga^2}{x(\mathrm{I} + ma^2)}$$

from which I may be calculated.

Alternative treatment: Let us suppose that when the wheel has moved downwards through a vertical distance h, the angular velocity of the wheel is ω, while $\dot{x}$ is the linear velocity of the mass-centre. Then the energy equation is,

$$mgh = \tfrac{1}{2}\mathrm{I}\omega^2 + \tfrac{1}{2}m\dot{x}^2,$$

or

$$mgx \sin \alpha = \tfrac{1}{2}\left[\mathrm{I}\left(\frac{\dot{x}}{a}\right)^2 + m\dot{x}^2\right] \quad [\because \ a\omega = \dot{x}] \quad . \quad . \quad \text{(vi)}$$

Differentiating this expression with respect to time

$$mg\dot{x} \sin \alpha = \left[\frac{\mathrm{I}}{a^2} + m\right]\dot{x}\ddot{x},$$

whence

$$\ddot{x} = \frac{mga^2 \sin \alpha}{\mathrm{I} + ma^2}$$

as before.

To determine the angle of the plane, two rectangular pieces of metal are arranged with their edges at right angles to the line of greatest slope in the plane. A long straight brass bar is mounted on top of these with its length parallel to the lines of greatest slope. The positions of the metal pieces are adjusted until the bar is horizontal, a condition which may be ascertained with the aid of a spirit level. Then the angle α may be deduced. [For each setting of the plane this part of the experiment should be repeated at least three times with the metal pieces in different positions and a mean value for α calculated.]

EXAMPLES III

3·01. Derive an expression for the moment of inertia of a body about any axis in terms of its moment of inertia about a parallel axis through its centroid and the distance between the axes.

Two metal spheres of different materials, one solid and the other hollow, have the same mass and external diameter and the same outward appearance. Describe and explain a simple method of finding which is the solid and which the hollow sphere, leaving the spheres intact. (G)

3.02. Define the moment of inertia about an axis, of a rigid body. Show that if I_x and I_y be the moments of inertia of a laminar body about two lines in its plane at right angles to one another, then the moment of inertia about a perpendicular axis through the intersection of these lines is $\mathrm{I}_z = \mathrm{I}_x + \mathrm{I}_y$.

Show that the radius of gyration of a circular plate whose radius is a cm. about an axis perpendicular to its plane bisecting a radius is

$$\kappa = \frac{\sqrt{3}}{2} a.$$

3·03. A uniform cylinder has a constant mass m. Its overall length is $2l$ and its radius of cross-section a. Find the ratio of l to a if the moment of inertia of the cylinder about an axis through its centre and normal to the length is a minimum. [$\sqrt{3}$: $2\sqrt{2}$]

3·04. Show that for any uniform triangular sheet ABC of mass m, the moment of inertia about an axis through A and normal to the sheet is

$$\frac{m}{12}[3b^2 + 3c^2 - a^2].$$

3·05. A large uniform sphere of lead, mass M and radius a, rests on a plane horizontal surface. A small bullet of mass m is fired horizontally with a velocity u into the sphere, the line of motion of the bullet being directed towards the centre of the sphere. Assuming that the sphere does not slide on the plane, show that it will be set rolling along the surface with a peripheral speed v, given by the equation.

$$v = \frac{5}{7}\frac{m}{M}u.$$

3·06. A uniform (solid) circular cylinder, of mass m, can rotate freely about its axis which is horizontal. A particle of mass m_0 hangs from the end of a light inextensible string coiled round the cylinder. When the system is allowed to move show that the particle descends with uniform acceleration $\dfrac{2m_0 g}{m + 2m_0}$.

3·07. A uniform cube swings about one of its edges which is horizontal and in the highest positions the centre of gravity is level with the axis of rotation. Prove that the thrust on the axis always has a value lying between 0·25W and 2·5W, where W is the weight of the cube.

3·08. Show that the kinetic energy of a four-wheeled truck travelling on a road with velocity u is equal to

$$\tfrac{1}{2}\left(M + 2m\frac{k^2}{a^2}\right)u^2,$$

where M is the total mass of the truck and its wheels, $m\kappa^2$ is the moment of inertia of each *pair* of wheels about its axle and a is the radius of each wheel.

If a force F propels the truck along a level road, show that the acceleration is

$$\frac{F}{\left(M + \dfrac{2m\kappa^2}{a^2}\right)}.$$

3·09. A plate is symmetrical about perpendicular axes OA and OB. An axis OC is drawn in the plane of OA and OB making an angle θ with OA. If the moments of inertia about OA and OB are I_1 and I_2 respectively, show that the moment of inertia about OC is

$$(I_1 \cos^2 \theta + I_2 \sin^2 \theta).$$

A wheel and axle can roll down parallel rails inclined at an angle α to the horizontal, the axle rolling on the rails with the wheel between them. Explain how the moment of inertia of the wheel and axle about the central axis normal to the plane of the wheel can be determined, deriving the formula required for the calculation. (G)

3·10. A cylindrical drum, radius a, is free to turn about its axis which is horizontal. One end of a light string coiled round the drum is attached to it while the other carries a mass m. If I is the moment of inertia of the drum about the axis of rotation, prove that when the system moves freely from rest the tension in the string is

$$\frac{\mathrm{I}mg}{(\mathrm{I} + ma^2)}.$$

3·11. Assume the earth to be a sphere of radius a and with a moment of inertia I about its axis of rotation; its angular velocity ω is constant only when all objects on its surface are at rest. A train of mass m starts from one pole and travels along a meridian with constant velocity v. Prove that when it reaches the other pole the angle turned through by the globe is

$$\frac{\pi\omega a}{v}\left(\frac{\mathrm{I}}{\mathrm{I} + ma^2}\right)^{\frac{1}{2}}.$$

3·12. Two masses m_1 and m_2 are in equilibrium on a wheel and axle. If the masses are interchanged, prove that, if friction and the inertia of the machine are neglected, the mass m_1 descends with acceleration

$$\frac{m_1(m_1 - m_2)g}{m_1^2 - m_1m_2 + m_2^2}.$$

3·13. A fly-wheel is composed of a uniform circular disc of mass m lb. and radius a ft. together with a mass M lb. distributed in a thin layer round its periphery. Prove that the couple which must be applied to the wheel to generate in t seconds an angular speed of N rev.min.$^{-1}$ is $3{\cdot}27 \times 10^{-3}\mathrm{N}a^2(\mathrm{M} + \frac{1}{2}m^2)t^{-1}$lb.-wt.ft. How will the above expression be modified if the disc is replaced by a sphere of mass m and radius a ft., the mass M lying in a diametral plane normal to the axis of rotation?

3·14. A wheel and axle rotates with negligible friction about a horizontal axis. The motion is brought about by the descent from rest of a mass m supported at the end of an inextensible thread wound round the axle of radius a. If the mass falls from rest through a distance z in time t, show that the moment of inertia of the wheel and axle about its axis of rotation is

$$ma^2(gt^2 - 2z)(2z)^{-1},$$

where g is the intensity of gravity.

3·15. Find the radius of gyration (κ) of a regular polygon with n sides, each of length a, about an axis through its centre and normal to its plane. Also obtain a value for κ when $n \to \infty$.

$$\left[\kappa^2 = \frac{1}{12}a^2\frac{2 + \cos 2\pi/n}{1 - \cos 2\pi/n}.\right]$$

3·16. Describe and explain how Atwood's machine may be used to determine the intensity of gravity at a given station. Indicate how the effect due to the inertia of the pulley may be (a) eliminated from the calculations, and (b) calculated and a correction applied.

CHAPTER IV

SIMPLE HARMONIC MOTION

Simple harmonic motion.—***When a particle moves under the action of a force which is directed towards a fixed point in the line of motion and which is directly proportional to the distance of the particle from the fixed point, the motion is said to be simple harmonic.***

Let x be the distance of the particle from the fixed point O, at a time t; its acceleration in the positive direction of x, i.e. in the direction of x increasing, is $\ddot{x}$, so that the force acting in this direction is $m\ddot{x}$. The differential equation for the motion is accordingly

$$m\ddot{x} = -\kappa x,$$

where κ is the force per unit displacement of the particle: the minus sign is necessary since the force is always opposite in sign to that of x, i.e. when x is positive the force is negative, and *vice versa*.

If we write $\kappa = m\beta$, the above equation becomes $\ddot{x} = -\beta x$, a solution to this being

$$x = \mathrm{A}\cos(\beta^{\frac{1}{2}}t + \alpha),$$

where A and α are arbitrary constants. [In passing we note that β is the acceleration per unit displacement of the particle.]

The motion is therefore periodic, i.e. the values of x and $\dot{x}$ recur whenever the ***phase*** $(\beta^{\frac{1}{2}}t + \alpha)$ increases by 2π. Let T be the period. Then if t_1 is the particular instant when the particle considered has a displacement x_1, we have

$$x_1 = \mathrm{A}\cos(\beta^{\frac{1}{2}}t_1 + \alpha).$$

and when the time is $(t_1 + \mathrm{T})$ the particle is next in the same state of motion. Hence

$$\cos(\beta^{\frac{1}{2}}t_1 + \alpha) = \cos[\beta^{\frac{1}{2}}(t_1 + \mathrm{T}) + \alpha],$$

$$\therefore\ \beta^{\frac{1}{2}}t_1 + \alpha + 2\pi = \beta^{\frac{1}{2}}(t_1 + \mathrm{T}) + \alpha,$$

and $$\mathrm{T} = \frac{2\pi}{\beta^{\frac{1}{2}}} = \frac{2\pi}{\sqrt{\text{acceleration per unit displacement}}}.$$

It will be noted that the periodic time is independent of the amplitude, A, of the motion; such vibrations are said to be ***isochronous.***

Example (i).—A particle of mass m is fixed to the mid-point of a wire stretched between fixed points. The tension in the wire is F. Find the period of small lateral oscillations.

Let y be the displacement of the particle from its position of rest at a given instant—cf. Fig. 4·01. If the displacement is small the tension in the wire will remain constant. Further, if the inertia of the wire is neglected, the motion of the particle is given by

$$m\ddot{y} = -2\mathrm{F}\sin\theta = -\frac{2\mathrm{F}y}{l},$$

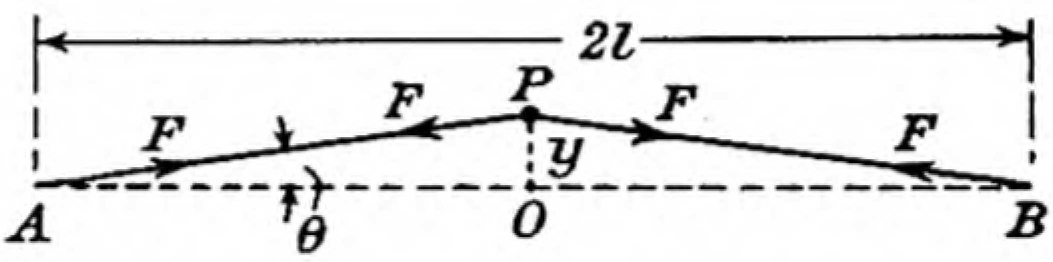


FIG. 4·01.—The oscillations of a particle at the centre of a stretched wire.

if $2l$ is the distance apart of the fixed points. The negative sign is used since the restoring force acting on the particle tends to diminish y.

Hence the period is $$2\pi\sqrt{\frac{ml}{2\mathrm{F}}}.$$

Example (ii).—Determine the period of the vertical oscillations of a body attached to a closely coiled helical spring.

In this problem we have to consider the motion of a body suspended from a fixed point by a helical spring of negligible mass. Let M be the mass of the body. When the spring is at rest, its lower end will take up some definite position. When a small additional mass m is attached to M, let h be the distance through which the lower end of the spring descends. Experiment shows that if a mass m had been removed from the total load carried by the spring the position of the lower end of the spring would have been raised by an amount h. It also shows that h is directly proportional to m.

If, therefore, the load is M and the spring is stretched by an amount y, the force tending to restore the load to its initial position is the resultant of the weight Mg acting vertically downwards and the upward pull $\left[\mathrm{M} + \frac{m}{h}\cdot y\right]g$, viz. $\frac{mgy}{h}$. The motion is therefore given by

$$\mathrm{M}\ddot{y} = -\frac{mgy}{h},$$

so that its period is $$\mathrm{T} = 2\pi\sqrt{\frac{\mathrm{M}h}{mg}}.$$

Now $\frac{\mathrm{M}h}{m}$ is the static extension of the spring due to the load M. Let it be called λ; then

$$\mathrm{T} = 2\pi\sqrt{\frac{\lambda}{g}},$$

so that the period is equal to that of a simple pendulum of length λ [cf. p. 106].

From the above it follows that by timing the oscillations of a loaded spring a value for the intensity of gravity may be obtained. In practice, however, a correction must be applied for the effect of the inertia of the spring itself. We may regard the effect as if it were due to an increase ΔM in the load carried; this is equivalent to an increase $\Delta\lambda$ in the extension due to the load.

Let T_1 and T_2 be the periodic times observed when the loads attached to the spring are M_1 and M_2 respectively. Then

$$T_1 = 2\pi\sqrt{\frac{\lambda_1 + \Delta\lambda}{g}}, \quad \text{and} \quad T_2 + 2\pi\sqrt{\frac{\lambda_2 + \Delta\lambda}{g}}.$$

Eliminating $\Delta\lambda$,

$$g(T_1^2 - T_2^2) = 4\pi^2(\lambda_1 - \lambda_2).$$

Alternatively, a graphical method may be used to eliminate $\Delta\lambda$.

Example (iii).—Determine the period of the vertical motion of a ship neglecting the inertia of the water displaced.

Let V be the volume of water displaced, ρ the density of the water, S the cross-sectional area of the ship at the water line, and suppose x is the small vertical displacement at a given instant. Then the equation of motion is

$$\rho V\ddot{x} = -g\rho Sx.$$

$$\therefore\ T = 2\pi\sqrt{\frac{V}{Sg}} = 2\pi\sqrt{\frac{z}{g}},$$

where z is the mean depth of immersion.

Theory of the simple pendulum.—A simple pendulum is really a mathematical conception and is supposed to consist of a bob, which may be treated as a particle, suspended from a fixed point by a string or cord of negligible mass so that it is constrained to move in a vertical circle under gravity.

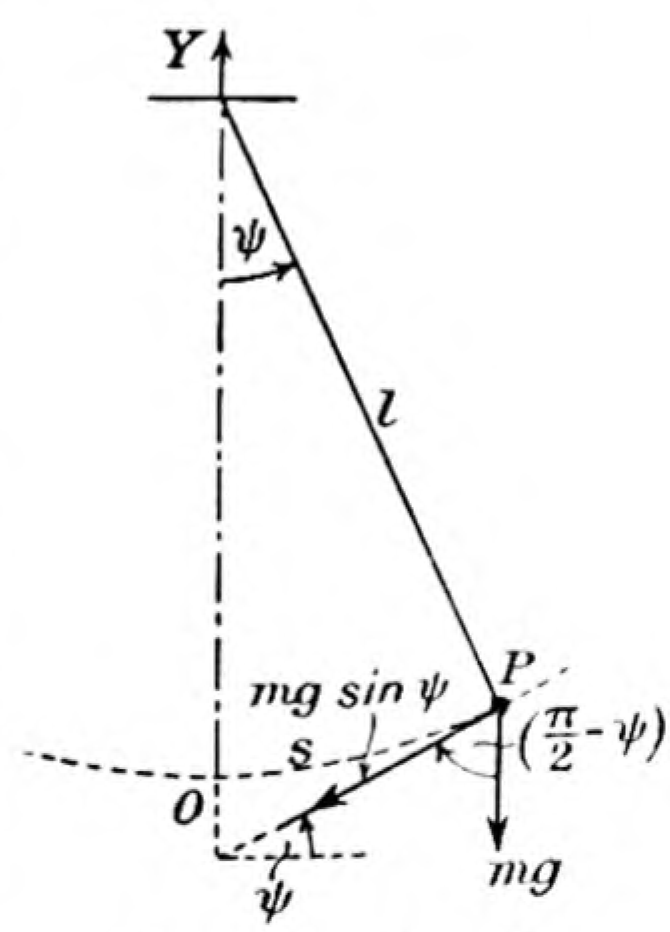


FIG. 4·02.—A simple pendulum.

Let s be the distance described by the bob, this distance being measured along the arc of the circle from O, Fig. 4·02, the lowest point of the swing. If ψ is the angle made by the string with the vertical OY through O, $s = l\psi$, where l is the length of the string. If m is the mass of the bob, the force acting upon it in the direction of s increasing is $-mg \sin \psi$ so that its acceleration in this direction is $-g \sin \psi$. The equation expressing the motion is

$$\ddot{s} = -g \sin \psi, \quad \text{or} \quad \ddot{\psi} = -\frac{g}{l}\sin \psi \quad . \quad . \quad \text{(i)}$$

If ψ is small this becomes

$$\ddot{\psi} = -\frac{g}{l}.\psi \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

so that the motion is simple harmonic, its period being $2\pi\sqrt{\frac{l}{g}}$.

Correction for finite arc of swing.—In 1749 BERNOULLI showed that the period of a simple pendulum having a finite arc of swing was greater than when the arc was infinitely small in the ratio $1 + \frac{1}{16}\psi_0^2$, where ψ_0 is the maximum angular displacement of the pendulum, provided this is not large. During any experimental determination of g by means of a simple pendulum the amplitude of swing decreases so that a further correction is necessary.

If the pendulum swings through a finite angle ψ_0 on each side of the vertical, so that when $\psi = \psi_0$, $\dot{\psi} = 0$, multiplying both sides of equation (i) by $\dot{\psi}$ and integrating, we get

$$\tfrac{1}{2}\dot{\psi}^2 = \frac{g}{l}\cos\psi + \kappa,$$

where κ is a constant. But $\dot{\psi} = 0$ when $\psi = \psi_0$, so that

$$\kappa = -\frac{g}{l}\cos\psi_0.$$

$$\therefore \ \tfrac{1}{2}\dot{\psi}^2 = \frac{g}{l}(\cos\psi - \cos\psi_0) \quad . \quad . \quad . \quad \text{(iii)}$$

Equation (iii) may be written

$$2\sqrt{\frac{g}{l}}\int_0^{\frac{T}{4}} dt = \int_{\psi=0}^{\psi=\psi_0} \frac{d\psi}{\sqrt{\sin^2\frac{\psi_0}{2} - \sin^2\frac{\psi}{2}}},$$

where t is the time measured from an instant when the pendulum passes through its position of static equilibrium, and the integration extends over a quarter period.

If we now introduce a new variable ϕ defined by the relation

$$\sin\phi = \frac{\sin\frac{1}{2}\psi}{\sin\frac{1}{2}\psi_0},$$

since $\psi_0 > \psi$, it follows that $\sin\phi$ is always less than unity, i.e. ϕ is real. The limits for ϕ corresponding to the values 0 and ψ_0 for ψ are 0 and $\frac{\pi}{2}$ respectively.

Hence

$$2\sqrt{\frac{g}{l}}\cdot\frac{T}{4}=\int_0^{\frac{\pi}{2}}\frac{\sin\frac{1}{2}\psi_0\cos\phi\,d\phi}{\frac{1}{2}\sqrt{1-\sin^2\phi\sin^2\frac{\psi_0}{2}}\cdot\sqrt{\sin^2\frac{\psi_0}{2}-\sin^2\phi\sin^2\frac{\psi_0}{2}}}$$

$$=\int_0^{\frac{\pi}{2}}\frac{2\,d\phi}{\sqrt{1-\sin^2\phi\sin^2\frac{\psi_0}{2}}}.$$

$$\therefore\ T=4\sqrt{\frac{l}{g}}\int_0^{\frac{\pi}{2}}\frac{d\phi}{\sqrt{1-\left(\frac{\psi_0}{2}\right)^2\sin^2\phi}},\quad\text{if }\psi_0\text{ is not large,}$$

$$=4\sqrt{\frac{l}{g}}\int_0^{\frac{\pi}{2}}\left[1+(-\tfrac{1}{2})\left\{-\left(\frac{\psi_0}{2}\right)^2\sin^2\phi\right\}+\frac{(-\frac{1}{2})(-\frac{3}{2})}{1.2}\cdot\frac{\psi_0^{\,4}}{16}\sin^4\phi+\ldots\right]$$

$$=4\sqrt{\frac{l}{g}}\left[\frac{\pi}{2}+\frac{1}{2}\cdot\frac{\psi_0^{\,2}}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}+\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\frac{\psi_0^{\,4}}{16}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}+\ldots\right],$$

since $$\int_0^{\frac{\pi}{2}}\sin^{2n}\phi.d\phi=\frac{2n-1}{2n}\cdot\frac{2n-3}{2n-2}\cdots\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}.$$

$$\therefore\ T=4\sqrt{\frac{l}{g}}\cdot\frac{\pi}{2}[1+\tfrac{1}{16}\psi_0^{\,2}],\qquad\text{if }\psi_0^{\,4}\to 0.$$

If $T_1=2\pi\sqrt{\dfrac{l}{g}}$, $\qquad T=T_1[1+\tfrac{1}{16}\psi_0^{\,2}]$.

$$\therefore\ T_1=T[1-\tfrac{1}{16}\psi_0^{\,2}].$$

If $\frac{1}{16}\psi_0^{\,2}=0{\cdot}0001$,

$$\psi_0^{\,2}=\tfrac{16}{10,000},\quad\text{or}\quad\psi_0=0{\cdot}040\text{ radian}=2^\circ\,18'.$$

Example.—When $g = 980$ cm.sec.$^{-2}$ the length of the seconds pendulum is 99·33 cm. Find the change in the number of oscillations per day if (*a*) the length is increased to 100·0 cm., (*b*) the intensity of gravity is increased to 981 cm.sec.$^{-2}$, and (*c*) both changes are made simultaneously.

If T is the period, we have

$$T^2=4\pi^2\frac{l}{g}.$$

Differentiating logarithmically, we get

$$2\frac{\delta T}{T}=\frac{\delta l}{l}-\frac{\delta g}{g}.$$

If N is the number of oscillations made per day by a pendulum with a period T,

$$NT = \text{constant}.$$

$$\therefore \frac{\delta N}{N} + \frac{\delta T}{T} = 0.$$

$$\therefore \frac{\delta N}{N} = \tfrac{1}{2}\left(\frac{\delta g}{g} - \frac{\delta l}{l}\right).$$

(*a*) If g is unaltered, we have approximately,

$$\frac{\Delta N}{N} = -\tfrac{1}{2}\frac{\Delta l}{l} = -\tfrac{1}{2}\cdot\frac{0{\cdot}67}{99{\cdot}3} \fallingdotseq -\frac{0{\cdot}335}{100}.$$

Now for a seconds pendulum, $N = \frac{1}{2} \times 24 \times 60 \times 60 = 43{,}200.$

$$\therefore \Delta N = -\frac{(24 \times 60 \times 60)}{2}\cdot\frac{0{\cdot}335}{100} = -144{\cdot}8.$$

(*b*) If l is not changed,

$$\frac{\Delta N}{N} = \tfrac{1}{2}\frac{\Delta g}{g} = \tfrac{1}{2}\cdot\frac{1}{980}.$$

$$\therefore \Delta N = 43{,}200 \div (2 \times 980) \fallingdotseq 22.$$

(*c*) When both changes occur simultaneously, we have

$$\Delta N = -144{\cdot}8 + 22 = -122{\cdot}8$$

Thus the pendulum makes per day 123 complete oscillations less than the seconds clock, i.e. it loses 246 seconds a day.

Systems with one degree of freedom.—Whenever the configuration of a system is known as soon as a single entity, called a *coordinate*, is known, then that system is said to have ***one degree of freedom***. For example, let us consider a rigid bar AB, Fig. 4·03, suspended by two inextensible strings from fixed points P and

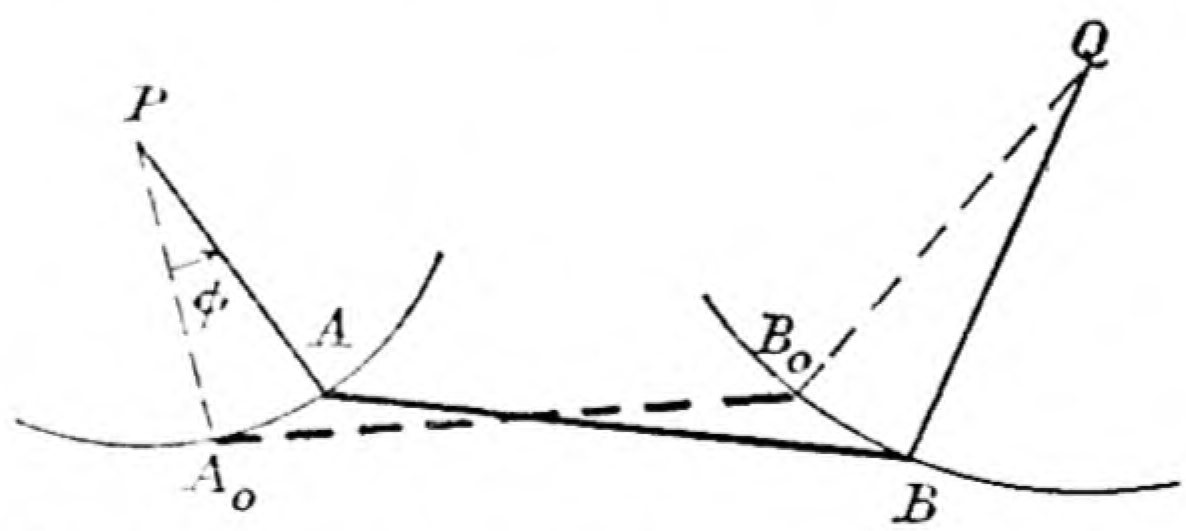


FIG. 4·03.—A system with one degree of freedom.

Q respectively. In the diagram A_0B_0 is the position of static equilibrium of the rod and it is assumed that *the strings are confined to the vertical plane through the fixed points* P *and* Q. Now let the system be displaced from its position of static equilibrium to a position AB; the two ends of the rod will move respectively on the

arcs of circles described about P and Q as centres and the configuration of the system is completely determined when ϕ, i.e. the $\widehat{A_0PA}$, is known; ϕ is the coordinate corresponding to the single degree of freedom which the system possesses under the conditions stated.

Potential and kinetic energies.—When a system with one degree of freedom is displaced from a position of ***stable equilibrium*** and then set free, it will vibrate about that position. Let x be the coordinate defining the single degree of freedom and let the origin be chosen so that x is zero when the system is in its position of static equilibrium.

Let the potential energy of the system be taken as zero in the above equilibrium position; in a displaced position let it be U. Since U is a minimum when $x = 0$, it follows that the curve showing how U varies with x must be concave with respect to a point on the U-axis. Such a curve is shown in Fig. 4·04 and as long as any displacement is small, the curve will not differ sensibly from a circular one. If P is the point (x, U), since PN is proportional to ON^2, we may write

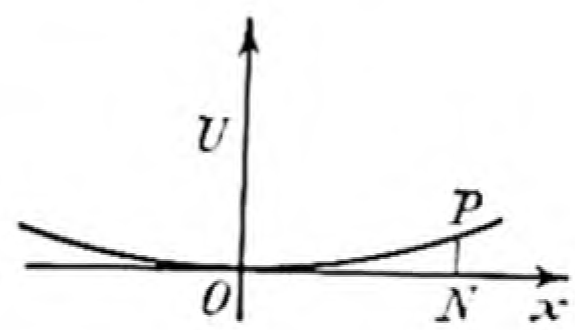


FIG. 4·04.—Potential energy a minimum in an equilibrium position.

$$\mathrm{U} = \alpha x^2,$$

where α is a constant provided x is kept small.

If v is the rate at which x increases with time, the velocity of every particle in the system will be proportional to v, so that the kinetic energy of the system may be written

$$\mathrm{W} = \beta v^2,$$

where β is independent of v, though it may depend on x. Now in all vibrating systems with which one is usually concerned, the velocity of every particle will be finite, so that if x is restricted to small values, the kinetic energy may be taken as that when $x = 0$, i.e. the kinetic energy in the position of static equilibrium. Hence when x is small, β may be considered constant.

The periodic time of a system with one degree of freedom.—Since the total energy of a vibrating system is constant (damping is assumed negligible) $\mathrm{U} + \mathrm{W} = \text{const.}$

$$\therefore \frac{d\mathrm{U}}{dt} + \frac{d\mathrm{W}}{dt} = 0, \quad \text{or} \quad \alpha x\dot{x} + \beta v\dot{v} = 0.$$

Since $\dot{v} = \ddot{x}$, this equation may be written $\ddot{x} + \dfrac{\alpha}{\beta}x = 0$, so that

the periodic time is given by

$$T = 2\pi\sqrt{\frac{\beta}{\alpha}}.$$

In words, it is often stated that

$$T = 2\pi\sqrt{\frac{\text{Kinetic energy for unit velocity}}{\text{Potential energy for unit displacement}}}.$$

Dimensionally this statement is incorrect; the correct statement is

$$T = 2\pi\sqrt{\frac{(\text{Kinetic energy})}{(\text{Velocity})^2} \div \frac{(\text{Potential energy})}{(\text{Displacement})^2}},$$

the dimensions of which are clearly those of time. In this expression the term displacement is used to denote the coordinate, viz. x, while the term velocity denotes its time rate of change; the terms, as used here, do not necessarily denote an actual displacement or an actual velocity, respectively.

Thus, for a simple pendulum, when the string makes an angle ψ with the vertical, the kinetic energy of the bob is $\frac{1}{2}m(l\dot{\psi})^2$, while its potential energy at the same instant is

$$\begin{aligned} mgl(1 - \cos\psi) &= mgl\,.\,2\sin^2\tfrac{1}{2}\psi \\ &= mgl\,.\,\tfrac{1}{2}\psi^2, \end{aligned}$$

when ψ is small. Thus according to the formula just established

$$\begin{aligned} T &= 2\pi\sqrt{\frac{\frac{1}{2}ml^2\dot{\psi}^2}{(\dot{\psi})^2} \div \frac{mgl(\frac{1}{2}\psi^2)}{(\psi)^2}} \\ &= 2\pi\sqrt{\frac{l}{g}}, \end{aligned}$$

which is $2\pi\sqrt{\frac{\beta}{\alpha}}$. For another illustration of this method let us consider the following problem.

The oscillations under gravity of a light helical spring when it is loaded.—Fig. 4·05(a) shows the spring when it is unloaded. When a load of mass M is carried by the spring let ξ_0 be the extension of the spring, cf. Fig. 4·05(b). If f is the force required to stretch the spring by unit length, for equilibrium.

$$f\xi_0 = Mg,$$

where g is the intensity of gravity. In this position the energy stored in the spring (this is considered as part of the potential

energy of the system) is equal to the work done on it, viz.

$$\int_0^{\xi_0} (f\xi)\, d\xi = \tfrac{1}{2} f\xi_0^{\,2}.$$

Now let the spring be extended and released so that it executes small vibrations in a vertical plane. If ξ is the displacement measured from the position of static equilibrium when the spring carries the mass M, cf. Fig. 4·05(c), the additional energy stored at this instant, is

$$\begin{aligned}\tfrac{1}{2} f[(\xi_0 + \xi)^2 - \xi_0^{\,2}] &= f\xi_0\xi + \tfrac{1}{2} f\xi^2 \\ &= \mathrm{M}g\xi + \tfrac{1}{2} f\xi^2.\end{aligned}$$

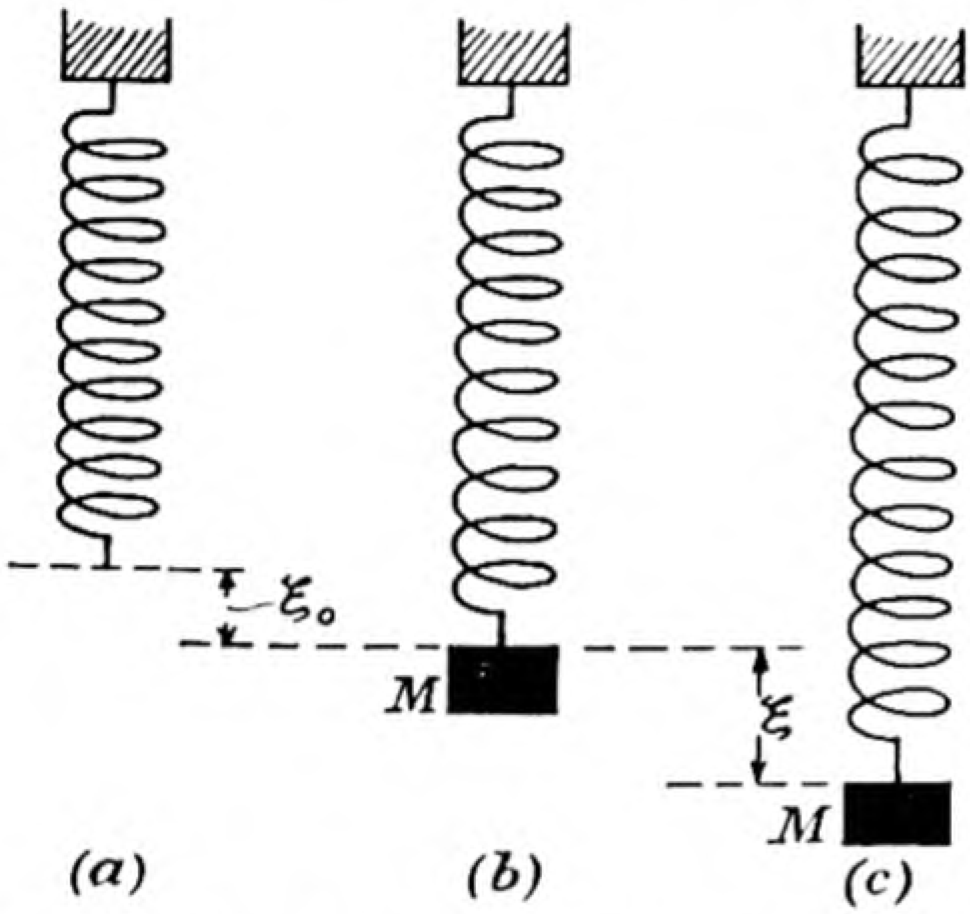


FIG. 4·05.—Oscillations under gravity of a light helical spring when it is loaded.

If the potential energy due to gravity is considered zero in the position of static equilibrium, then the potential energy corresponding to the position ξ is $-\mathrm{M}g\xi$. Hence the total potential energy is

$$-\mathrm{M}g\xi + \mathrm{M}g\xi + \tfrac{1}{2} f\xi^2 = \tfrac{1}{2} f\xi^2.$$

The kinetic energy is $\tfrac{1}{2}\mathrm{M}\dot{\xi}^2$. Hence

$$\tfrac{1}{2}\mathrm{M}\dot{\xi}^2 + \tfrac{1}{2} f\xi^2 = \text{const.}$$

$$\therefore\ \ddot{\xi} + \frac{f}{\mathrm{M}}\,\xi = 0,$$

i.e.

$$\mathrm{T} = 2\pi\sqrt{\frac{\mathrm{M}}{g}},$$

which agrees with the relation

$$\mathrm{T} = 2\pi\sqrt{\frac{(\text{Kinetic energy})}{(\text{Velocity})^2} \div \frac{(\text{Potential energy})}{(\text{Displacement})^2}}.$$

Equation to a cycloid.—A cycloid is the curve traced out by a point on the circumference of a circle which rolls along a straight line. To determine its equation, let us consider a circle of radius a rolling along the line $y = 2a$—cf. Fig. 4·06. Let the initial position of the circle be such that its centre lies at C_1, a point on the axis Oy. Let P be the point on the circumference of the circle which is to trace out the required curve. Let the initial position of P be P_1 at the origin of coordinates. Now let the circle roll along the straight

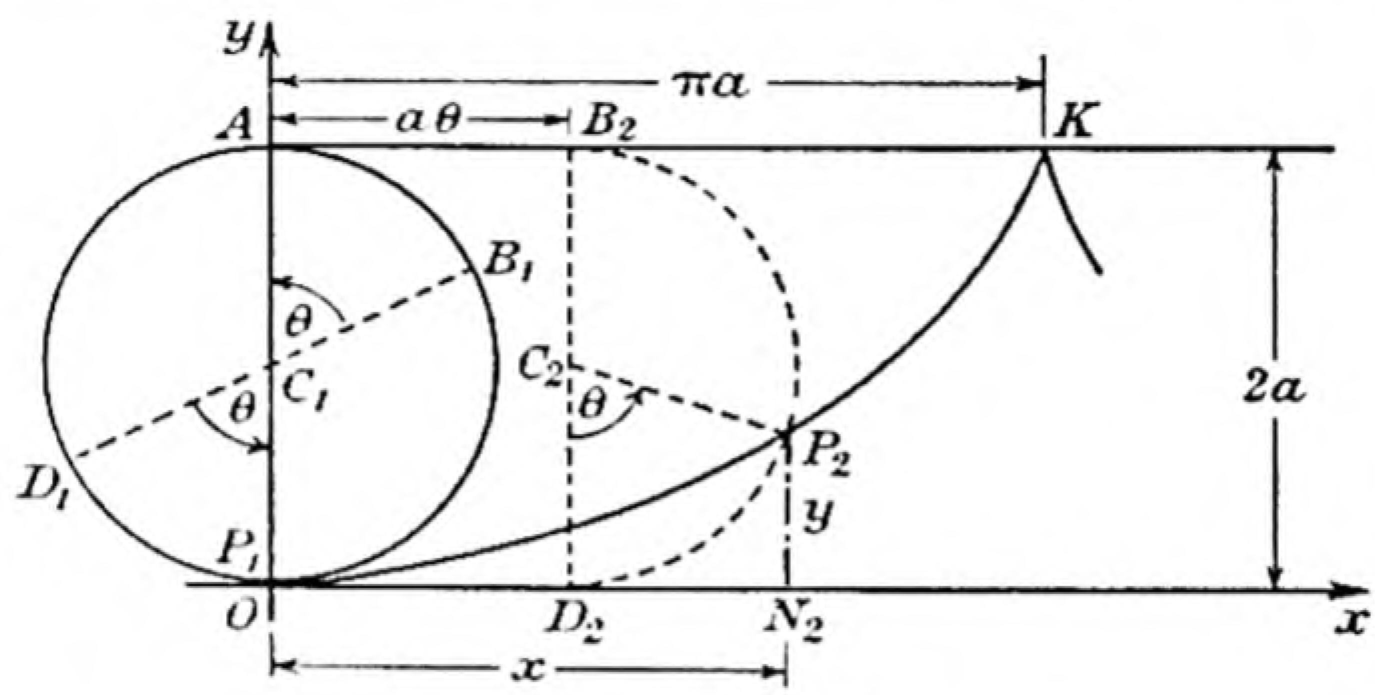


FIG. 4·06.—A cycloid.

line $y = 2a$ until its centre is at C_2, and B_1, a point on the circumference of the circle such that $A\widehat{C_1}B_1 = \theta$, is at B_2 on the line along which the circle rolls. Then $C_1C_2 = a\theta$, and D_1, the point diametrically opposite B_1, lies at D_2 on Ox. Let the coordinates of P_2, the position of the tracing point, be (x, y). Then

$$\begin{aligned} x &= ON_2 \qquad [N_2 \text{ is the projection of } P_2 \text{ on } Ox] \\ &= C_1C_2 + C_2P_2 \sin\theta \\ &= a\theta + a\sin\theta = a(\theta + \sin\theta). \end{aligned}$$

Also

$$\begin{aligned} y &= P_2N_2 = C_2D_2 - C_2P_2\cos\theta \\ &= a(1 - \cos\theta). \end{aligned}$$

When $\theta = \pi$, the tracing point is at K, the point $(\pi a, 2a)$. The tracing point then moves along that portion of the curve which is the image of OP_2K in a line through K parallel to the y axis. K is a cusp.

The cycloidal pendulum.—The motion of a simple pendulum is simple harmonic only if the amplitude is sufficiently small. It is, however, possible to constrain a particle moving under gravity so that the motion is strictly simple harmonic, provided that the amplitude does not exceed a certain finite limit. The differential equation which has to be satisfied for all values of ψ is

$$\ddot{s} = -g\sin\psi,$$

for this is the equation giving the motion of a particle constrained to move without friction in any curve if ψ is the angle made with the horizontal by the tangent to the curve at a point distance s along it. If this equation is to represent simple harmonic motion, the acceleration must be $-\mu s$, where μ is a constant. Hence

$$s = \frac{g}{\mu} \sin \psi.$$

To determine the nature of this curve, we proceed as follows: Let $s = c \sin \psi$. Then $ds = c \cos \psi \, d\psi$.
But

$$\frac{dy}{dx} = \tan \psi, \qquad \frac{dy}{ds} = \sin \psi, \qquad \text{and} \qquad \frac{dx}{ds} = \cos \psi.$$

$$\therefore \frac{dy}{d\psi} = \frac{dy}{ds}\frac{ds}{d\psi} = c \cos \psi \sin \psi.$$

$$\therefore y = \tfrac{1}{2}c \sin^2 \psi + \text{A}, \qquad \text{where A is a constant of integration.}$$

$$= \tfrac{1}{4}c[1 - \cos 2\psi], \qquad \text{choosing } y = 0 \text{ when } \psi = 0.$$

Also

$$\frac{dx}{d\psi} = \frac{dx}{ds}.\frac{ds}{d\psi} = c \cos^2 \psi = \tfrac{1}{2}c(1 + \cos 2\psi).$$

$$\therefore x = c\left[\frac{\psi}{2} + \frac{\sin 2\psi}{4}\right] + \text{B}, \quad \text{where B is a constant of integration.}$$

$$= \tfrac{1}{4}c[2\psi + \sin 2\psi], \qquad \text{if } x = 0 \text{ when } \psi = 0.$$

Thus the curve represented by $s = c \sin \psi$ is a cycloid, the diameter of the generating circle being $\frac{1}{2}c$.

An important property of a cycloid.—We have just seen that the length s of the arc of a cycloid, measured from the origin, is expressed by

$$s = c \sin \psi,$$

when the equation to the cycloid is

$$\left.\begin{aligned} x &= \tfrac{1}{4}c(2\psi + \sin 2\psi) \\ y &= \tfrac{1}{4}c(1 - \cos 2\psi) \end{aligned}\right\}.$$

Let P, Fig. 4·07(a), be a point on the above curve, the vertex being at O, the origin of coordinates, while K is the first cusp. Then ρ, the radius of curvature at P, is given by

$$\rho = \frac{ds}{d\psi} = c \cos \psi.$$

Since the diameter of the rolling circle is $\frac{1}{2}c$, it follows that $\rho = 2 . \frac{c}{2} \cos \psi = 2\text{CP}$, where C is the point of contact of the circle with the line along which the latter rolls. If PC is produced to Q such that PC = CQ, then Q is the centre of curvature at the point P. The coordinates of the point Q are therefore

$$x = \tfrac{1}{4}c(2\psi + \sin 2\psi) - c \cos \psi \sin \psi = \tfrac{1}{4}c(2\psi - \sin 2\psi),$$
$$y = \tfrac{1}{4}c(1 - \cos 2\psi) + c \cos \psi \cos \psi = \tfrac{1}{4}c(2 + 2 \cos^2 \psi).$$

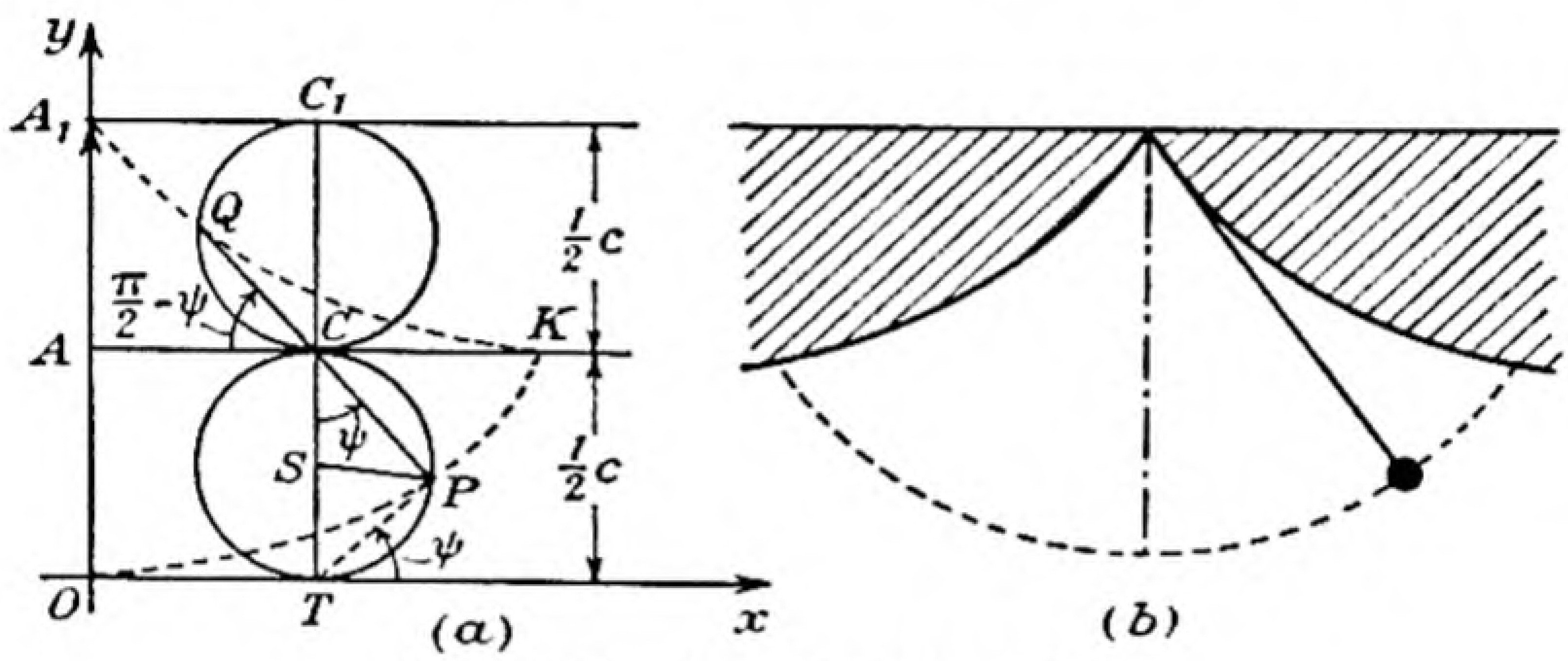


FIG. 4·07.—A cycloidal pendulum.

Now let the origin of coordinates be transferred to the point $(-\frac{1}{4}\pi c, \frac{1}{2}c)$, i.e. the first cusp to the original cycloid on its negative side. Let X and Y be the coordinates of Q referred to rectangular axes through this new origin and parallel to the axes Ox, Oy. Then

$$\begin{aligned} \text{X} &= \tfrac{1}{4}c(2\psi - \sin 2\psi) + \tfrac{1}{4}\pi c \\ &= \tfrac{1}{4}c\left[2\left(\psi + \frac{\pi}{2}\right) + \sin 2\left(\psi + \frac{\pi}{2}\right)\right], \end{aligned}$$

and

$$\begin{aligned} \text{Y} &= \tfrac{1}{4}c[2 + 2 \cos^2 \psi - 2] = \tfrac{1}{4}c[1 + \cos 2\psi] \\ &= \tfrac{1}{4}c\left[1 - \cos 2\left(\psi + \frac{\pi}{2}\right)\right]. \end{aligned}$$

Hence the locus of Q is a cycloid equal in size to the original cycloid.

From the diagram it is clear that QC makes an $\widehat{\text{QCA}} = \left(\frac{\pi}{2} - \psi\right)$ with the horizontal. Hence $\text{KQ} = c \sin \left(\frac{\pi}{2} - \psi\right) = \text{QP}$.

The cycloidal pendulum in practice.—The above property of a cycloid is really the one which gives it its importance in the study of simple harmonic motion. It has already been shown that the motion of a simple pendulum is only simple harmonic if the amplitude

is small; when the amplitude is finite the period is different from that of a pendulum of the same length but whose amplitude is small. Hence a clock beating two seconds for a definite angle of swing will have a different period if the amplitude is altered, i.e. the clock will gain or lose. The importance of cycloidal motion lies in the fact that if a particle can be constrained to move under gravity on a cycloid whose axis is vertical, then the period will be independent of the amplitude, i.e. the timekeeping powers of a particle moving without friction in a cycloid are unimpeachable.

In practice, the simplest method by which a particle may be constrained to move in the above manner is one first proposed by HUYGENS (1629–95). The particle is suspended from a fixed point by a fine cord of length $A_1P = 2c$ in such a way that it wraps itself on the cheeks of two metal jaws, the shape of these being given by the equation $s = c \sin \psi$—cf. Fig. 4·07(*b*).

Later designers have proceeded differently in endeavours to obtain a pendulum whose period at a fixed temperature shall be invariable. They have tried to keep the amplitude constant by applying a suitable control force to the pendulum so that the energy losses due to friction and air-damping are just counter-balanced by the energy supplied by the control.

The oscillations of a sphere on a concave mirror.—When a steel sphere is displaced from its position of equilibrium on the surface of a concave spherical mirror and then set free, it will *roll* backwards and forwards since, when the oscillations are sufficiently small, the friction between the surfaces in contact is great enough to ensure that there is no ***sliding***. By observing the periodic times of the small oscillations of the sphere on the curved mirror, a value for g may be found. The theory of the method is as follows:—

Let O, Fig. 4·08(*a*), be the centre of curvature of the mirror, of radius r, C the centre of the sphere of radius a, A_0 the point on the mirror vertically below O, and P the instantaneous point of contact between the sphere and the mirror. Let $\psi = A_0OP$. Then if A is the position of the point on the sphere which is in contact with A_0 when the sphere is in its lowest position, the arc PA is equal to the arc A_0P since the motion is one of pure rolling. Hence, if ϕ is the angle indicated,

$$a\phi = r\psi.$$

If CA produced cuts OA_0 in B, and if $\theta = O\widehat{B}C$, then since CA is a fixed line in the sphere, ϕ is the angle through which any radius of the sphere has turned since it left its lowest position, i.e. after time t.

Now $$\phi = \psi + \theta, \quad \text{i.e. } \theta = \frac{r\psi}{a} - \psi = \left(\frac{r-a}{a}\right)\psi.$$

To determine the periodic time let us calculate the energy of the system. If the potential energy is reckoned zero for the lowest position of the sphere, we have, if m is the mass of the sphere

$$\begin{aligned} \text{P.E.} &= mg(r-a)(1-\cos\psi) \\ &= \tfrac{1}{2}mg(r-a)\psi^2, \quad \text{if } \psi \to 0. \end{aligned}$$

Expressed in terms of θ this is $\dfrac{1}{2}\dfrac{mga^2}{r-a}\cdot\theta^2$.

The kinetic energy is calculated by using the fact that since P is

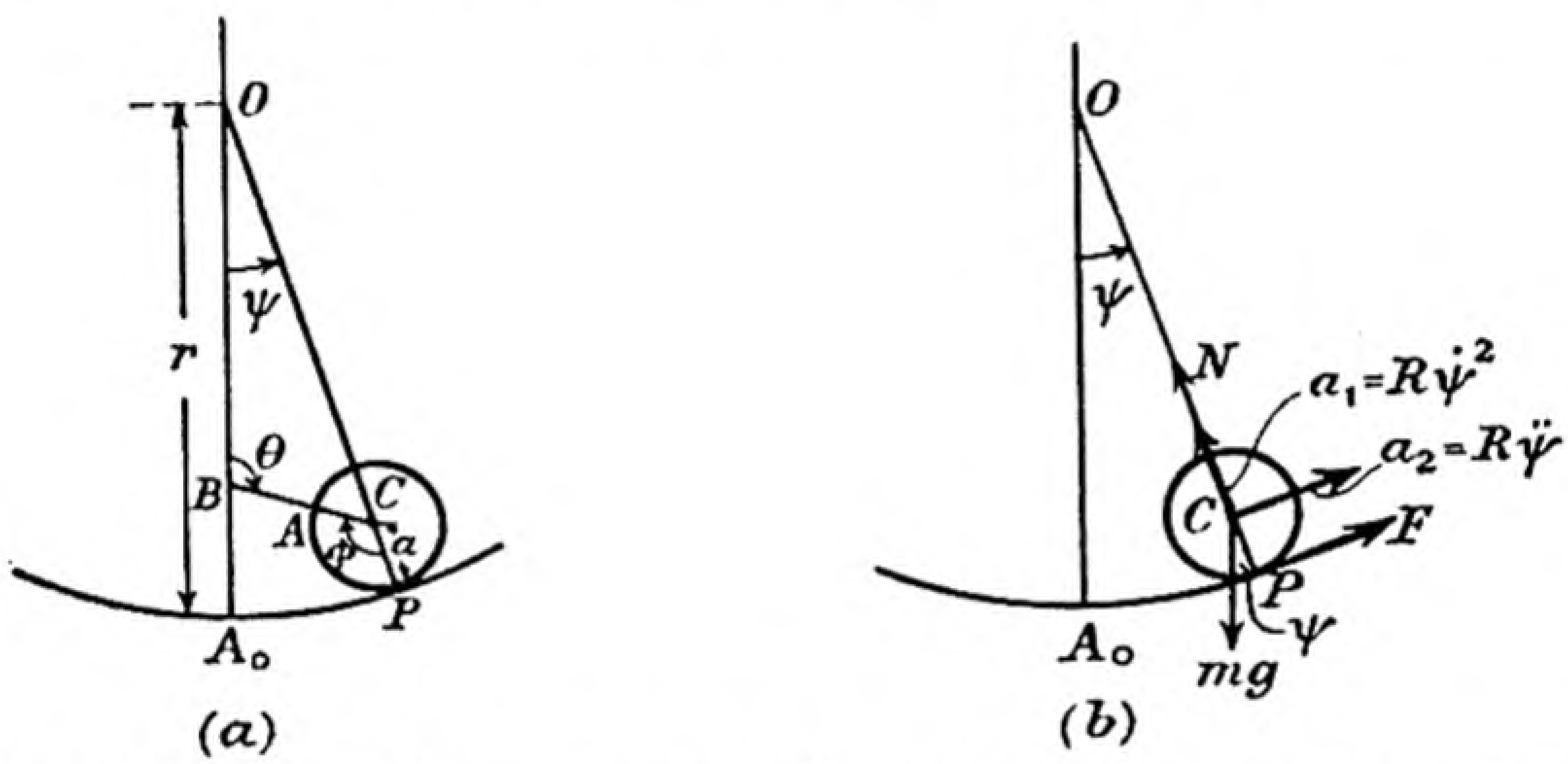


FIG. 4·08.—Oscillations under gravity of a sphere on a concave mirror.

an instantaneous centre of rest, the sphere is at that instant rotating about an axis through P perpendicular to the plane A_0OP. If I is the moment of inertia of the sphere about a horizontal axis through C, its moment of inertia about a parallel axis through P is $(\text{I} + ma^2)$.

$$\therefore \ \text{K.E.} = \tfrac{1}{2}(\text{I} + ma^2)\dot{\theta}^2,$$

since θ is the angle which a fixed line CA in the sphere makes with the fixed line OA_0 in space, i.e. $\omega = \dot{\theta}$.

$$\begin{aligned} \therefore \ \text{T} &= 2\pi\sqrt{\frac{\text{K.E.}}{(\dot{\theta})^2} \div \frac{\text{P.E.}}{\theta^2}} \\ &= 2\pi\sqrt{\tfrac{1}{2}(\text{I} + ma^2) \div \frac{1}{2}\frac{mga^2}{r-a}} \\ &= 2\pi\sqrt{\frac{7}{5}\left(\frac{r-a}{g}\right)}, \qquad [\because \ \text{I} = \tfrac{2}{5}ma^2] \end{aligned}$$

The length of the simple equivalent pendulum is therefore $\frac{7}{5}(r-a)$.

N.B. The above expression may also be obtained from considerations of the motion of a sphere on an inclined plane. For such motion we have proved, cf. p. 96

$$\ddot{x} = \frac{a^2}{a^2 + \kappa^2} g \sin \alpha.$$

When the sphere is on the mirror and in a position defined by ψ, we may regard it as instantaneously on an inclined plane in which $\alpha = \psi$, so that

$$-(r - a)\ddot{\psi} = \frac{a^2}{a^2 + \kappa^2} . g\psi = \tfrac{5}{7} g\psi.$$

[The minus sign occurs because the motion was *down* the plane when the equation of motion was derived; now, when ψ increases the motion is *up* the tangent plane at P.]

$$\therefore \ \mathrm{T} = 2\pi \sqrt{\frac{7}{5} . \frac{(r - a)}{g}}, \quad \text{as before.}$$

Again, by the principle of the conservation of energy, we have

$$mg(r - a)(1 - \cos \psi) + \tfrac{1}{2} m(\kappa^2 + a^2)\dot{\theta}^2 = \text{constant},$$

since P is an instantaneous centre of rest and $\dot{\theta}$ is the angular velocity of the sphere, cf. p. 116. Now, as before, $a\phi = r\psi$ and $\phi = \theta + \psi$, so that $a\theta = (r - a)\psi$. Differentiating the energy equation with respect to time, we have

$$g(r - a) \sin \psi . \dot{\psi} + (\kappa^2 + a^2)\dot{\theta}\ddot{\theta} = 0.$$

Eliminating ψ and $\dot{\psi}$ from this equation by using $a\theta = (r - a)\psi$, we have, if $\psi \to 0$,

$$(\kappa^2 + a^2)\ddot{\theta} + \left(\frac{a^2}{r - a}\right) g\theta = 0,$$

which with $\kappa^2 = \frac{2}{5}a^2$, gives, as before

$$\mathrm{T} = 2\pi \sqrt{\frac{7}{5} . \frac{(r - a)}{g}}.$$

Finally it should be noted that the period of oscillations may be determined by making use of the equations of motion. If we assume that the reaction at P is a single force so that no frictional couple acts on the sphere, by taking moments about P, we have

$$m(a^2 + \kappa^2)\ddot{\theta} + mga \sin \psi = 0,$$

which when $\psi \to 0$ becomes

$$(a^2 + \kappa^2) \frac{\mathrm{R}}{a} \ddot{\psi} + ag\psi = 0, \qquad [\because \ a\theta = \mathrm{R}\psi]$$

which, with $\kappa^2 = \frac{2}{5}a^2$, gives

$$\mathrm{T} = 2\pi \sqrt{\frac{7}{5} . \frac{\mathrm{R}}{g}} = 2\pi \sqrt{\frac{7}{5} . \frac{(r - a)}{g}}.$$

The reaction between a sphere and the concave spherical surface on which it rolls.—To determine the least value of the coefficient of friction which will ensure that the sphere will not slide when in motion on a concave spherical surface, let α be the greatest value of ψ, and N and F the normal and tangential components of the reaction at P, Fig. 4·08(*b*), the point of contact between the sphere and surface on which it rolls. Then, by the energy principle, we have if $(r - a) = R$,

$$mg\mathrm{R}(1 - \cos\psi) + \tfrac{1}{2}m(\kappa^2 + a^2)\,\dot{\theta}^2 = mg\mathrm{R}(1 - \cos\alpha) + 0, \qquad \text{(i)}$$

since the sphere is momentarily at rest when its angular displacement is α.

Also, since the components of the acceleration of C *towards* O and at right angles to this are, cf. p. 60,

$$a_1 = -(\ddot{\mathrm{R}} - \mathrm{R}\dot{\psi}^2) = \mathrm{R}\dot{\psi}^2, \qquad [\because \mathrm{R} \text{ is constant}]$$

and $$a_2 = \mathrm{R}\ddot{\psi} + 2\dot{\mathrm{R}}\dot{\psi},$$

and since the linear acceleration of the centre of the sphere is the same as if all external forces acted at that point, we have

$$m\mathrm{R}(\dot{\psi})^2 = \mathrm{N} - mg\cos\psi, \qquad \text{(ii)}$$

and $$m\mathrm{R}\ddot{\psi} = \mathrm{F} - mg\sin\psi. \qquad \text{(iii)}$$

If in equation (i) we use the fact that $a\theta = R\psi$ or $a\dot{\theta} = R\dot{\psi}$, we have, after differentiation with respect to time,

$$g\mathrm{R}\sin\psi\,.\,\dot{\psi} + (\kappa^2 + a^2)\left(\frac{\mathrm{R}}{a}\right)^2\dot{\psi}\ddot{\psi} = 0,$$

or $$\sin\psi = -\frac{\kappa^2 + a^2}{ga^2}\mathrm{R}\ddot{\psi} \qquad \text{(iv)}$$

Eliminating $\ddot{\psi}$ from equations (iii) and (iv), we find

$$\mathrm{F} = mg\left(1 - \frac{a^2}{\kappa^2 + a^2}\right)\sin\psi = \tfrac{2}{7}mg\sin\psi.$$

Also from (i) and (ii), we find after some reduction

$$\mathrm{N} = \tfrac{1}{7}mg(17\cos\psi - 10\cos\alpha).$$

The greatest value of F is $\frac{2}{7}\,mg\sin\alpha$ and the least value of N occurs when $\psi = \alpha$; in such circumstances $\mathrm{N} = mg\cos\alpha$. Hence the greatest value of $\frac{\mathrm{F}}{\mathrm{N}}$ is $\frac{2}{7}\tan\alpha$ and so if μ, the coefficient of friction exceeds $\frac{2}{7}\tan\alpha$, the sphere will always roll and never slide.

Experiment.—A value for the intensity of gravity may be obtained by observing the period of a sphere rolling on a concave mirror. First determine the radius of curvature of the front surface of the mirror by an optical method. Then determine the period in the usual way, not forgetting to use a fiducial mark past which the transit of the sphere may be observed and timed. A convenient mark for the above purpose consists of a piece of black cotton stretched horizontally above the centre of the sphere when at rest. If the eye of the observer is in the plane containing the cotton and its image, formed by reflexion in the mirror, the timing of a definite event and its repetition is assured.

For success in this experiment it is essential that the surfaces of the sphere and of the mirror be free from grease so that the sphere rolls without slipping. They must be cleaned with a detergent; afterwards they should not be touched by hand; the sphere may be held in a piece of clean filter paper when necessary to move it.

Example.—A uniform circular hoop is suspended from three fixed points by three strings each of length λ. The strings are vertical and the plane of the hoop horizontal. Determine the period of a small rotational oscillation about the vertical through the centre.

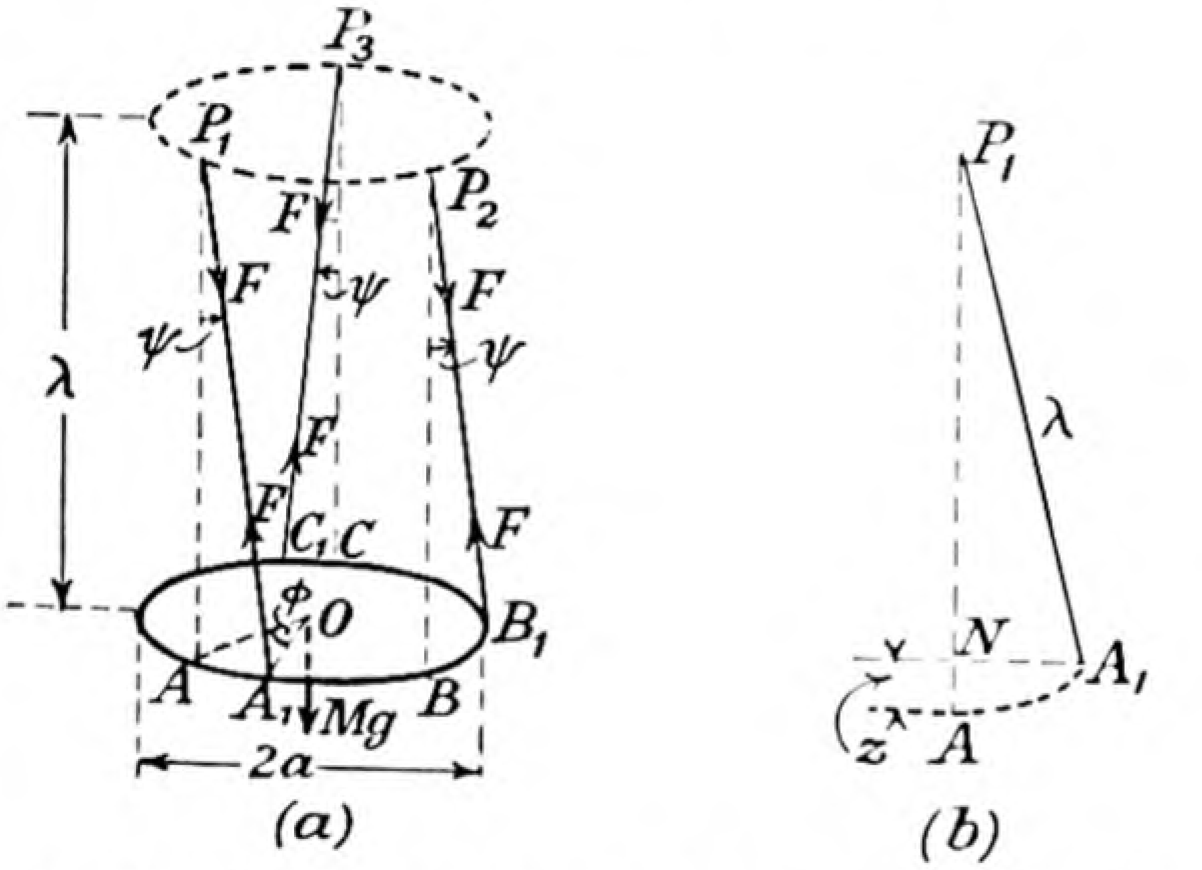


FIG. 4·09.—Small oscillations of a circular hoop suspended by three vertical strings of equal length. [In (a) the small 'rise' of the hoop is not shown.]

If F, Fig. 4·09(a), is the tension in each string, and M the mass of the hoop, we have

$$3F\cos\psi = Mg,$$

where g is the intensity of gravity and ψ the angle each string makes with the vertical. Since ψ is small we may write

$$3F = Mg.$$

Suppose that each radius of the hoop is displaced through an angle ϕ at some instant. Then, since ϕ is small when ψ is small,

$$\lambda\psi = a\phi,$$

a being the radius of the hoop the moment about the axis of rotation of the forces tending to increase ϕ is

$$-3(F\sin\psi)a \doteqdot -Mg\frac{a^2\phi}{\lambda}.$$

But this is $I\ddot{\phi} = M\kappa^2\ddot{\phi}$, where I and κ are respectively the moment of inertia and the radius of gyration of the hoop about the axis of rotation. Hence, since $\kappa \doteqdot a$ if the hoop is thin,

$$\ddot{\phi} + \frac{g}{\lambda}\phi = 0.$$

The motion is therefore simple harmonic, its period being $2\pi\sqrt{\dfrac{\lambda}{g}}$.

Alternative method.—Let us suppose that when the displacement of each supporting thread from the vertical is ψ, the centre O rises a distance z. Then since ψ is small the arc AA_1 may be taken as the side NA_1 of the right-angled ΔP_1A_1N, when

$$\lambda^2 = a^2\phi^2 + (\lambda - z)^2,$$

or

$$2\lambda z = a^2\phi^2,$$

if $\dfrac{z}{2\lambda}$ is small in comparison with unity. The energy principle gives, at once,

$$\tfrac{1}{2}I\dot{\phi}^2 + Mgz = \text{constant},$$

or

$$\tfrac{1}{2}Ma^2\dot{\phi}^2 + Mg\,\frac{a^2\phi^2}{2\lambda} = \text{constant}.$$

Differentiating with respect to t, we obtain

$$\ddot{\phi} + \frac{g}{\lambda}\phi = 0,$$

so that, as before,

$$T = 2\pi\sqrt{\frac{\lambda}{g}}.$$

It will be noticed that T is independent of a; it is also independent of the number of strings in excess of two used to support the hoop and the manner in which they are arranged round its periphery; in this latter instance the tensions in the strings will be different from each other, viz. $F_1, F_2, F_3 \ldots$, where

$$F_1 + F_2 + F_3 + \ldots = Mg.$$

Example.—A uniform circular disc, mass M and radius a, is suspended with its plane horizontal by three symmetrically situated vertical threads each of length λ, attached to the rim. These threads are assumed inelastic and to have negligible mass and rigidity. The disc carries another disc of mass m, one of the faces of this disc being in contact with the larger disc. What is the period of small oscillations under gravity of the above system about a vertical axis through the centre of the large disc? Describe and explain how, by using the above arrangement,* the theorem of parallel axes may be tested experimentally.

* The method is due to Dr. E. J. Irons.

The system is shown diagrammatically in Fig. 4·10(a). Let F_1, F_2 and F_3 be the tensions in the strings when the centre of gravity of the disc m is at a distance r from the vertical axis about which the system rotates. If ψ is the angle each thread makes with the vertical when each radius in the larger disc has moved through an angle ϕ, then

$$(F_1 + F_2 + F_3) \cos \psi = (M + m)g.$$

The couple tending to increase ϕ is

$$-(F_1 + F_2 + F_3) \sin \psi . a = (M + m)ga \tan \psi$$

$$\fallingdotseq (M + m)ga^2 . \frac{\phi}{\lambda},$$

since $a\phi = \lambda\psi$. The equation of motion is therefore

$$I\ddot{\phi} + (M + m)ga^2 . \frac{\phi}{\lambda} = 0,$$

where I is the moment of inertia of the whole system about its axis of

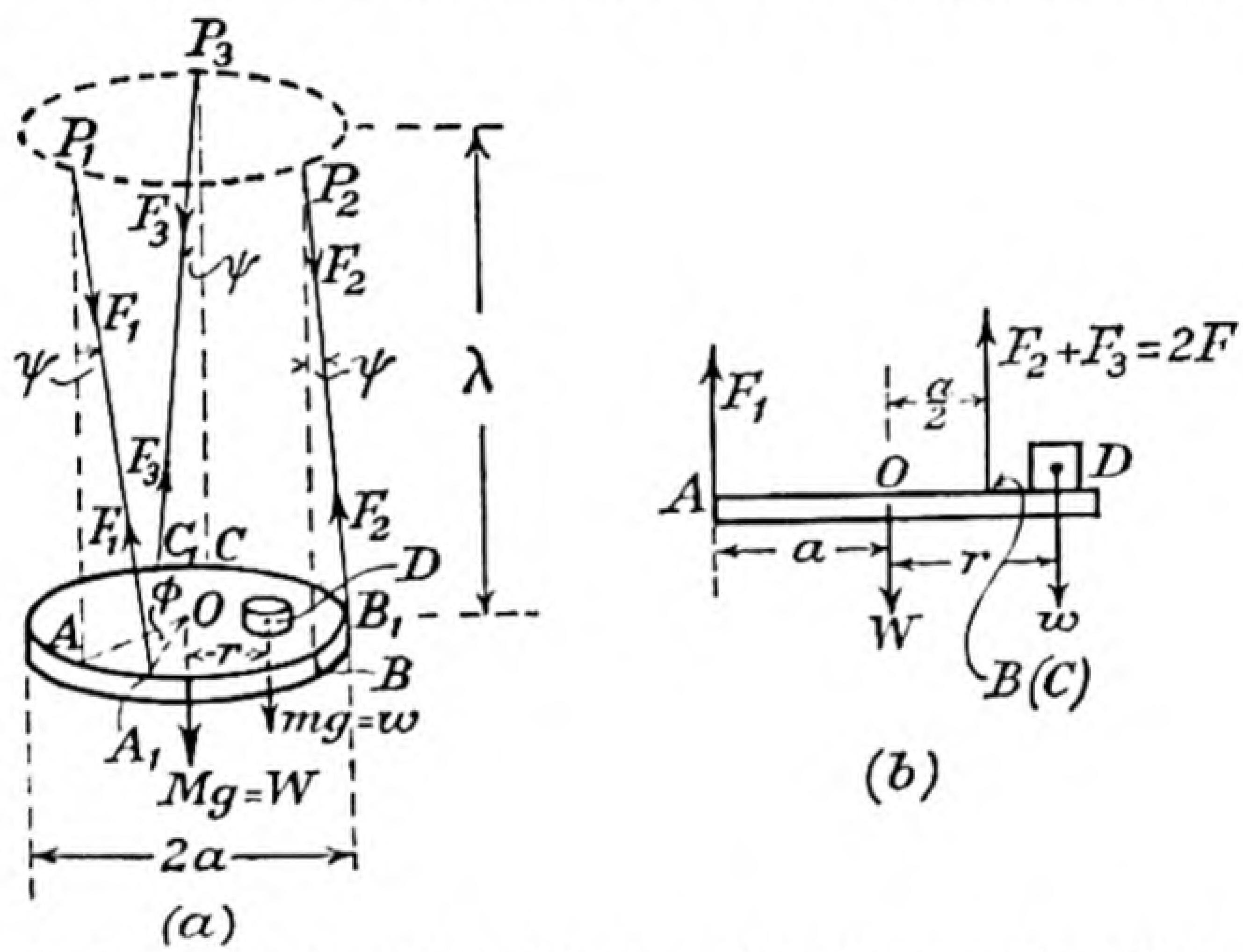


FIG. 4·10.—Experimental verification of the theorem of parallel axes.

rotation. This moment is equal to

$$\tfrac{1}{2}Ma^2 + m(\kappa^2 + r^2),$$

where κ is the radius of gyration of the smaller disc about a vertical axis through its centroid.

Thus $\dfrac{\ddot{\phi}}{\phi}$ is constant, i.e. the motion is simple harmonic and its period T given by

$$T^2 = \frac{4\pi^2[\frac{1}{2}Ma^2 + m(\kappa^2 + r^2)]\lambda}{(M + m)ga^2}.$$

This equation may be written

$$\frac{(M + m)ga^2}{4\pi^2\lambda m} . T^2 = (\kappa^2 + r^2) + \tfrac{1}{2}\frac{M}{m}.a^2.$$

If therefore, after a suitable set of observations has been made, $y = \dfrac{(M + m)ga^2}{4\pi^2\lambda m}.T^2$ is plotted against $x = r^2$, the graph should be a

straight line with a slope unity and making an intercept on the y-axis equal to $\frac{1}{2}\frac{M}{m}(a^2 + \kappa^2)$. If the disc m is replaced by a cylinder with its axis vertical and cut so that its mass is M, the calculations are less tedious.

If these deductions are confirmed, then the theorem of parallel axes, cf. p. 70, which has been used in deriving the time period, will have been verified.

It should be noticed that under certain conditions it is impossible for the large disc to remain horizontal. Thus, in Fig. 4·10(b), let the small disc or cylinder D be arranged with its axis vertical and passing through a point on the diameter AO of the larger disc. Then on, account of symmetry, the tensions in the threads through B and C will be equal, although, by proceeding as follows, we do not make use of this fact. Taking moments of forces about the axis BC we have, since this axis is at a distance $\frac{1}{2}a$ from O,

$$F_1(a + \tfrac{1}{2}a) + w(r - \tfrac{1}{2}a) = W(\tfrac{1}{2}a).$$

$$\therefore\ F_1 = \frac{2}{3a}[\tfrac{1}{2}Wa + w(\tfrac{1}{2}a - r)].$$

Now if F_1 is zero the equilibrium of the disc becomes unstable. This occurs when

$$r = \frac{W + w}{w}.\frac{a}{2} = \frac{M + m}{m}.\frac{a}{2}.$$

In practice any such tendency towards instability may be avoided by using two equal small cylinders and arranging them symmetrically on a diameter through O and on opposite sides of O.

[If each point on the disc rises through a distance z, the energy equation is easily shown to be

$$\tfrac{1}{2}I\dot{\phi}^2 + (M + m)gz = \text{constant},$$

where $$z \doteq \frac{a^2\phi^2}{2\lambda},$$ [cf. p. 120.]

i.e. the equation of motion is

$$\tfrac{1}{2}[\tfrac{1}{2}Ma^2 + m(\kappa^2 + r^2)]2\dot{\phi}.\ddot{\phi} + \frac{(M + m)g.a^2.2\phi.\dot{\phi}}{2\lambda} = 0,$$

which is as before. Hence, etc.]

The compound pendulum.—A rigid body of any shape and internal structure which is free to rotate about a fixed horizontal axis, the only external forces being those due to gravity and the reaction of the axis on the body, constitutes a compound pendulum.

Suppose that the mass of the pendulum is m and that it is oscillating about a fixed horizontal axis through O, Fig. 4·11. Let G be the mass-centre and r the distance of G from O. Let ψ be the angle OG makes with the vertical when the pendulum is displaced from its position of static equilibrium. Then the moment of the external forces about O and tending to increase ψ is $-mgr \sin \psi$. But this is equal to the rate of change of the angular momentum of the pendulum, viz. $(I + mr^2)\ddot{\psi}$ or $m(\kappa^2 + r^2)\ddot{\psi}$, where I is the moment of inertia of the body about an axis through G parallel to

the axis of rotation and κ the corresponding radius of gyration. [$(I + mr^2)$ is the moment of inertia of the body about the axis of suspension since this is at a distance r from G—cf. p. 70.] The motion of the pendulum is therefore given by

$$(I + mr^2)\ddot{\psi} + mgr \sin \psi = 0.$$

or $$(\kappa^2 + r^2)\ddot{\psi} + gr \sin \psi = 0 \quad . \quad \text{(i)}$$

If ψ is limited in magnitude so that $\sin \psi$ may be replaced by ψ, then

$$(\kappa^2 + r^2)\ddot{\psi} + gr\psi = 0. \quad . \quad . \quad \text{(ii)}$$

Hence the motion is simple harmonic and the period is given by

$$T = 2\pi\sqrt{\frac{\kappa^2 + r^2}{rg}} \quad . \quad . \quad . \quad \text{(iii)}$$

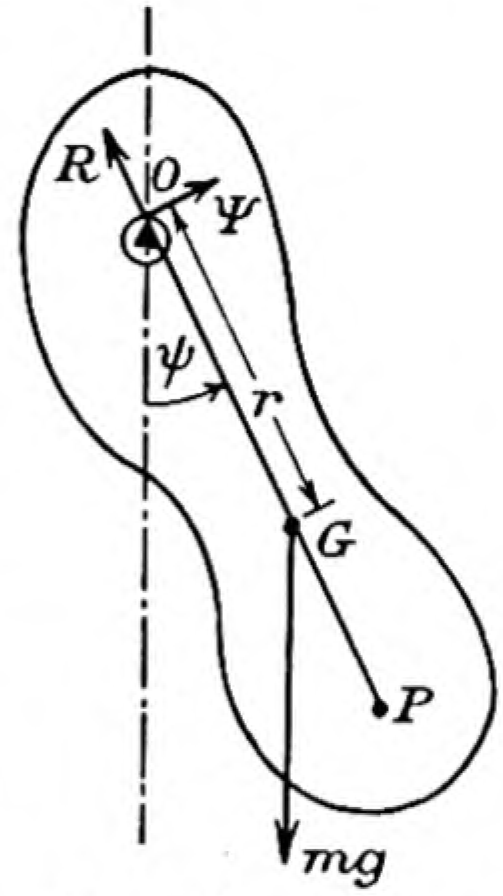


FIG. 4·11.—A compound pendulum.

The period is therefore equal to that of a simple pendulum of length $\left(r + \frac{\kappa^2}{r}\right)$. If the straight line OG is produced to P so that OP $= l$, the length of the ***equivalent simple pendulum***, P is known as the ***centre of oscillation***, while O is termed the ***centre of suspension***. Then

$$OP = l = \frac{\kappa^2 + OG^2}{OG},$$

i.e. $$OP.OG = \kappa^2 + OG^2,$$

or $$OG(OP - OG) = \kappa^2,$$

which may be written

$$OG.GP = \kappa^2.$$

This expression shows that P and O are such that if the body were suspended from an axis through P parallel to that through O, then O would be the centre of the oscillation. The centre of oscillation and of suspension are said to be convertible. This property of a compound pendulum was discovered by HUYGENS.

Now the expression for the period may be written

$$r^2 - r.\frac{T^2 g}{4\pi^2} + \kappa^2 = 0 \quad . \quad . \quad . \quad \text{(iv)}$$

Since the sum and products of the roots of this quadratic equation in r are each positive, both roots are positive, i.e. for a given value of T, this equation gives, in general, two ***positive*** values for r, viz. r_1 and r_2, i.e. for any particular value of T there are ***two positions for O on the same side of G***. It follows at once that $k^2 = r_1 r_2$.

Similarly, if the pendulum is suspended from an axis on the other side of G, this axis being perpendicular to and cutting OG produced, exactly the same quadratic equation will be obtained for a given value of T. Therefore, on any chosen line through G, there are four points about which the period of oscillation of the pendulum is the same, the axis of rotation being always normal to the chosen line.

Suppose h is the distance between two of these points on opposite sides of G and at unequal distances from it. The distance between them is equal to the sum of the roots of (iv), i.e.

$$h = \frac{gT^2}{4\pi^2}.$$

It will be noted that h is the length of a simple pendulum whose periodic time is T; this is the so-called **simple equivalent pendulum.**

These remarks may be illustrated by Fig. 4·12. Since it is only the distance of the centre of suspension from the centre of gravity G which determines the time period, it follows at once that the pendulum shown will have the same period for any point of suspension on either of two circles defined by their radii r_1, r_2 and common centre G. Let O_1 and O_2 be two such points on a straight line through G and in the plane of the diagram.

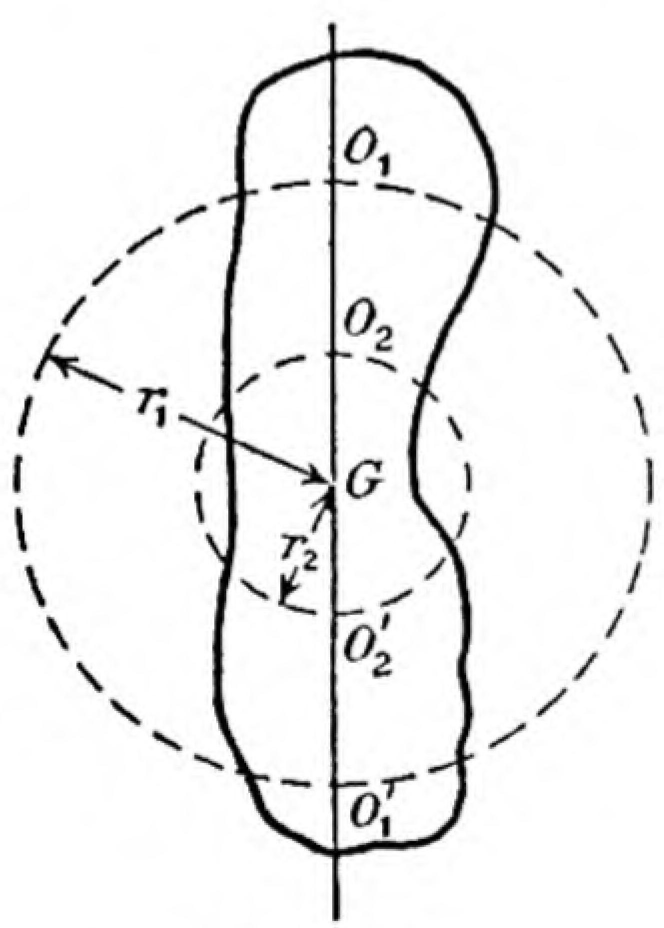


FIG. 4·12.

Since another pair of points O_1' and O_2' given by the intersection of O_1O_2G produced with the above circles also fulfil the same conditions, there will always be, in general, four points on a given straight line through G for which the pendulum will have the same periodic time. Now because

$$GO_1 + GO_2' = r_1 + r_2,$$

it follows that O_1O_2' (or O_2O_1') is the length of the simple equivalent pendulum.

Experiment.—**To determine a value for the intensity of gravity by means of a compound pendulum.** In this experiment use is made of the facts established above. The pendulum consists of a rectangular brass bar about a metre long which may be suspended on a knife-edge at different points along its length—for the purpose of description we regard the bar as a lamina. If a series of holes has been drilled in the bar so that their centres lie on a line passing through the centre of gravity of the bar, the suspension of the bar in the above manner is made more easy—cf. Fig. 4·13(a).

It is essential that the knife-edge should be level. This usually consists of a piece of hard steel ground to a sharp edge and fixed in a wooden

board, A, Fig. 4·13(*b*). The heads of two round-headed screws are fixed on the under side of the wood while an adjustable levelling screw, S, is placed behind. When the bar is placed on the knife-edge it is usual to find that it wobbles as it swings. While the bar is swinging the levelling screw is adjusted until the bar moves only in a vertical plane. It may be necessary to repeat this procedure at every hole.

By means of a chronometer determine the period of the bar when it is suspended from every other hole in turn, using an accurate method of timing. Pins P_1 and P_2 attached to the pendulum are used in

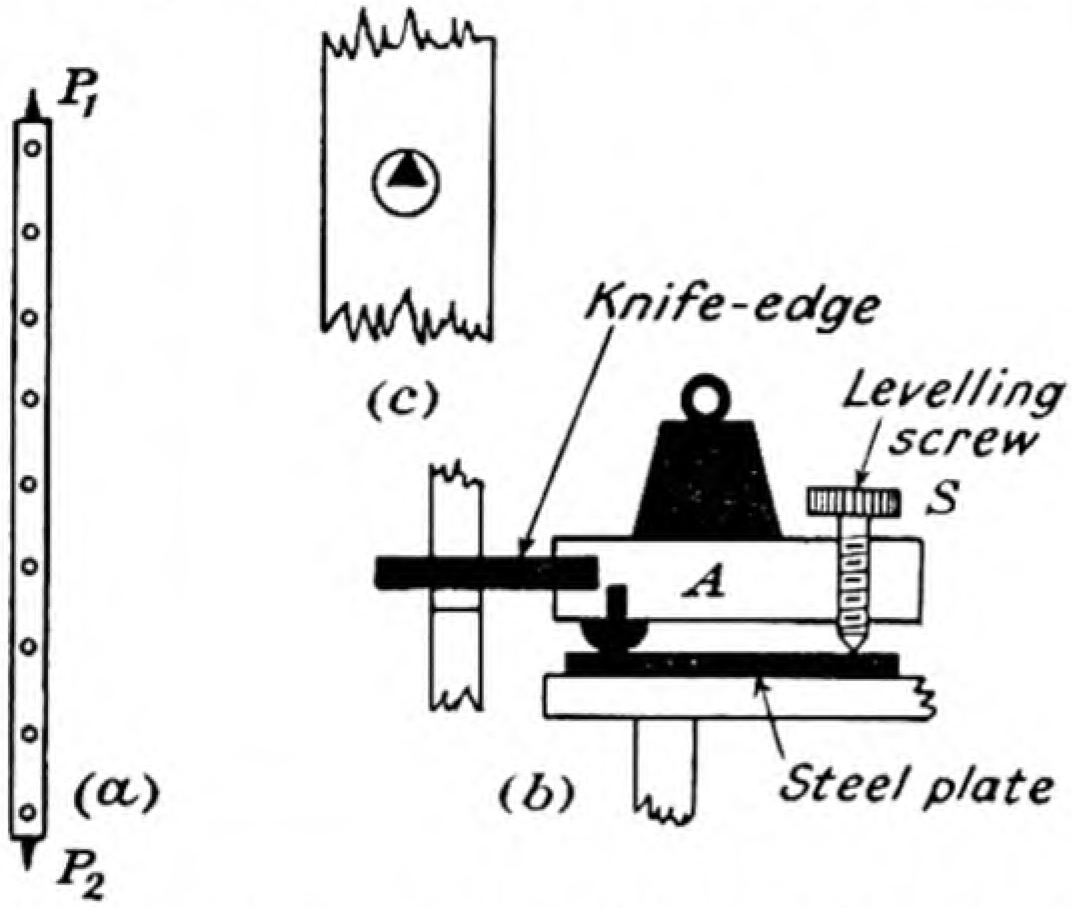


FIG. 4·13.—A compound pendulum used to find '*g*'.

conjunction with a fiducial mark to determine the number of complete oscillations made by the pendulum in a measured time interval. Then measure the distance of each point of suspension from one end of the bar.

Use the observations thus obtained to plot a curve with periodic times as ordinates and the distances of the point of suspension from one end of the bar as abscissae. The curve will be similar to that shown in Fig. 4·14(*a*). The values for the periodic times when these approach

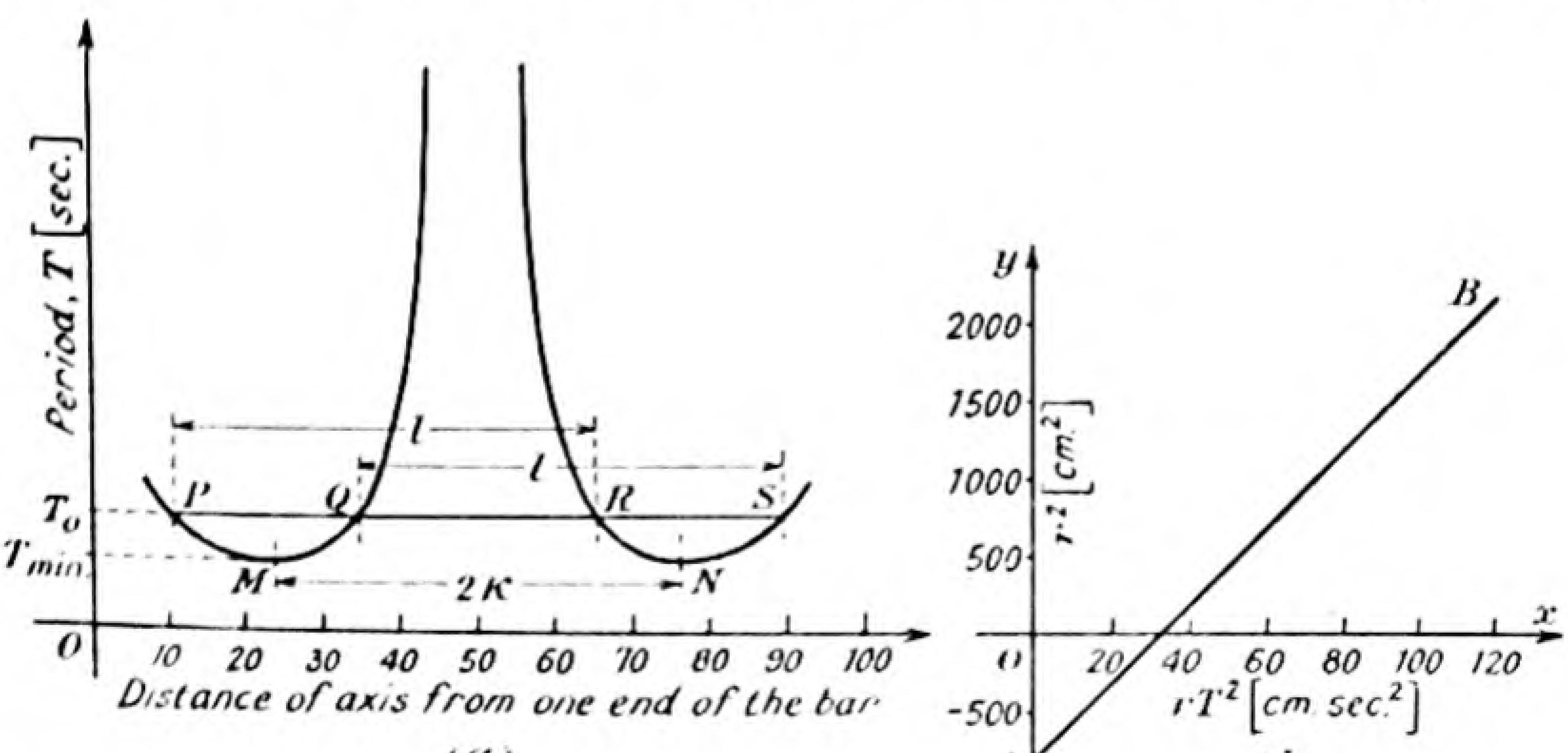


FIG. 4·14.—Graphical methods of dealing with the observations obtained on the periods of a compound pendulum.

the minima indicated, should then be further investigated by using every hole bored in the bar in the corresponding regions.

Let PQRS be a straight line drawn parallel to the x-axis to cut the curve in the points indicated. Now it has already been shown that if we can locate two points in a rigid body, which are on opposite sides of its centre of gravity and not placed symmetrically with respect to it, such that the periodic time when the bar is suspended in turn from each has the same value, then the distance between these points is equal to the length of a simple pendulum having a period equal to that of the bar when thus suspended. Thus P, R and Q, S are two pairs of such points; hence PR and QS are each equal to l, the length of a simple pendulum whose period is represented by the intercept SP makes on Oy, say T_0 seconds.

The intensity of gravity is therefore given by $g = 4\pi^2 \dfrac{l}{T_0^{\,2}}$.

Measure off the lengths of the simple equivalent pendulum for several different values of the period, and thus obtain a mean value for the intensity of gravity.

The condition that the roots of equation (iv) [cf. p. 123] should be equal, say α, is that

$$\left(\frac{T^2 g}{4\pi^2}\right)^2 - 4\kappa^2 = 0.$$

The particular value for r is then $\alpha = \dfrac{1}{2}\left(\dfrac{T^2 g}{4\pi^2}\right)$, so that $\alpha = \kappa$. If MN is the common tangent to the curve under consideration then $\frac{1}{2}\mathrm{MN} = \kappa$, the radius of gyration for the bar about an axis normal to its plane and passing through its mass-centre G. The period is then a minimum.

This method of dealing with the observations can never yield an accurate value for the intensity of gravity, or for the radius of gyration required, since a curve drawn freehand can never be as good as a straight line for making deductions like these and, in addition, since the minima are rather flat the points M and N cannot be located precisely.

A graphical method for obtaining a value for 'g' from the observations taken in this experiment is due to Ferguson,† but in using this method it is necessary to locate, by the simple method of balancing, the position of the centre of gravity of the bar so that values for r may be obtained. Now equation (iv) (p.123) may be written

$$\frac{rT^2}{4\pi^2}\cdot g = r^2 + \kappa^2,$$

so that if we plot $y = r^2$ and $x = rT^2$, a straight line whose slope is $\dfrac{g}{4\pi^2}$ and whose intercept on the y-axis is $-\kappa^2$ will be obtained. Hence both g and κ may be determined—cf. Fig. 4·14(b).

Owen's bar pendulum.—In an experiment with a bar pendulum described by Owen‡ many of the defects in the apparatus used in the above experiment are eliminated. The many troubles in the usual form of bar pendulum arise from the presence of the holes drilled along its length. These holes serve as points of suspension so that only a rough degree of accuracy in the timing of the oscillations can be expected owing to the short length of the bearing and the

† *Science Progress*, **22**, 461, 1928.

‡ *Proc. Phys. Soc.*, **51**, 456, 1939.

consequent tendency to wobble; moreover it is impossible to ensure that the axes of the holes are always normal to the plane of the bar.

These defects may be eliminated by the use of a carriage, C, Fig. 4·15, which slides along the bar and can be clamped at any point, and which supports rigidly a knife-edge, K, projecting on either side of the carriage and filed away in the interior. The knife-edges are supported on a pair of glass plates G_1, G_2 mounted on either side of the platform A of a wall-bracket B, the platform between the plates being cut away to allow free passage of the pendulum and carriage. The bracket should, of course, be set by means of a plumb line so that the glass bearing surfaces are in a horizontal plane.

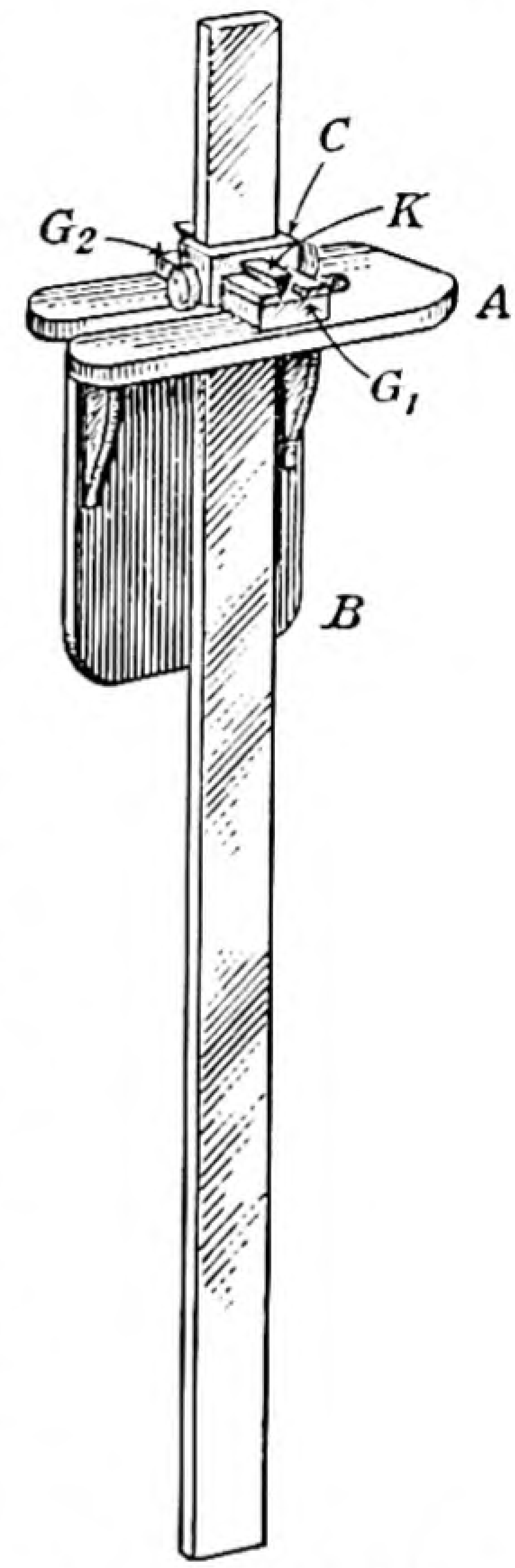


Fig. 4·15.—Owen's bar pendulum.

The pendulum is now apparently complicated by the presence of the carriage, which (*a*) may add to the gravitational couple, and (*b*) must add a term to the moment of inertia of the system. The effect of (*a*) is made quite negligible by adjusting the carriage until its centre of gravity coincides with the line of the knife-edge. The effect of (*b*) is rendered negligible by reducing the moment of inertia of the carriage about the axis of rotation to a value well below 1 part in 10,000 of that of the bar for any position of the carriage, while adequate strength and rigidity is permitted in the carriage, and may thus to this approximation be neglected.

Values for g and for κ may then be found as in the usual form of experiment, or better, by means of the method described in the appendix.

***Experiment.*—To determine the moment of inertia of an irregular solid about a given axis.** It will be assumed that, for the purpose of this experiment, there is provided a piece of lead or brass of uniform thickness and in the form of a quadrilateral. Also, that it is required to find its moment of inertia about that axis through its mass-centre which is normal to its plane. Four holes are drilled near the corners of the metal sheet and it is suspended on a horizontal knife-edge in turn from each of these holes so that its plane is vertical, and that it swings freely under gravity. If T is the time of swing, then

$$T = 2\pi\sqrt{\frac{I}{mgr}},$$

where I is the moment of inertia of the sheet about a horizontal axis through the point of suspension. If m is the mass of the metal, r the distance of the point of support from the mass-centre, κ the radius of gyration about a horizontal axis through the mass-centre, then

$$T = 2\pi\sqrt{\frac{\kappa^2 + r^2}{gr}}, \qquad \text{since } I = m(\kappa^2 + r^2).$$

Hence κ may be determined, and from it the required moment of inertia.

To measure the different times of swing accurately we shall make use of the method of coincidences. Approximate values for the times of 50 complete swings when the lamina is suspended in turn from each of the four holes are first obtained. A simple pendulum is then constructed so that its period shall be about the mean period of the four periodic times determined above. The period of this pendulum is then determined accurately so that it becomes a standard of time, known in seconds.

The simple pendulum is then placed in front of the metal sheet and arranged so that the upper fixed end is level with the knife-edge as shown in Fig. 4·16. A slit in a piece of cardboard, mounted in front of the two pendulums, has stretched across it a piece of black cotton. This is arranged to be coplanar with the string of the simple pendulum when at rest, and a vertical line on the metal passing through the point of support. The system should be viewed from a distance of two metres or more, or a telescope used, to avoid parallax errors. The two pendulums are set swinging together, and it will be found convenient to make the amplitudes approximately equal. If both pendulums have the same period and commence to oscillate together, they will appear to move as one when viewed as above. If the periods are not identical, the pendulums soon get out of step. If this motion is examined carefully it will be noticed that at some instant the line on the metal, the string of the simple pendulum and the vertical reference line are coplanar, and that the two objects are moving in the same direction. This event does not occur again until one pendulum has lost or gained a whole swing with respect to the other.

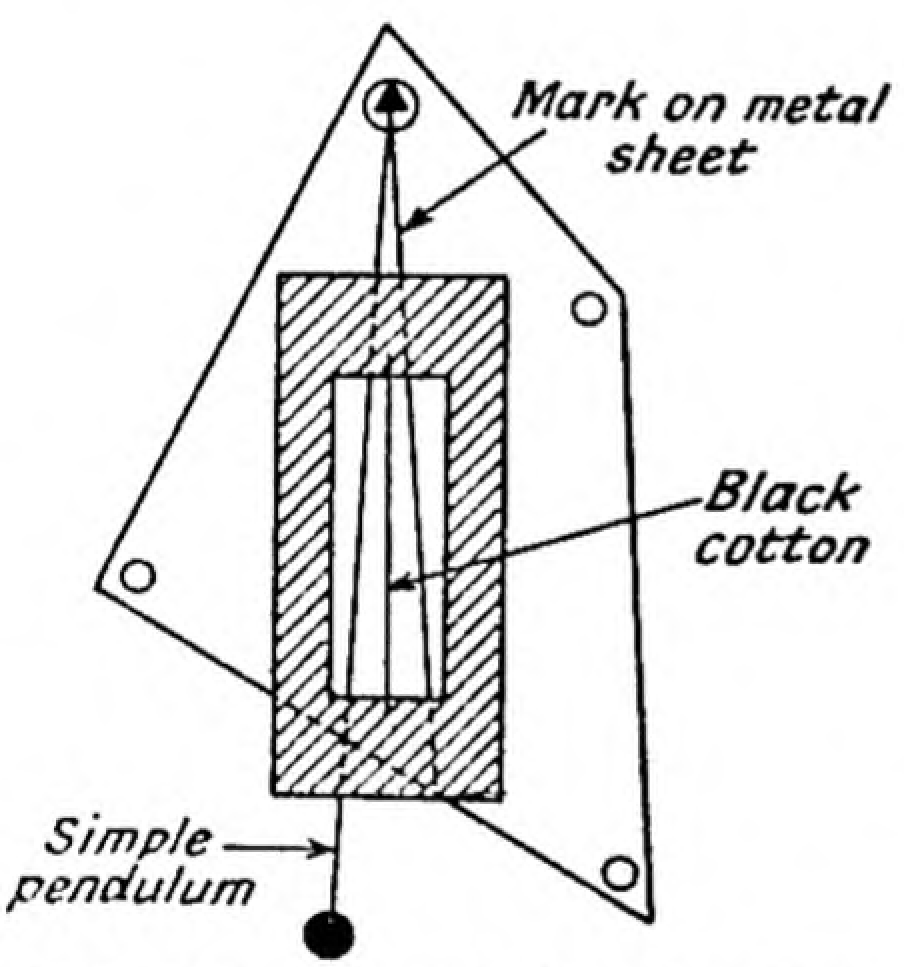


FIG. 4·16.—Moment of inertia of an irregular lamina—experimental determination.

Suppose the simple pendulum of period T_1 makes n_1 complete swings during the interval of time in which the metal sheet makes $(n_1 \pm 1)$ complete swings. If T is the period of the metal sheet

$$n_1 T_1 = (n_1 \pm 1)T,$$

so that T may be calculated. The correct sign to be used is ascertained by noting which pendulum gains on the other.

The above is repeated at each hole and then the radius of gyration for the metal sheet about a horizontal axis through its mass-centre and normal to its plane is calculated.

Example.—Determine when the time period of a component pendulum is a minimum.

The position of the axis of rotation of a compound pendulum of mass M is adjusted so as to give the time period of the pendulum its minimum value T_0. A piece of metal of mass m but of negligible size is then attached at the centre of gravity of the pendulum. Derive an expression for (*a*) the new time period for the same position of the axis, (*b*) the new minimum value of the time period.

Before loading we have, with the usual notation,

$$\frac{g}{4\pi^2} rT^2 = r^2 + \kappa^2.$$

Differentiating with respect to r, we have

$$\frac{g}{4\pi^2}\left[T^2 + 2rT\frac{dT}{dr}\right] = 2r.$$

$$\therefore\ 2rT\frac{dT}{dr} = \frac{8\pi^2}{g}r - T^2 = \frac{8\pi^2}{g}r - 4\pi^2\left(\frac{r^2+\kappa^2}{rg}\right).$$

Hence $\dfrac{dT}{dr} = 0$ when $2r^2 - (r^2 + \kappa^2) = 0$, i.e. $r = \kappa$.

To show that the period is then a minimum, we have

$$\frac{dT}{dr} = \frac{4\pi^2}{g}\frac{1}{T} - \frac{T}{2r}.$$

$$\therefore\ \frac{d^2T}{dr^2} = \frac{4\pi^2}{g}\left(-\frac{1}{T^2}\right)\frac{dT}{dr} - \frac{1}{2}\left[\frac{1}{r}\frac{dT}{dr} - \frac{T}{r^2}\right].$$

and when $\dfrac{dT}{dr} = 0$, this expression is $\dfrac{1}{2}\dfrac{T}{r^2}$, an essentially positive quantity. Hence the period is a minimum and we have

$$T_0 = 2\pi\sqrt{\frac{2\kappa}{g}}.$$

The added mass has no effect on I_G, the moment of inertia about a horizontal axis through G.

$$\therefore\ I_G = M\kappa^2 = (M + m)\kappa_1^2,$$

where κ_1 is the new radius of gyration.

(*a*). $\therefore$ New time period about the same axis $= 2\pi\sqrt{\dfrac{\kappa_1^2 + \kappa^2}{\kappa g}}$

$[\because r = \kappa]$

$$= T_0\sqrt{\frac{2M + m}{2(M + m)}}.$$

(*b*). New minimum period $= 2\pi\sqrt{\dfrac{2\kappa_1}{g}} = T_0\left(\dfrac{M}{M + m}\right)^{\frac{1}{4}}.$

The compound pendulum; reaction on a fixed axis.—Let a compound pendulum be oscillating in a plane about a fixed horizontal axis through O, as in Fig. 4·11 [cf. p. 123], and let OG, where G is the centre of gravity, make an angle ψ with the vertical. Let G be at a fixed distance r from O. Let R and Ψ' be the components of the reaction of the axle on the pendulum in directions along and at right angles to GO—we assume that there is no frictional couple.

Now the motion of G is the same as if the whole mass of the pendulum were concentrated at G and if all the external forces acted at G, the lines of action of the forces being parallel to those along which the forces actually act.

Now the radial and transverse components of the acceleration of G are, in general, $\ddot{r} - r\dot{\psi}^2$ and $r\ddot{\psi} + 2\dot{r}\dot{\psi}$ respectively. In this instance, when r is constant, they become $-r\dot{\psi}^2$ and $r\ddot{\psi}$. Hence resolving forces along and perpendicular to OG, we have

$$m(-r\dot{\psi}^2) = mg\cos\psi - \mathrm{R} \quad . \quad . \quad . \quad \text{(i)}$$

$$m(r\ddot{\psi}) = \Psi' - mg\sin\psi \quad . \quad . \quad . \quad \text{(ii)}$$

Also [cf. p. 123],

$$\ddot{\psi} = -\frac{gr}{\kappa^2 + r^2}\sin\psi, \quad . \quad . \quad . \quad \text{(iii)}$$

where κ is the radius of gyration about an axis through G parallel to the axis of suspension.

From (ii) and (iii)

$$\Psi' = mg\left[1 - \frac{r^2}{\kappa^2 + r^2}\right]\sin\psi = mg.\frac{\kappa^2}{\kappa^2 + r^2}.\sin\psi \quad . \quad \text{(iv)}$$

To determine R it is necessary to know $\dot{\psi}$. This is obtained from the energy equation which, in this instance, is

$$\tfrac{1}{2}m(r^2 + \kappa^2)\dot{\psi}^2 + mgr(1 - \cos\psi) = \mathrm{A},$$

where A is a constant. To determine A we use the fact that if the pendulum has an amplitude ψ_0, so that $\dot{\psi} = 0$ when $\psi = \psi_0$, then

$$\mathrm{A} = mgr(1 - \cos\psi_0),$$

r being a constant under the conditions contemplated.

$$\therefore\ \dot{\psi}^2 = \frac{2gr(\cos\psi - \cos\psi_0)}{r^2 + \kappa^2}.$$

$$\therefore\ \mathrm{R} = mg\left[\cos\psi + \frac{2r^2}{(r^2 + \kappa^2)}(\cos\psi - \cos\psi_0)\right].$$

Centre of percussion.—When a single impulse applied to a rigid body, which is free to turn about a fixed axis, produces no impulsive

reaction on this axis, the line of the impulse is called the ***line of percussion*** and the point in which this line meets the plane through the centre of gravity and the axis of rotation is called the ***centre of percussion.***

As a simple case consider a thin uniform rod OA, Fig. 4·17, suspended freely from one end and struck by a horizontal blow at a point Q. Let P be the impulse of the blow and call OQ $= x$. Let $2a =$ OA and let ω_0 be the instantaneous angular velocity acquired by the rod. Let X be the impulsive reaction upon the rod of the axis about which it rotates.

Now G begins to move with linear velocity $a\omega_0$ and the momentum of the system equals the momentum of the whole mass m supposed collected at G.

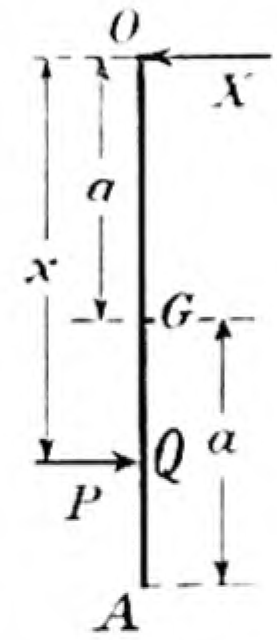


FIG. 4·17.—centre of percussion.

$$\therefore \text{ Resultant impulse} = \text{P} - \text{X}$$
$$= \text{change in linear momentum}$$
$$= ma\omega_0.$$

Also the moment of momentum of the rod about O immediately after it has received the blow is given by

$$\text{P}x = (m\kappa^2)\omega_0, \qquad \text{where } \kappa^2 = \tfrac{4}{3}a^2.$$

$$\therefore \text{ X} = \text{P} - ma\omega_0 = \text{P}\left(1 - \frac{a}{\kappa^2}x\right).$$

Hence X is zero, i.e. the impulsive reaction at O is zero if $x = \dfrac{\kappa^2}{a} = \dfrac{4}{3}a$ and then OQ is the length of the equivalent simple pendulum, i.e. the impulse at O is zero if the blow is delivered at the centre of oscillation, i.e. the centre of percussion with regard to the fixed axis of rotation coincides with the corresponding centre of oscillation. [The jar which is sometimes felt in the handle of a bat does not occur if the ball strikes the bat at the centre of percussion.]

Experimental determination of the centre of percussion of a bar pendulum.—From the remarks made in the previous paragraph it follows that if, when a bar pendulum is suitably suspended, the centre of percussion can be determined experimentally, then a value for the length of the simple equivalent pendulum may be found. An experimental arrangement,† designed by C. A. Haywood for this purpose, is shown in Fig. 4·18(a). It enables one to apply the same impulse to different points on one side of a compound pendulum and to measure the reaction at the support. The condition for no reaction is found by interpolation. The support bracket, A, consists of a piece of hard aluminium sheet, 0·13 cm. thick, 9 cm. wide and 32 cm. high; an extra length is bent at right angles so that the bracket thus formed may be screwed to a piece of wood, H, which, when clamped to a bench top,

† Private communication.

provides the necessary overhang. The knife-edge B, on which the pendulum swings, consists of two pieces of mild steel fastened together, one on either side of A. One of these extends further than the other and is 'sharpened' to provide the actual knife-edge on which the pendulum may swing. A fairly stiff bristle, not too long, is attached by soft wax to the end of B remote from the pendulum and presses lightly on the smoked surface of a glass plate, G, held in a retort clamp.

The pendulum C is a bar of aluminium with a square cross-section of about 4 cm.^{2}; it is 60 cm. long and a hole, 2·5 cm. from its upper end is

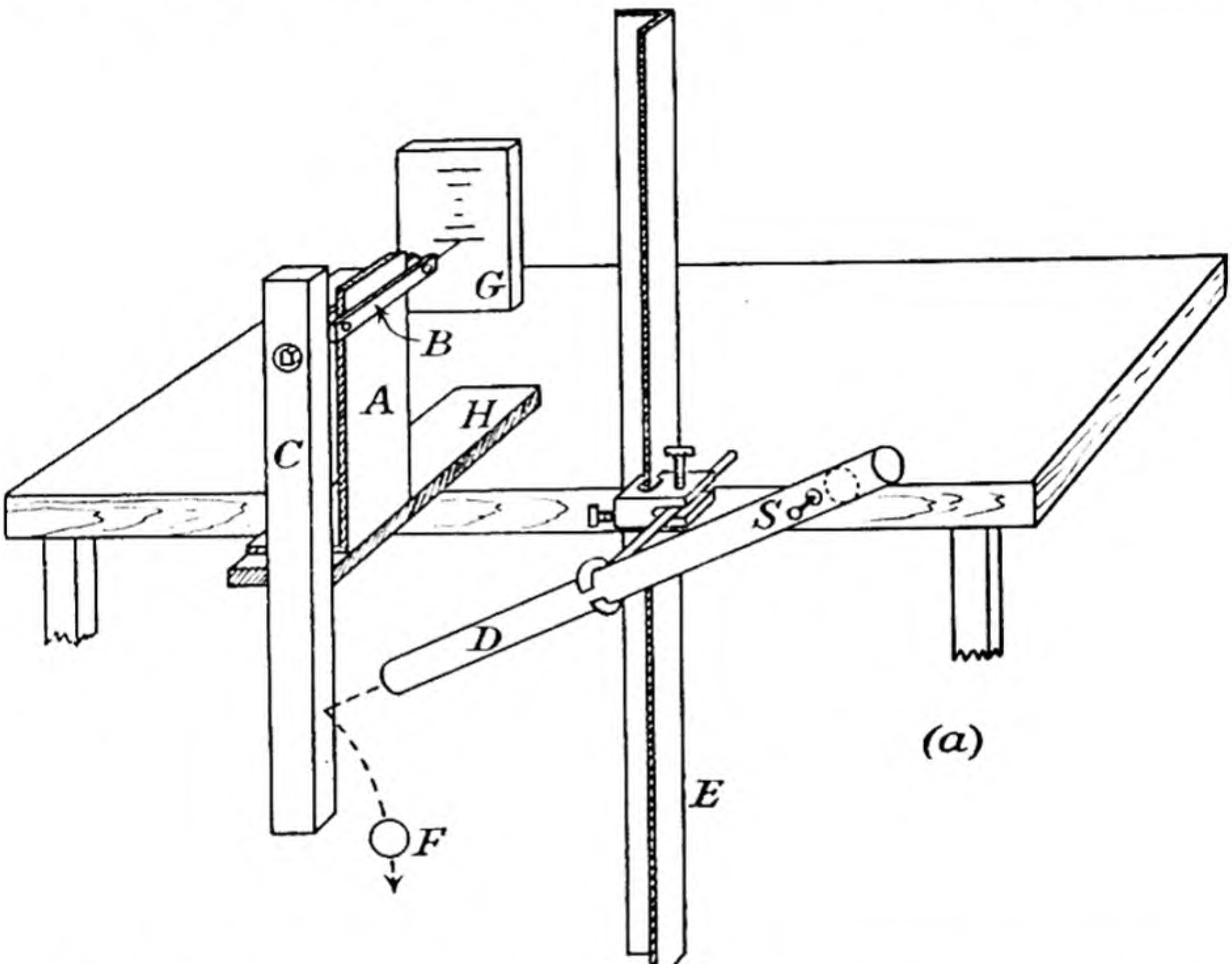


FIG. 4·18(*a*).—Experimental determination of the centre of percussion of a bar pendulum.

drilled to take the knife-edge. Similar holes may be provided so that the experiment may be repeated for different values of the natural period. The knife-edge may be lowered nearly to the base of A when the natural period is determined; this is only required if a value for gravity is contemplated.

The impulse is applied by rolling a 2 cm. diameter steel ball F down a brass tube D, 2·2–2·3 cm. in internal diameter and 45–50 cm. long. This tube is held in a retort clamp in a box-type retort boss which is clamped to a length of angle iron E with a web of 2·5 cm. Thus the tube D may be raised or lowered while still remaining with its axis at the same angle, about 35°, to the horizontal. In this way a constant impulse may be applied at different points along one side of the pendulum. The 'impact-face' of C has a strip of white paper gummed along it and a piece of carbon paper is also attached on top of the gummed strip. When the experiment is complete the carbon paper is removed and the series of dots on the white paper indicates where

impacts have occurred. The plate G is also moved between impacts so that a series of short scratch marks on it may be obtained.

The direction of the initial reaction should be observed so that it may be plotted algebraically against the distance of the centre of impact from the knife-edge; the centre of percussion is given by the intersection of the graph with the x-axis, cf. Fig. 4·18(b). Since the damping is constant for the initial reaction on the support, the lengths of the lines drawn upon G are proportional to this reaction.

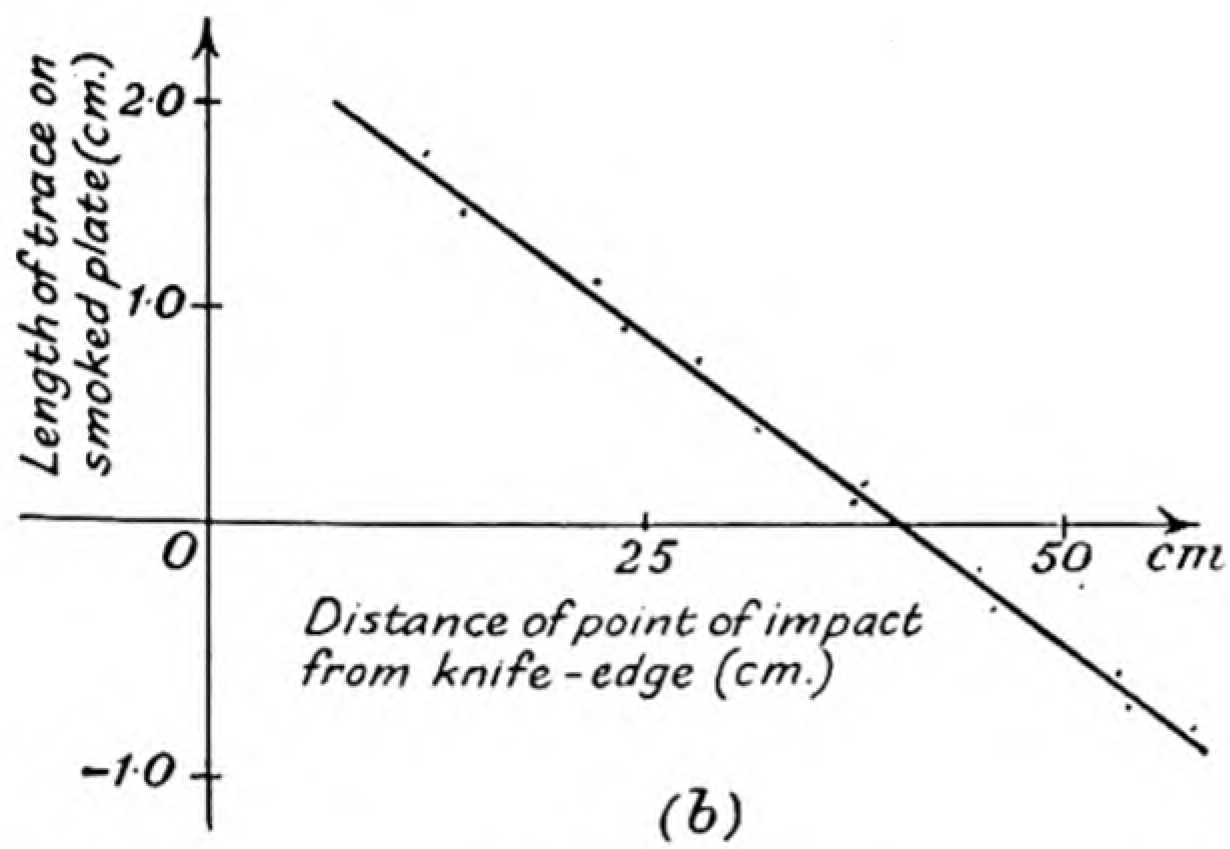


FIG. 4.18(b).

Finally, it must be pointed out that the frequency of oscillation of the knife-edge is very much greater than that of the pendulum but as the centre of percussion is approached the pendulum builds up a larger amplitude due to the impact. This is 'coupled' eventually to the knife-edge and a slow swing, with large amplitude, results; it is much longer than the initial reaction 'judder'. Consequently, under these circumstances and preferably throughout the experiment, the plate G must be removed from contact with the bristle before the slow oscillation occurs.

[Instead of using a bristle, a concave mirror (or etc.) may be attached to the plate A and the reaction measured by the first deflexion of the spot of light on the scale arranged in the normal manner, cf. p. 379.]

The oscillations of a uniform bar rolling on a horizontal cylinder.—In Fig. 4·19(a), let PQ be the central cross-section of a uniform rectangular bar resting on a cylinder whose centre is C. Suppose A is the point on the bar in contact with the cylinder at O when the bar is not oscillating. Let G be the centre of gravity of the bar, while $2a$ and $2b$ are its length and depth respectively. We shall further assume that the axis of the bar is in a vertical plane normal to the axis of the cylinder. When the bar is displaced so that B is the point of contact between the cylinder and the bar, let $\widehat{OCB} = \psi$. Through G draw a vertical line to meet a horizontal line through B in L; this is shown in greater detail in Fig. 4·19(b), where G_1 is the 'rest-position' of G and G_2 a 'displaced-position'.

Then the moment of the external forces acting on the bar and tending to restore it to its position of equilibrium is

$$\begin{aligned} mg.\mathrm{LB} &= mg[\mathrm{AB}\cos\psi - \mathrm{AG}\sin\psi] \\ &= mg[r\psi\cos\psi - b\sin\psi] \\ &= mg(r-b)\psi, \end{aligned}$$

if ψ is so small that ψ^2 and all higher powers of ψ may be neglected

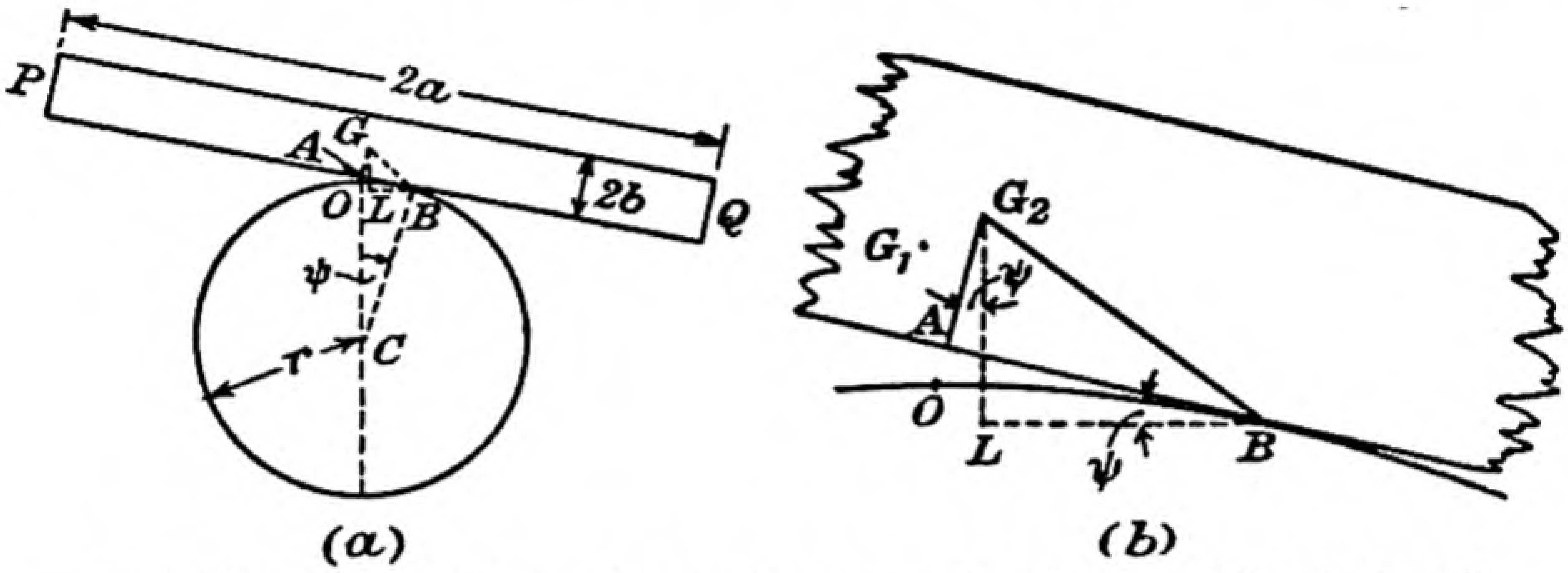


FIG. 4·19.—Oscillations of a uniform bar on a horizontal cylindrical surface.

in comparison with ψ. The motion of the bar is expressed by the equation

$$\mathrm{I}\ddot{\psi} + mg(r-b)\psi = 0,$$

where I is the moment of inertia of the bar about an axis through B, parallel to that of the cylinder. But $\mathrm{I} = m(\kappa^2 + \mathrm{GB}^2)$, where κ is the radius of gyration for the bar about a horizontal axis through G normal to the plane of the diagram. Also

$$\begin{aligned} \mathrm{GB}^2 &= \mathrm{GA}^2 + \mathrm{AB}^2 \\ &= b^2 + r^2\psi^2. \end{aligned}$$

If $r^2\psi^2$ is negligible in comparison with b^2, the equation for the motion becomes

$$\ddot{\psi} + g\left(\frac{r-b}{\kappa^2+b^2}\right)\psi = 0.$$

The motion is therefore simple harmonic, the period T being given by

$$\mathrm{T} = 2\pi\sqrt{\frac{\kappa^2+b^2}{g(r-b)}}.$$

But $\kappa^2 = \frac{1}{3}(a^2+b^2)$, so that

$$\mathrm{T} = 2\pi\sqrt{\frac{a^2+4b^2}{3g(r-b)}}.$$

Experiment.—We may make use of the oscillations of a bar on a cylinder as investigated above to find a value for the intensity of gravity. For this purpose the bar should be about a metre long, 2 cm.

wide, and 1 cm. deep. The cylinder may consist of a wheel about 20 cm. in diameter. To obtain a good result the surfaces of the wheel and bar where they come into contact with one another should be very clean. In order that observations may be made on the occurrence of some definite event a horizontal white line should be drawn on the end of the bar and the time measured for this to move an integral number of times in a chosen direction past a piece of black cotton placed in a horizontal position close to it.

Now an examination of the equation for the periodic time shows that it is necessary to know the distance of the centre of gravity of the bar from its lower side. This is not easily determined. Suppose, however, that observations are made to determine the period when first one face and then the opposite face of the bar rests on the cylinder. If b_1 and b_2 be the distances of G from these faces and T_1 and T_2 the corresponding times of oscillation, then

$$r - b_1 = \frac{4\pi^2}{3gT_1^2}(a^2 + 4b_1^2)$$

$$= \frac{4\pi^2}{3gT_1^2}.a^2, \qquad \text{if } \frac{2b_1}{a} \text{ is small.}$$

Similarly
$$r - b_2 = \frac{4\pi^2}{3gT_2^2}.a^2.$$

$$\therefore\ g = \frac{2\pi^2a^2}{3(r-b)}\left(\frac{1}{T_1^2} + \frac{1}{T_2^2}\right).$$

A method for determining the centre of gravity of a body.—It will be assumed that the body is in a uniform gravitational field, i.e. the lines of action of the weights of all its constituent particles are parallel. In practice, this means that the following analysis will only be applicable to bodies which are not very large. Let A, Fig. 4·20, be an irregular plane lamina situated in the plane xOy. Suppose that the gravitational field acts vertically downwards. Consider a small element of area δA, surrounding P, the point (x, y). If σ is the mass per unit area of the lamina, the moment of the weight of the above element about Oy is $\sigma g.\delta A.x$, where g is the intensity of gravity. For all such particles the total moment of their weights about Oy is $\Sigma\, \sigma gx.\delta A$. Now the centre of gravity G, $(\bar{x}, \bar{y})$, will be such that if the whole weight of the lamina acted at G, the moment about Oy of this weight would be $\Sigma\, \sigma gx.\delta A$. Hence

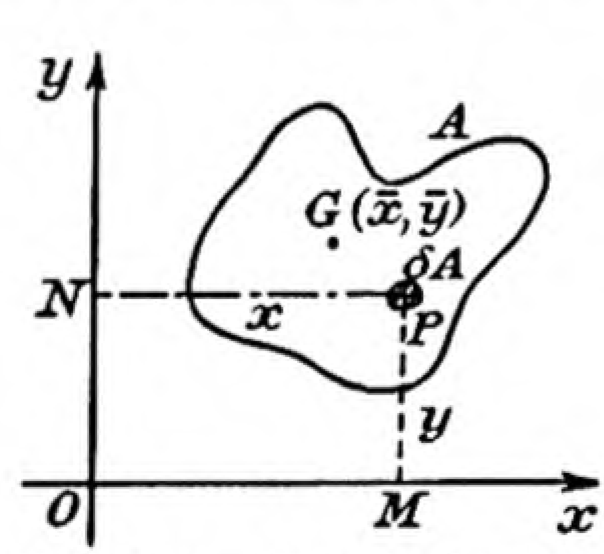


FIG. 4·20.—Centre of gravity of a body in a uniform gravitational field.

$$(\Sigma\, \sigma g.\delta A)\bar{x} = \Sigma\, \sigma gx.\delta A$$

or
$$\bar{x} = \frac{\Sigma\, x.\delta A}{\Sigma\, \delta A} = \frac{\Sigma\, x.\delta A}{A}.$$

Similarly,
$$\bar{y} = \frac{\Sigma\, y\,.\,\delta A}{\Sigma\, \delta A} = \frac{\Sigma\, y\,.\,\delta A}{A},$$

and the method may at once be extended to the case of a solid body.

If the distribution of matter in a body is continuous we may use the notation of the calculus and write

$$\bar{x} = \frac{\int\!\!\int x\,.\,dx\,.\,dy}{\int\!\!\int dx\,.\,dy} = \frac{\int\!\!\int x\,.\,dx\,.\,dy}{A},$$

and
$$\bar{y} = \frac{\int\!\!\int y\,.\,dx\,.\,dy}{\int\!\!\int dx\,.\,dy} = \frac{\int\!\!\int y\,.\,dx\,.\,dy}{A},$$

where $\delta x\,.\,\delta y$ is the area of the small element at P.

In the problems we have to deal with in physics, it will be found that, in general, there is an axis of symmetry, so that the evaluation of the above integrals is much simplified. The method will be illustrated by the following problems.

To determine the position of the centre of gravity of a uniform circular arc.—Let AOB, Fig. 4·21, be the arc of radius a,

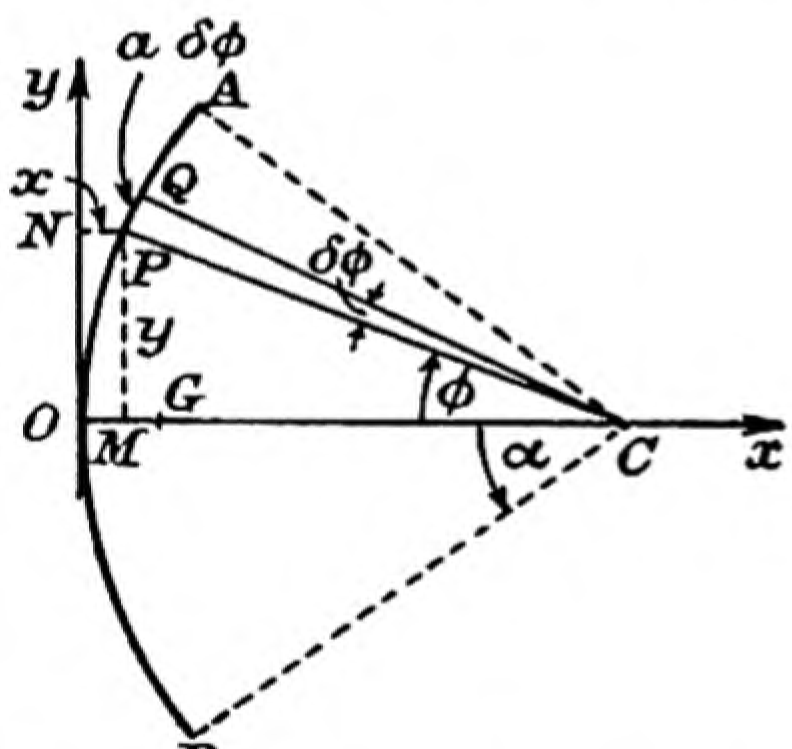


Fig. 4·21.—Centre of gravity of a uniform circular arc.

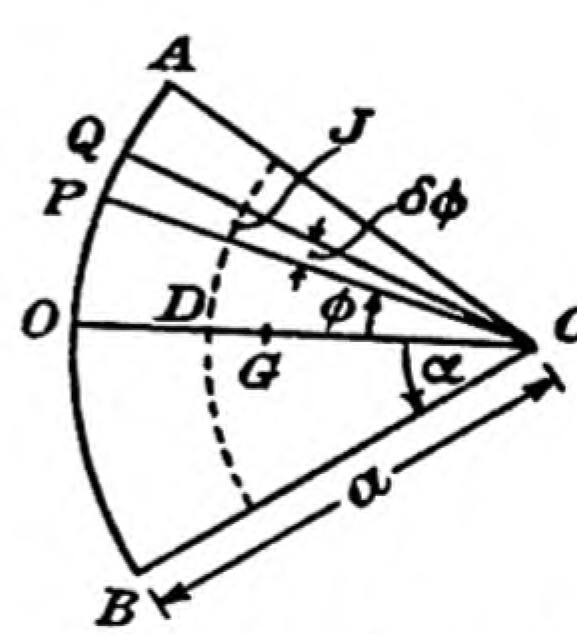


Fig. 4·22.—Centre of gravity of a uniform circular sector.

of mass μ per unit length and let it subtend an angle 2α at its centre of curvature C. Take axes Ox, Oy as shown. Consider a small element PQ, of mass μ PQ, its position being (x, y). Let ϕ be $\widehat{\text{OCP}}$. Then $\widehat{\text{PCQ}} = \delta\phi$, and $\text{PQ} = a\,\delta\phi$.

Now
$$x = \text{NP} = \text{OC} - \text{MC},$$

where M and N are the projections of P on the coordinate axes. Hence

$$x = a(1 - \cos\phi).$$

$\therefore$ the moment of the weight of this element about Oy is

$$(\mu g a . \delta\phi)a(1 - \cos\phi).$$

$$\therefore\ \bar{x} = \frac{\int_{-\alpha}^{\alpha} \mu g a^2(1 - \cos\phi)\,d\phi}{\mu g \int_{-\alpha}^{\alpha} a\,d\phi}$$

$$= \left[2a\int_0^{\alpha}(1 - \cos\phi)\,d\phi\right] \div 2\int_0^{\alpha} d\phi$$

$$= a\left[\frac{\alpha - \sin\alpha}{\alpha}\right],$$

and since the arc is symmetrical with respect to Ox, $\bar{y} = 0$.

[N.B. It is more usual to determine the position of the centre of gravity of an arc with respect to C, but in the problems which follow it is the position of the centre of gravity with respect to O which will be required; it has been determined directly by the above analysis.]

To determine the position of the centre of gravity of a uniform circular sector.—Let AOBC, Fig. 4·22, be such a sector of radius a, of mass σ per unit area, and let it subtend an angle 2α at C, its centre of curvature. Let P and Q be adjacent points on the circumference of the arc, such that $\widehat{OCP} = \phi$ and $\widehat{PCQ} = \delta\phi$. Then the area of this triangular element is $\frac{1}{2}PQ.QC = \frac{1}{2}a^2\,\delta\phi$. Its centre of gravity will be at J, where $CJ = \frac{2}{3}a$. The sector under consideration may therefore be replaced by a series of particles each of mass $\frac{1}{2}a^2\sigma.\delta\phi$ uniformly distributed along an arc of radius $\frac{2}{3}a$. If this arc cuts OC in D, we have

$$DG = \tfrac{2}{3}a\left[\frac{\alpha - \sin\alpha}{\alpha}\right] = \tfrac{2}{3}a\left[1 - \frac{\sin\alpha}{\alpha}\right].$$

$$\therefore\ OG = OD + DG = \tfrac{1}{3}a + \tfrac{2}{3}a - \tfrac{2}{3}a.\frac{\sin\alpha}{\alpha}$$

$$= a\left[1 - \tfrac{2}{3}.\frac{\sin\alpha}{\alpha}\right].$$

To determine the position of the centre of gravity of a uniform circular segment.—Let AOB in Fig. 4·23 be a segment of a circular disc cut off by a chord AB. Take axes Ox, Oy as shown, and let C be the centre of the circle of which AOB is an arc. Let P and Q be points on the periphery of the segment such that $\widehat{OCP} = \phi$ and $\widehat{PCQ} = \delta\phi$. Through P and Q draw straight lines perpendicular to Ox to cut this axis in M and N and the arc OB in P_1 and Q_1, respectively. Let 2α be the angle $\widehat{ACB}$. Consider the element PQQ_1P_1. If (x, y) are the coordinates of P, the height of this strip is $2y$, and its width

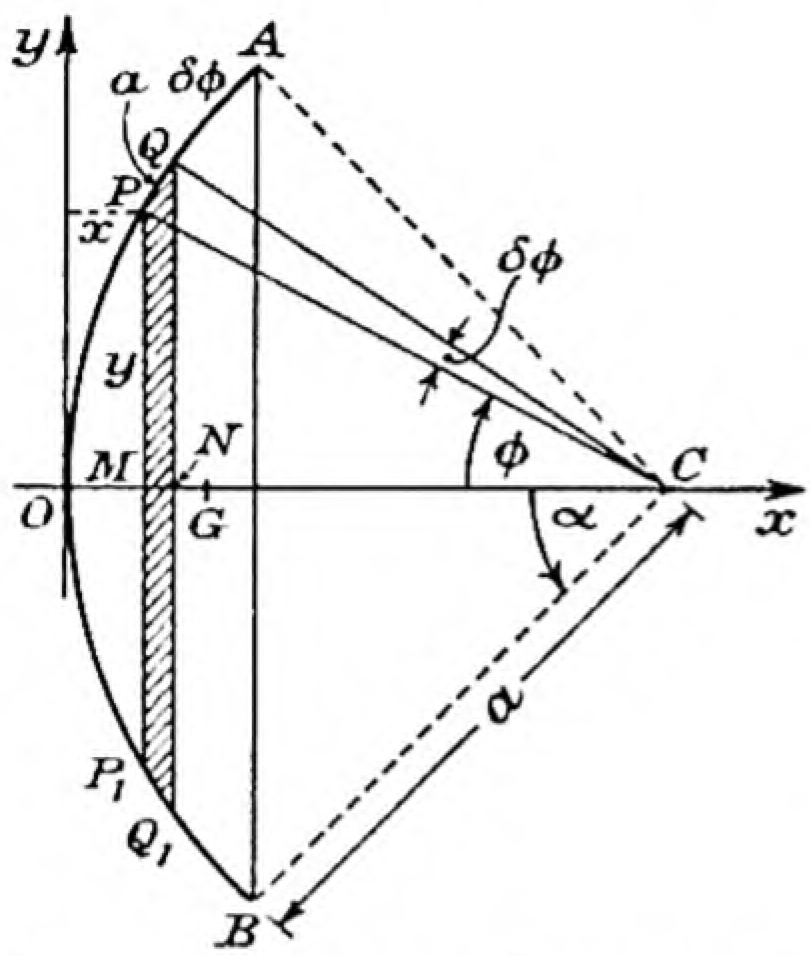


FIG. 4·23.—Centre of gravity of a uniform circular segment.

$$MN = PQ \cos\left(\frac{\pi}{2} - \phi\right)$$

$$= a \sin\phi . \delta\phi.$$

Now $y = a \sin\phi$, and the shortest distance of every point in the strip PQQ_1P_1 from the y-axis is equal to $OM = a(1 - \cos\phi)$. The moment of the weight of this strip about OY, if g is the intensity of gravity acting normally to the plane of the diagram, is

$$2a \sin\phi . a \sin\phi . \delta\phi . \sigma g a(1 - \cos\phi), \qquad [\sigma = \text{mass per unit area}]$$
$$= 2\sigma g a^3 \sin^2\phi(1 - \cos\phi)\,\delta\phi.$$

The area of the segment is $\frac{1}{2}a^2(2\alpha - \sin 2\alpha)$. The position of the centre of gravity is therefore given by

$$\bar{x} = \frac{2\int_0^\alpha a^3(\sin^2\phi - \sin^2\phi\cos\phi)\,d\phi}{\frac{1}{2}a^2(2\alpha - \sin 2\alpha)}$$

$$= \frac{4a\left[\int_0^\alpha \sin^2\phi\,d\phi - \int_0^{\sin\alpha} \sin^2\phi\,d(\sin\phi)\right]}{2\alpha - \sin 2\alpha}$$

$$= \frac{2a}{\alpha - \sin\alpha\cos\alpha}\left[\frac{\alpha - \frac{1}{2}\sin 2\alpha}{2} - \frac{\sin^3\alpha}{3}\right]$$

$$= a - \tfrac{2}{3}a\left[\frac{\sin^3\alpha}{\alpha - \sin\alpha\cos\alpha}\right],$$

and $\bar{y} = 0$, since the segment is symmetrical about Ox.

We are now in a position to continue our study of the periods of oscillation of certain compound pendulums.

Small oscillations of a uniform circular arc about a horizontal axis through its vertex.—In Fig. 4·24(a), let AOB be a uniform circular arc of mass μ per unit length, subtending an angle 2α at C, its centre of curvature, and oscillating under the influence of gravity about a horizontal axis through its vertex O. We require an expression for I, the moment of inertia of this arc about the axis of suspension. Consider the small element PQ of length $a.\delta\phi$, where $\delta\phi$ is the $\widehat{PCQ}$, ϕ being the $\widehat{OCP}$. The mass of this element

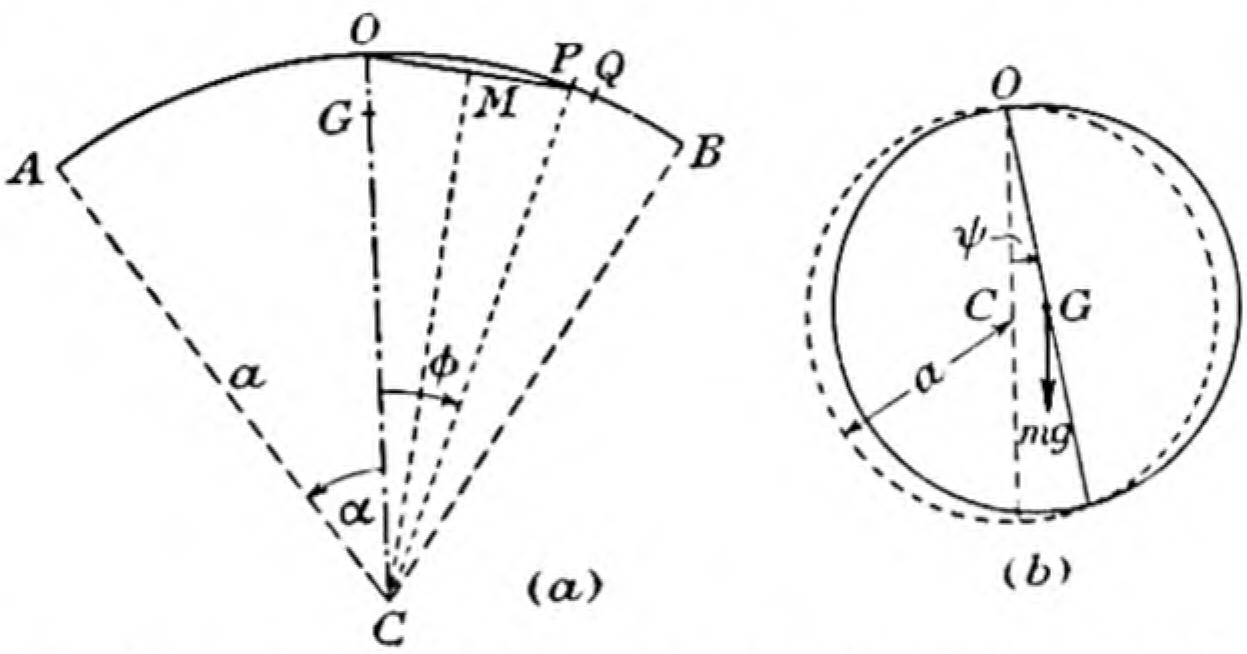


FIG. 4·24.—Small oscillations under gravity of (a) a uniform circular arc and (b) a uniform circular hoop, about a horizontal axis through the vertex.

is $\mu a.\delta\phi$, and its moment of inertia about the axis of suspension is $(\mu a.\delta\phi.)\text{OP}^2 = \mu a.\delta\phi(2\text{OM})^2$, where M is the midpoint of OP. But $\text{OM} = a \sin \frac{1}{2}\phi$. Hence, the moment of inertia of the complete arc about the axis of suspension is given by

$$\begin{aligned} \text{I} &= 2\int_0^\alpha \mu a\, \text{OP}^2\, d\phi = 8\mu a^3 \int_0^\alpha \sin^2 \frac{\phi}{2}\, d\phi \\ &= 4\mu a^3 \int_0^\alpha (1 - \cos \phi)\, d\phi \\ &= 4\mu a^3[\alpha - \sin \alpha]. \end{aligned}$$

The motion of the pendulum, for small angular displacements, is therefore expressed by

$$\text{I}\ddot{\psi} + mgh\psi = 0,$$

where ψ is the angular displacement of the pendulum at a given instant, h the distance of the centre of gravity from O, and $m = 2\mu a\alpha$ is the mass of the pendulum. This may be written

$$4\mu a^3[\alpha - \sin \alpha]\ddot{\psi} + 2\mu g a\alpha\left[a\left(\frac{\alpha - \sin \alpha}{\alpha}\right)\right]\psi = 0.$$

$$\therefore\ \ddot{\psi} + \frac{g}{2a}\,\psi = 0.$$

The motion is therefore simple harmonic with a period the same as that of a simple pendulum whose length is equal to the diameter of the circle of which the arc forms a part, i.e.

$$T = 2\pi\sqrt{\frac{2a}{g}}.$$

We notice that this time is independent of the length of the arc, so that the period of a complete circular hoop oscillating under gravity in a vertical plane about a horizontal axis through a point on its circumference will be given by the same equation. This fact may be verified as follows.

Let O, Fig. 4·24(b), be the point of suspension of a circle of radius a and mass μ per unit length. Its total mass is therefore $2\pi a\mu$, and its moment of inertia about an axis through its centre and normal to its plane is $(2\pi a \,.\, \mu)a^2 = 2\pi\mu a^3$. Then I, its moment of inertia about its axis of suspension, is $2\pi a\mu(a^2 + a^2)$, since a is the distance between the axis of suspension and the axis through C about which the moment of inertia has been calculated. If ψ is the small angular displacement of the pendulum at a given instant, its motion is determined by

$$I\ddot{\psi} + mga\psi = 0,$$

i.e.

$$4\pi\mu a^3\ddot{\psi} + 2\pi\mu g a^2\psi = 0.$$

$$\therefore\ T = 2\pi\sqrt{\frac{2a}{g}},$$

as before.

The oscillations of a physical balance.—For simplicity it will be assumed that the balance has equal arms of length a and all three

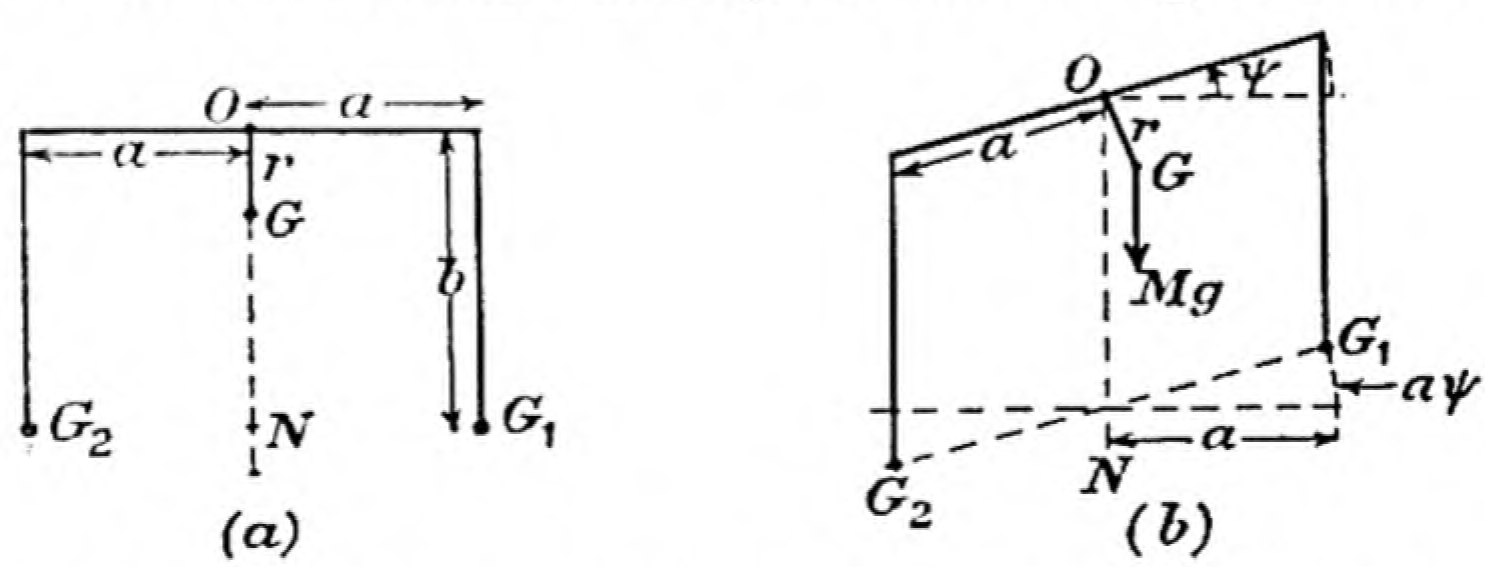


Fig. 4·25.—Oscillations of a physical balance when it is loaded.

knife-edges are coplanar. Let M be the mass of the beam and its immediate attachments and suppose that G, the centre of gravity of this system, cf. Fig. 4·25(a), lies at a distance r below the fulcrum O. Let G_1 and G_2 be the centres of gravity of the scale pans; these points are at a vertical distance b below O when the balance

is in the position of static equilibrium. When the beam is swinging freely, let ψ be an instantaneous small angular displacement of the beam from its equilibrium position; cf. Fig. 4·25(b). Since the beam and pans do not form a rigid system we cannot equate the rate of change of angular momentum to the restoring couple in order to obtain a value for the period but proceed as follows by considering the total energy of the system.

If κ is the radius of gyration of the beam about the horizontal axis through G and normal to the plane of the diagram, the kinetic energy of the beam is $\frac{1}{2}M(r^2 + \kappa^2)\dot{\psi}^2$. Now the displacement of the centre of gravity of each pan is $a\psi$ and is independent of b. When this displacement is changing the velocity of each pan is $a\dot{\psi}$, i.e. its kinetic energy is $\frac{1}{2}\mu a^2\dot{\psi}^2$, where μ is the mass of the pan. If the position of static equilibrium of the beam is taken as the position of zero potential energy, then when the tilt of the beam is ψ, the potential energy of the beam will be $Mgr(1 - \cos\psi) \simeq \frac{1}{2}Mgr\psi^2$. Since the centre of gravity of one pan rises by an amount equal to that through which the second pan falls, the total change in potential energy associated with the pans is zero. Hence the total energy of the system is

$$\tfrac{1}{2}M(r^2 + \kappa^2)\dot{\psi}^2 + \tfrac{1}{2}Mgr\psi^2 + 2\{\tfrac{1}{2}\mu a^2\dot{\psi}^2\},$$

and this is constant. Differentiating with respect to time we obtain

$$\{M(r^2 + \kappa^2) + 2\mu a^2\}\ddot{\psi} + Mgr\psi = 0,$$

i.e. the motion is simple harmonic and the periodic time is given by

$$T_0 = 2\pi\sqrt{\frac{M(r^2 + \kappa^2) + 2\mu a^2}{Mgr}}.$$

Now when each pan carries a concentrated load of mass m, the position of the centre of gravity of each pan and its load will vary but since b does not enter into the expression for the period, this will be given by

$$T = 2\pi\sqrt{\frac{M(r^2 + \kappa^2) + 2(\mu + m)a^2}{Mgr}},$$

when each pan is equally loaded.

To verify these conclusions the period of a balance was measured. Then from the supports for the pan two equal 100 gm. masses were suspended by short threads; the centres of the masses were at a distance b_1 below the knife edges. Finally the masses and threads were placed in the balance pans at a distance b_2 below the knife edges. Let T_1 and T_2 be the periods. With

$2a = (13{\cdot}1 \pm 0{\cdot}1)$ cm., $\quad T_0 = (9{\cdot}50 + 0{\cdot}06)$ sec.

$b_1 = (12{\cdot}5 \pm 0{\cdot}2)$ cm., $\quad T_1 = (13{\cdot}15 + 0{\cdot}06)$ sec.

$b_2 = (24{\cdot}2 \pm 0{\cdot}2)$ cm., $\quad T_2 = (13{\cdot}08 + 0{\cdot}06)$ sec.

A comparison of the values of T_1 and T_2 shows that the period is independent of the quantity we have called b.

The bifilar suspension.—This consists of a heavy uniform rod or bar suspended by two cords or threads of equal length and symmetrically arranged so that the rod may make small oscillations in a horizontal plane under gravity. We shall consider the two cases which may arise.

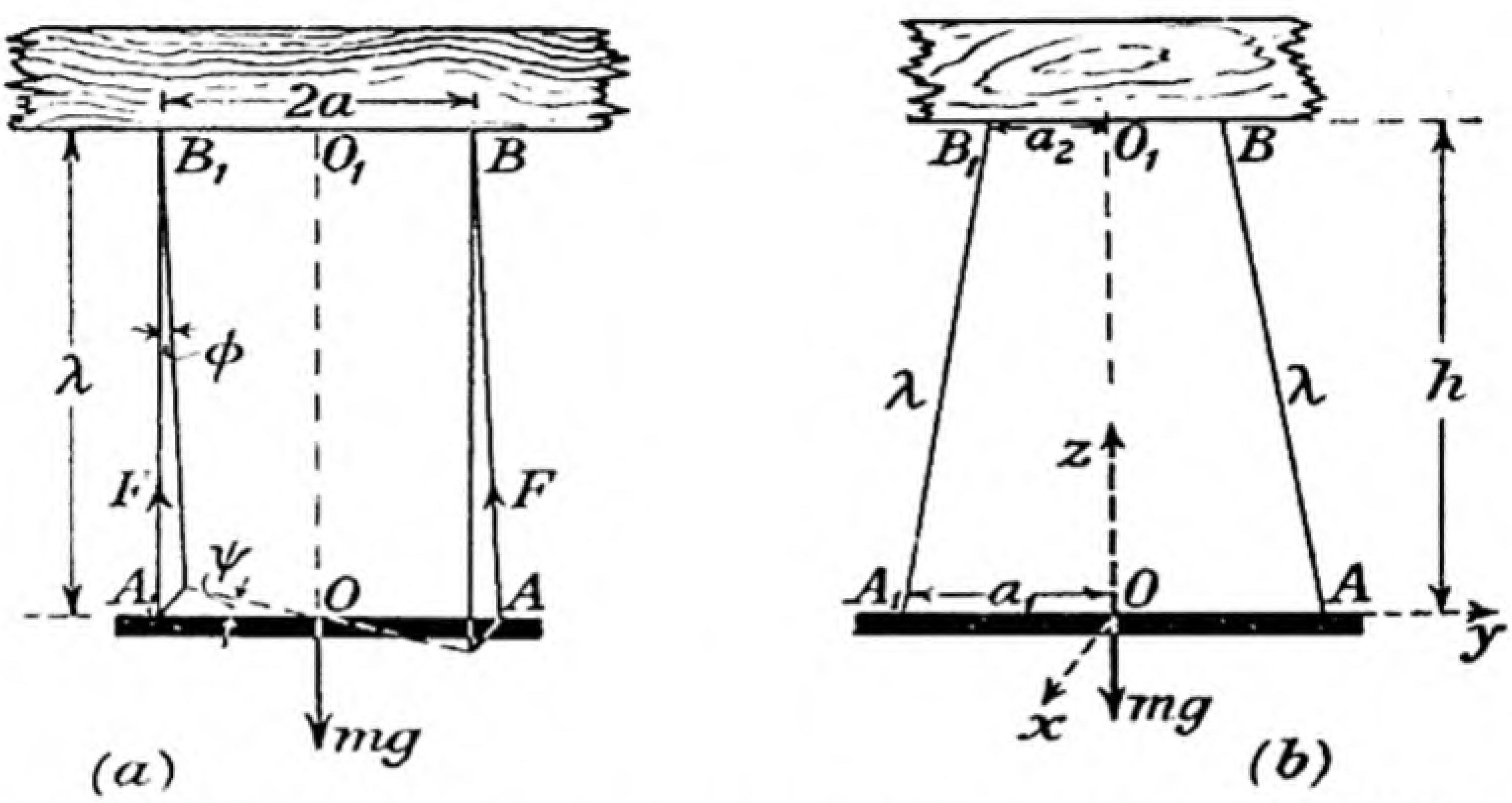


FIG. 4·26.—Bifilar suspensions, (a) parallel cords, (b) non-parallel cords.

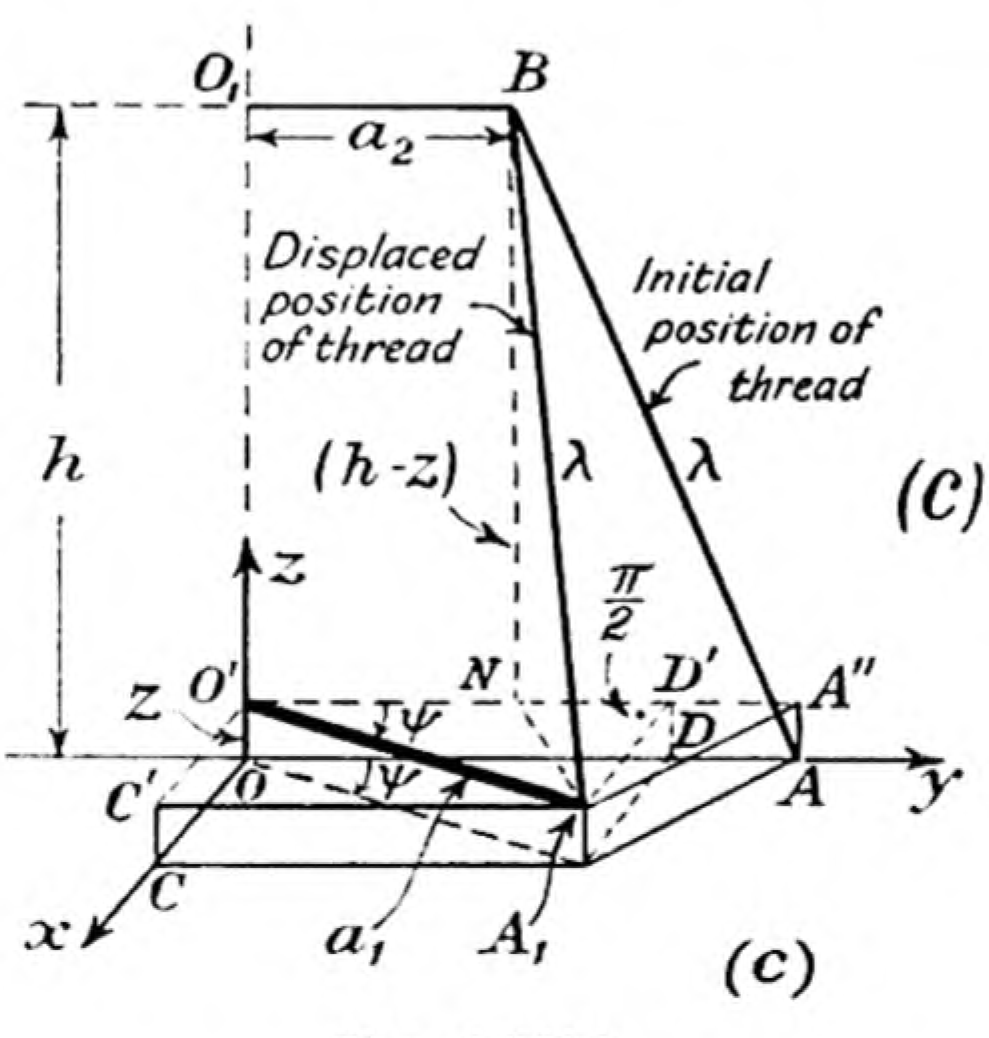


FIG. 4·26(c).

(a) ***Parallel cords:*** Let us suppose that each cord is of length λ, the distance between them being $2a$. Let them support a brass cylinder or bar AA_1, Fig. 4·26(a), of uniform cross-section and density. If the cords are attached at their upper ends B,B_1, to a horizontal beam, then the bar may be made to oscillate under gravity in a horizontal plane (if the cords are long and the angular

displacement small), about a vertical axis OO_1, which passes through the centre of gravity of the bar.

Suppose that the bar rotates about the above axis through a small angle ψ; then the displacement of the lower end of each cord is $a\psi$ and ϕ, the angle of inclination of each cord to the vertical, is given by

$$\lambda\phi = a\psi.$$

When the bar is in the position defined by ϕ, let F be the tension in each cord; if m the mass of the bar and g the intensity of gravity, we have

$$mg = 2F \cos \phi \doteqdot 2F,$$

if the cords are always approximately vertical.

When the cords are at an angle ϕ to the vertical, F sin ϕ is the force at each end of the cylinder where it is attached to the cord. These forces act at right angles to the axis of the cylinder and tend to restore the latter to its position of equilibrium. The moment of these forces about the axis of rotation is $(2F \sin \phi)a$, or $2F\phi a$, when ϕ is small. Since this couple, which may be written as $\frac{mga^2}{\lambda}\cdot\psi$, tends to diminish ψ, the motion of the cylinder is expressed by

$$I\ddot{\psi} + \frac{mga^2\psi}{\lambda} = 0,$$

where I is the moment of inertia of the rod about a vertical axis through its centre of gravity. If κ is the radius of gyration of the bar about this same axis, $I = m\kappa^2$, and the above equation becomes

$$\ddot{\psi} + \frac{ga^2}{\kappa^2\lambda}\cdot\psi = 0,$$

so that T, the periodic time, is given by

$$T = 2\pi\frac{\kappa}{a}\sqrt{\frac{\lambda}{g}}.$$

(*b*) ***Non-parallel cords:*** When the cords are of equal length but not parallel, let $2a_1$ be their separation where they are attached symmetrically to the bar and $2a_2$ their separation where they are fixed to the horizontal beam supporting the system. Let the coordinates of A referred to the axes indicated in Fig. 4·26(*b*) be initially (O, a_1, O). When the system rotates through a small angle ψ about the vertical axis OO_1, which is the straight line joining the centre of AA_1 and BB_1, these coordinates become

$$a_1 \sin \psi, \quad a_1 \cos \psi, \quad z,$$

where z is the distance through which every point in the bar rises.

Since the length λ of each string remains constant, we have, if $h = OO_1$ and using Fig. 4·26(*c*),

$$h^2 + (a_1 - a_2)^2 = \lambda^2 = BN^2 + [(A_1D')^2 + A_1N^2]$$
$$= (h - z)^2 + a_1{}^2 \sin^2 \psi + (a_1 \cos \psi - a_2)^2.$$

If z^2 is small and neglected, the above equation reduces to

$$hz = a_1 a_2(1 - \cos \psi)$$
$$= \tfrac{1}{2} a_1 a_2 \psi^2,$$

if ψ is small. Hence the increase in potential energy of the bar is

$$mgz = \tfrac{1}{2} \frac{a_1 a_2}{h} g \,.\, \psi^2.$$

The energy equation is therefore

$$\tfrac{1}{2} m\kappa^2 \dot{\psi}^2 + \frac{1}{2} \frac{a_1 a_2}{h} g\psi^2 = \text{constant}.$$

Differentiating with respect to time, we get

$$m\kappa^2 \ddot{\psi} + \frac{a_1 a_2}{h} g\psi = 0.$$

Hence the motion is simple harmonic with a period T, where

$$T = 2\pi \frac{\kappa}{\sqrt{a_1 a_2}} \cdot \sqrt{\frac{h}{g}}.$$

Example.—A uniform rod, of length $2a$ and mass m, hangs in a horizontal position being supported by two light vertical strings each of length l. The lower end of each string is attached to an end of the rod while the upper end is attached to a fixed point in a suitable support. The rod is given an angular velocity ω about a vertical axis through its centre. Show that it will rise a distance $\dfrac{a^2\omega^2}{6g}$, where g is the intensity of gravity. Find also the period of small oscillations about the position of static equilibrium and examine two other possible modes of vibration of the system.

Let AB, Fig. 4·27(*a*), be the initial position of the rod, the strings AP_1 and BP_2 being vertical. Let A_1B_1 be the position of the rod when it has risen through a vertical distance z and its axis rotated through an angle θ, and let ϕ be the inclination of the strings to the vertical at the same instant; let C and D be the projections of the points P_1 and P_2 on the horizontal plane through A_1B_1. The energy equation is

$$\tfrac{1}{2} m\dot{z}^2 + \tfrac{1}{2} m\kappa^2 \dot{\theta}^2 + mgz = mg(0) + \tfrac{1}{2} m\kappa^2\omega^2,$$

where the symbols have their usual meanings.

Now $z = l(1 - \cos\phi)$ and $A_1C = 2a \sin \tfrac{1}{2}\theta = l \sin \phi$,

$$\therefore\ \dot{z} = l \sin\phi \,.\, \dot{\phi} = \tan\phi \,.\, a \cos \tfrac{1}{2}\theta \,.\, \dot{\theta}$$
$$= \frac{a^2 \sin\theta \,.\, \dot{\theta}}{[l^2 - 4a^2 \sin^2 \tfrac{1}{2}\theta]^{\frac{1}{2}}}.$$

$$\therefore\ \tfrac{1}{2} a^2 \dot{\theta}^2 \left[\frac{a^2 \sin^2\theta}{l^2 - 4a^2 \sin^2 \tfrac{1}{2}\theta} + \frac{1}{3}\right] = \frac{1}{2}\frac{a^2}{3} \,.\, \omega^2 - gz.$$

This gives the angular velocity in any position of the rod. The rod is at rest instantaneously when $\dot{\theta} = 0$, i.e. the rod will rise through a vertical distance given by

$$[z]_{\dot{\theta}=0} = \frac{1}{6}\cdot\frac{a^2\omega^2}{g}.$$

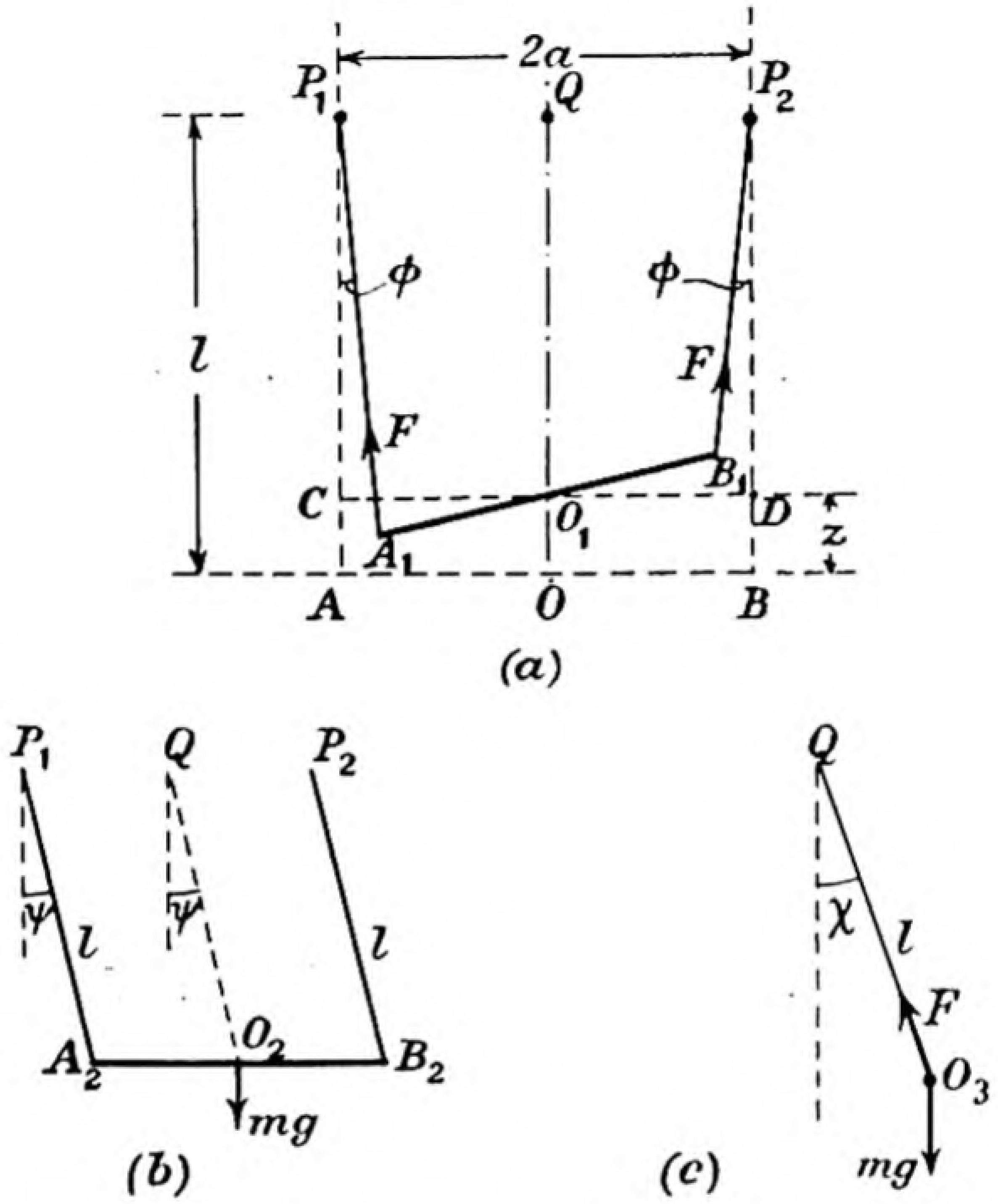


FIG. 4·27.—Three possible modes of vibration of a uniform rod supported horizontally by two equal vertical strings.

For a small oscillation about a vertical axis through O, the centre of the rod AB, if F is the tension in each string,

$$2F\cos\phi = mg,$$

and the equation of motion is

$$m.\frac{a^2}{3}.\ddot{\theta} + 2F\sin\phi.a = 0.$$

If θ is small, ϕ is also small, and then $l\phi = a\theta$; also $2F = mg$.

$$\therefore\ m.\frac{a^2}{3}\ddot{\theta} + mg.\frac{a^2}{l}.\theta = 0.$$

Hence the period, T_1, is given by

$$T_1 = 2\pi\sqrt{\frac{l}{3g}},$$

which agrees with the result obtained by using the general formula given on p. 143.

The rod may also move in a vertical plane through its points of support, P_1P_2. Fig. 4·27(*b*) shows the rod when it is thus displaced through

an angle ψ from its position of static equilibrium. Since the moment of inertia of the rod about a horizontal axis through Q, the mid-point of P_1P_2 is, by the theorem of parallel axes, $m(\frac{1}{3}a^2 + l^2)$, the equation of motion is

$$I\ddot{\psi} + mgl \sin \psi = 0,$$

so in the usual way we find for this mode of motion that the period, T_2, of small oscillations is given by

$$T_2 = 2\pi \sqrt{\frac{\frac{1}{3}a^2 + l^2}{gl}}.$$

The rod may also make oscillations when it moves in such a way that, at any given instant, each point in the rod is at the same distance from the vertical plane through P_1P_2. If χ is the inclination of each string to the vertical, cf. Fig. 4·27(c), which is an end-on view of the system, the tension F in a string is given by

$$2F \cos \chi = mg,$$

while the equation of motion is

$$I\ddot{\chi} + 2F \sin \chi = 0,$$

where I is the moment of inertia of the rod about an axis through Q and parallel to the axis of the rod, i.e. $I = ml^2$. If χ is small, we have at once

$$\ddot{\chi} + \frac{g}{l}\chi = 0,$$

i.e.

$$T_3 = 2\pi\sqrt{\frac{l}{g}},$$

where T_3 is the period in this instance; the system behaves as a simple pendulum of length l.

Example.—(a). A uniform rod, of length $2a$ and mass m, is freely attached to a fixed support at its end A and swings as a compound pendulum in a vertical plane through A. Determine the frequency of small oscillations.

(b). If a particle of mass βm is attached to the rod at B, prove that the frequency of its oscillations is reduced in the ratio

$$\sqrt{\frac{1 + 2\beta}{1 + 3\beta}} : 1.$$

(c). If the rod is the pendulum of a clock which keeps good time, but loses one-third of a minute a day when the particle is attached, prove that

$$3\beta(3N^2 - 6N + 1) = 6N - 1,$$

where N is the number of minutes in a day.

(a). In this instance the period of the pendulum is given by

$$T = 2\pi\sqrt{\frac{r^2 + \kappa^2}{rg}} = 2\pi\sqrt{\frac{a^2 + \frac{1}{3}a^2}{ag}} = 2\pi\sqrt{\frac{4}{3}\frac{a}{g}} = 4\pi\sqrt{\frac{a}{3g}}.$$

[In an examination this formula should be obtained from first principles; it is obtained here from the formula for the period of a compound pendulum since the method has been explained, in detail, on several occasions.]

(*b*). The system is shown in Fig. 4·28. If ψ is the angular displacement of the rod at a given instant, the equation of motion is

$$\mathrm{I}\ddot{\psi} + \Gamma = 0,$$

where I is the moment of inertia of the rod about a horizontal axis through A and Γ is the restoring couple due to the external forces acting on the rod. In the usual way we have

$$\left[m\left(\frac{a^2}{3} + a^2\right) + \beta m.4a^2\right]\ddot{\psi} + (mga + \beta mg.2a)\psi = 0,$$

provided ψ is small. Hence the period is given by

$$\mathrm{T} = 2\pi\sqrt{\frac{\frac{4}{3}a^2 + 4\beta a^2}{(a + 2a\beta)g}} = 2\pi\sqrt{\frac{4(1 + 3\beta)}{3(1 + 2\beta)}\cdot\frac{a}{g}}.$$

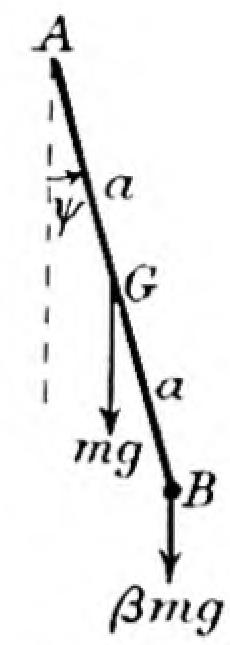


FIG. 4·28.

$\therefore$ Frequency is reduced in the ratio $\sqrt{\dfrac{1 + 2\beta}{1 + 3\beta}} : 1.$

(*c*). Let t be the period in seconds when the pendulum keeps good time, i.e. $t = 2$. Then its period in minutes is given by

$$\frac{t}{60} = 2\pi\sqrt{\frac{4a}{3g}} \times \frac{1}{60}.$$

If there are N minutes in a day and the clock loses one-third of a minute in that time, the ratio of the frequencies is

$$\frac{\dfrac{\mathrm{N}}{2} - \dfrac{1}{6}}{\dfrac{\mathrm{N}}{2}} = \sqrt{\frac{1 + 2\beta}{1 + 3\beta}}.$$

$$\therefore \left(\frac{3\mathrm{N} - 1}{3\mathrm{N}}\right)^2 = \frac{1 + 2\beta}{1 + 3\beta}.$$

$$\therefore 3\beta(3\mathrm{N}^2 - 6\mathrm{N} + 1) = 6\mathrm{N} - 1.$$

EXAMPLES IV

4·01. Find the position of the centre of mass of a uniform circular arc.

A uniform chain of weight W is attached to the highest point A of a fixed smooth sphere and rests on the surface, its free end being level with the centre. Prove that the horizontal pull on A is $\dfrac{2\mathrm{W}}{\pi}$.

4·02. Find the centre of gravity of a triangular lamina bounded by the straight lines $x = a$, $y = mx$ and $y = 0$, the surface density at each point varying as the square of its distance from the origin. Also determine the moment of inertia about the line $x = 0$.

$$\left[\bar{x} = \tfrac{4}{5}a,\ \bar{y} = \tfrac{3}{5}ma\frac{m^2 + 2}{m^2 + 3};\ \tfrac{2}{3}\mathrm{M}a^2.\right]$$

4·03. Find the centroid of a rod in which the linear density varies as the distance from one end. Show that the moment of inertia of the rod about its centre is $\frac{1}{12}ma^2$, where a is the length of the rod and m its mass.

4·04. In the first quadrant of the circle $x^2 + y^2 = a^2$, the surface density varies at each point as xy. Show that the centre of gravity of this quadrant is given by

$$\bar{x} = \bar{y} = \tfrac{8}{15}a,$$

and that the moment of inertia about either coordinate axis is $\frac{1}{3}ma^2$, where m is the mass of the quadrant.

4·05. Use the theorem of Pappus to show that when a semicircular arc of radius a revolves about the diameter joining its extremities, the position of its centroid is $\bar{x} = 0$, $\bar{y} = \dfrac{2a}{\pi}$, these being referred to rectangular axes. Ox coincides with the above diameter and O is its mid-point.

Also show for a uniform lamina in the form of a semicircular area of radius a, that $\bar{x} = 0$, $\bar{y} = \dfrac{4a}{3\pi}$.

4·06. A uniform lamina in the form of an equilateral triangle of mass m hangs in a horizontal position suspended by three equal vertical light strings each of which passes through an angular point in the triangle. The length of each string is $2a$, where $2a$ is the diameter of the circumscribing circle. Prove that the couple required to keep the lamina at a height $2(1 - n)a$ above its initial position where, $0 < n < 1$, is $mga\sqrt{1 - n^2}$.

4·07. Explain what is meant by saying that a certain quantity varies in a simple harmonic manner. Define ***amplitude***, ***frequency*** and ***phase***.

A particle oscillates along a straight line with s.h.m. Its greatest acceleration is $5\pi^2$ cm.sec.$^{-2}$ and when its distance from its zero position is 4 cm. its velocity is 3π cm.sec.$^{-1}$. Calculate the period of oscillation of the particle, and its amplitude. [2 sec., 5 cm.]

4·08. An aluminium disc 10 cm. in diameter and 0·5 cm. thick is supported symmetrically with its plane horizontal by a suspension consisting of two fine threads each 25 cm. long, the upper ends being 4 cm. and the lower 3 cm. apart. Find the time of oscillation for small angular displacements of the disc about a vertical axis. [The density of aluminium may be taken as 2·72 gm.cm.$^{-3}$.]

What would the period be if the disc were supported in a horizontal plane by three parallel strings each 25 cm. long, the ends of the strings being arranged symmetrically round the edge of the disc?

[2·05 sec., 0·71 sec.]

4·09. A test-tube, weighted with lead shot, floats in water with a length of 15 cm. immersed. When displaced vertically from its equilibrium position the period of oscillation is found to be 0·89 sec. Show how a value for this period could have been predicted and suggest reasons why the observed period is not exactly equal to the calculated period. [0·78 sec.]

4·10. A body is oscillating vertically under gravity at the end of a light helical spring. The body is cylindrical in shape and its axis is vertical; it is surrounded by a coaxial cylinder with a small clearance between the two cylindrical surfaces. This space is filled with a thin oil so that the motion of the cylinder is subjected to a resistance varying as the speed. If x is the distance of the body below its equilibrium position at time t (sec.) the equation of motion is $\ddot{x} + 0{\cdot}1\dot{x} + 4x = 0$. Obtain a value for the period of the motion and find how many complete oscillations occur while the amplitude is reduced to one third of its initial value. [3·14 sec., 70]

4·11. From successive observations on the 'turning points' of a balance pointer, show how its rest position may be deduced. The scale across which the end of a balance pointer swings has its zero at one end. Successive turning points occur at 11, 3 and 8 divisions. Deduce the rest point. [6·08.]

4·12. A light rough circular cylinder of radius a rests on a horizontal plane with a particle of mass M attached to its lowest point and a string of length l, carrying a particle of mass m, attached to its highest point. Show that the system is stable if $M > m$.

Show that the period of small oscillations of the system is

$$2\pi\left[\frac{l}{g}\left\{1 + \frac{4a}{l}\cdot\frac{m}{M - m}\right\}\right]^{\frac{1}{2}},$$

and determine the ratio of the excursions of the cylinder to the excursions of the bob of the pendulum. Discuss the fact that though the system has two degrees of freedom it has only one free period.

4·13. Investigate the condition that the time period of a compound pendulum shall be a minimum.

The position of the axis of rotation of a compound pendulum of mass M is adjusted so as to give the time period of the pendulum its minimum value T_0. A piece of metal of mass m but of negligible size is then attached at the centre of gravity of the pendulum. Derive expressions for (a) the new time period for the same position of the axis and (b) the new minimum value of the time period.

[Minimum when $r = \kappa$;

(a) $\sqrt{\frac{1}{2}\left(1 + \frac{M}{M + m}\right)}\,.\,T_0$, ($b$) $\left(\frac{M}{M + m}\right)^{\frac{1}{4}}T_0$]

4·14. A circular ring of uniform thickness has a mass m, the internal and external radii being a and b respectively. Show that the moment of inertia of the ring about an axis through its centre and normal to its plane is $0{\cdot}5m(a^2 + b^2)$.

The ring is allowed to oscillate in its own plane which is vertical about an axis perpendicular to the plane of the ring and passing through a point on its outer edge. Obtain an expression for the period of oscillation for small amplitudes. Find values for the length of the equivalent simple pendulum (i) as $a \to 0$, (ii) as $a \to b$.

$\left[2\pi\sqrt{\frac{a^2 + 2b^2}{bg}}\right.$; (i) $2b$, (ii) $3b$.]

4·15. Five circular discs, of the same thickness and density, are fastened together as shown in Fig. 4·29. The smaller outer discs rest

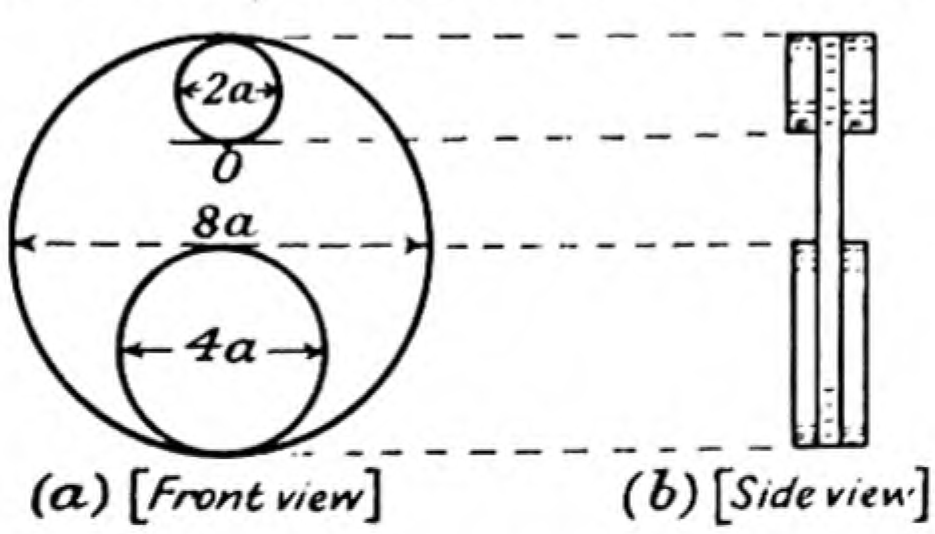


FIG. 4·29.

on two parallel horizontal rails which are rough enough to prevent slipping. If the radii of the discs are a, $2a$ and $4a$, as shown, prove that the periodic time of small oscillations under gravity is $\frac{\pi}{4}\sqrt{339\frac{a}{g}}$, where g is gravity.

Chapter V

THE INTENSITY OF GRAVITY

The intensity of gravity.†—In this chapter it is proposed to discuss methods which have been used to determine accurately the intensity of gravity and the manner in which it varies over the surface of the earth. The usual laboratory methods for determining the value for this intensity have been described in an earlier chapter and elsewhere‡; a discussion of these will not be repeated now. It is necessary, however, to examine the effect of the earth's rotation on a measured value for the intensity of gravity.

The effect of the earth's rotation on a measured value for the intensity of gravity.—Consider a body of mass m at a point A on the earth's surface, cf. Fig. 5·01(a), and in latitude ϕ. Let r be the least distance of A from the axis about which the earth rotates, i.e. $r = \mathrm{AN} = a \cos \phi$, where N is the projection of A on the axis and a is the earth's radius. The acceleration of the body due to the earth's rotation is $\omega^2 r = \omega^2 a \cos \phi$ and it is directed along AN. Now the earth's attraction on the body at A is independent of the earth's rotation, i.e. it is always directed along AC, where C is the centre of the earth; this force is mg, where g is the intensity of gravity, and, in general, this is not a quantity that can be measured directly. Let g_ϕ be the measured value for the intensity of gravity at A, i.e. mg_ϕ is the weight of the body. Consider Fig. 5·01(b) in which, on a much-exaggerated scale, there is shown a simple pendulum with its bob at A. The forces on A, when the pendulum is at rest relative to the earth, are the gravitational attraction mg, directed towards C, and the tension, $\mathrm{F} = mg_\phi$, in the string. These two forces must provide a resultant which gives to the body an acceleration $\omega^2 a \cos \phi$ along AN. From a triangle of forces we have

$$(g_\phi)^2 = g^2 + (\omega^2 a \cos \phi)^2 - 2g \,.\, \omega^2 a \cos \phi \,.\, \cos \phi,$$

and since $\omega^2 a$ is very small compared with g, we may neglect the term in ω^4 and using $(1 + x)^{0\cdot 5} = 1 + \frac{1}{2}x$, when $x \to 0$, obtain

$$g_\phi = g\left[1 - 2\,\frac{\omega^2 a}{g}\cos^2 \phi\right]^{0\cdot 5} = g - \omega^2 a \cos^2 \phi$$
$$= g - \mu \cos^2 \phi,$$

† Formerly called the acceleration due to gravity; cf. *I.P.*, p. 27.

‡ Cf. *I.P.*, pp. 30 and 45.

where $\mu = \omega^2 a$, the maximum value for the diminution in the measured value for the intensity of gravity. The diminution is zero when $\phi = 90°$, i.e. $g_{90} = g$. We may therefore write

$$g_\phi = g_{90} - \mu \cos^2 \phi.$$

Now $$a = 6{\cdot}37 \times 10^8 \text{ cm.},$$

and $$\omega = 2\pi \div (24 \times 3600)$$
$$= 7{\cdot}27 \times 10^{-5} \text{ radian.sec.}^{-1}$$
$$\therefore\ \mu = 3{\cdot}37 \text{ cm.sec.}^{-2}.$$

Hence the acceleration of a freely falling body in latitude ϕ is less than its value at the poles by $3{\cdot}37 \cos^2 \phi$ cm.sec.$^{-2}$.

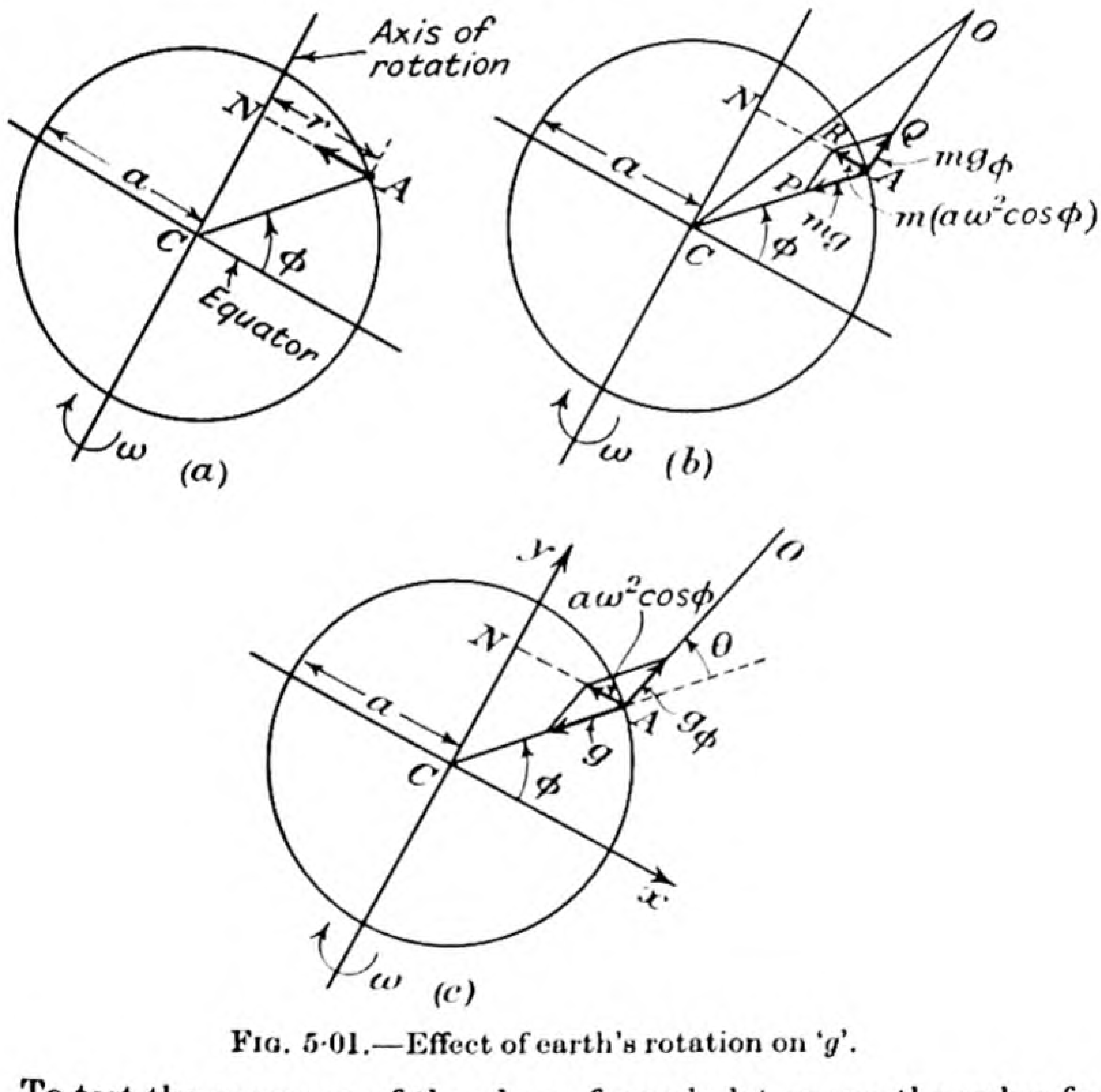


FIG. 5·01.—Effect of earth's rotation on 'g'.

To test the accuracy of the above formula let us use the value for g_{45}, viz. 980·62 cm.sec.$^{-2}$, and calculate g_{90} and g_0, comparing the values so obtained with the accepted values. We have

$$g_{45} = 980{\cdot}62 \text{ cm.sec.}^{-2} = g_{90} - 3{\cdot}37 \cos^2 45°$$
$$= g_{90} - 1{\cdot}69.$$
$$\therefore\ g_{90} = 982{\cdot}31 \text{ cm.sec.}^{-2}.$$

Hence $$g_0 = 982.31 - 3{\cdot}37 = 978{\cdot}94 \text{ cm.sec.}^{-2}.$$

The corresponding accepted values are 983·21 cm.sec.$^{-2}$ and 978·04 cm.sec.$^{-2}$. The above suggests at once that a better formula expressing the variation of the intensity of gravity with latitude is

$$g_\phi = (983{\cdot}21 - 5{\cdot}17 \cos^2 \phi) \text{ cm.sec.}^{-2}.$$

The difference must be attributed in part, at least, to the fact that the simple treatment given above regards the earth as a sphere and, moreover, the earth is certainly not homogeneous.

To determine the angle which a pendulum at rest relative to the earth makes with the radius passing through its bob, it is convenient to use Fig. 5·01(c), which is similar to Fig. 5·01(b), except that accelerations, instead of forces, are indicated. Then the acceleration $(a \cos \phi)\omega^2$ along AN is the resultant of g and g_ϕ. Considering the components of the accelerations parallel to Ox and Oy, respectively, we have

$$g \cos \phi - g_\phi \cos (\phi + \theta) = a\omega^2 \cos \phi, \qquad \text{(i)}$$

and

$$g \sin \phi = g_\phi \sin (\phi + \theta), \qquad \text{(ii)}$$

where θ is the inclination required. Eliminating g_ϕ, we have

$$\frac{\cos \phi - \dfrac{a\omega^2}{g}}{\sin \phi} = \frac{\cos (\phi + \theta)}{\sin (\phi + \theta)}.$$

Expanding the terms in $(\phi + \theta)$ and cross-multiplying, we get

$$\sin \theta \left(1 - \frac{a\omega^2}{g} \cos^2 \phi\right) = \frac{a\omega^2}{2g} . \sin 2\phi . \cos \theta.$$

Since $\dfrac{a\omega^2}{g}$ is small in comparison with unity,

$$\theta \simeq \tan^{-1} \left(\frac{a\omega^2}{2g} \sin 2\phi\right).$$

By eliminating $(\phi + \theta)$ from equations (i) and (ii), we get

$$g_\phi^2[\sin^2 (\phi + \theta) + \cos^2 (\phi + \theta)]$$
$$= g^2(\cos^2 \phi + \sin^2 \phi) - 2ga\omega^2 \cos^2 \phi + a^2\omega^4 \cos^2 \phi.$$

Neglecting the term in ω^4, we obtain, as before

$$g_\phi = g\left(1 - 2\frac{a\omega^2}{g} \cos^2 \phi\right)^{\frac{1}{2}},$$

$$\simeq g\left(1 - \frac{a\omega^2}{g} \cos^2 \phi\right).$$

The experiments of Borda and Cassini.—In 1792, BORDA and CASSINA, working in Paris, carried out a series of experiments with the aid of a simple pendulum to determine the intensity of gravity at Paris. But before entering into a discussion of their work it is necessary to examine the extent to which a pendulum consisting of a bob of finite size differs from an ideal simple pendulum, and the effect of a knife-edge and its adjuncts—the whole being termed the 'head' of the pendulum—on the period of the pendulum.

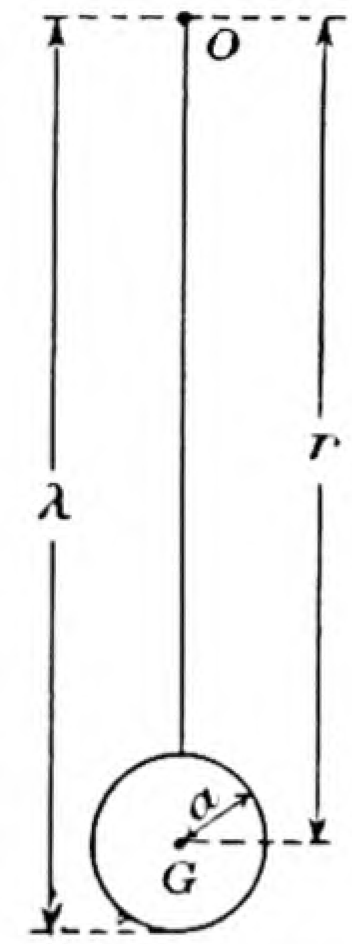


FIG. 5·02.—Simple pendulum with bob of finite size.

A simple pendulum: correction for finite size of bob.—Let a homogeneous sphere of radius a be suspended from a fixed point O, Fig. 5·02. Let G be the centre of the bob: call OG $= r$, and let λ be the distance from O to the lowest point in the bob when at rest. Then l, the length of the simple equivalent pendulum, is given by

$$l = r + \frac{\kappa^2}{r},$$

where κ is the radius of gyration for the bob about a horizontal axis through G. Since $\kappa^2 = \frac{2}{5}a^2$, we have

$$l = r + \frac{2}{5}.\frac{a^2}{r} = (\lambda - a) + \frac{2}{5}.\frac{a^2}{(\lambda - a)}.$$

The period is therefore given by

$$\mathrm{T} = 2\pi\left(1 + \frac{\psi_1\psi_2}{16}\right)\sqrt{\frac{(\lambda - a) + \frac{2}{5}.\frac{a^2}{(\lambda - a)}}{g}},$$

if we introduce the correction for the fact that the amplitude is finite and not quite constant but decreases from ψ_1 to ψ_2 during the time interval from which the period is deduced, i.e. ψ_0, cf. p. 107, is the geometric mean of ψ_1 and ψ_2.

To examine the fact due to the finite size of the bob let us consider (i) $\lambda = 105$ cm., (ii) $\lambda = 405$ cm. and assume a to be 5 cm. in each instance. Then $\frac{2}{5}\cdot\frac{a^2}{\lambda - a}$ is (i) 0·1 cm. and (ii) 0·025 cm. The effective length is therefore increased by 1 in 10^3 and 1·6 in 10^4 respectively, i.e. the corrections are by no means negligible in precision determinations of the intensity of gravity.

The period of the knife-edge and its effect on that of the simple pendulum.—Figs. 5·03(a) and (b) suggest a method of clamping the wire to the knife-edge. The position of G_1, the centre of gravity of the 'head' of the pendulum, may be adjusted by altering the position of the screw-head S. The period is increased as G_1 approaches from below the axis of suspension.

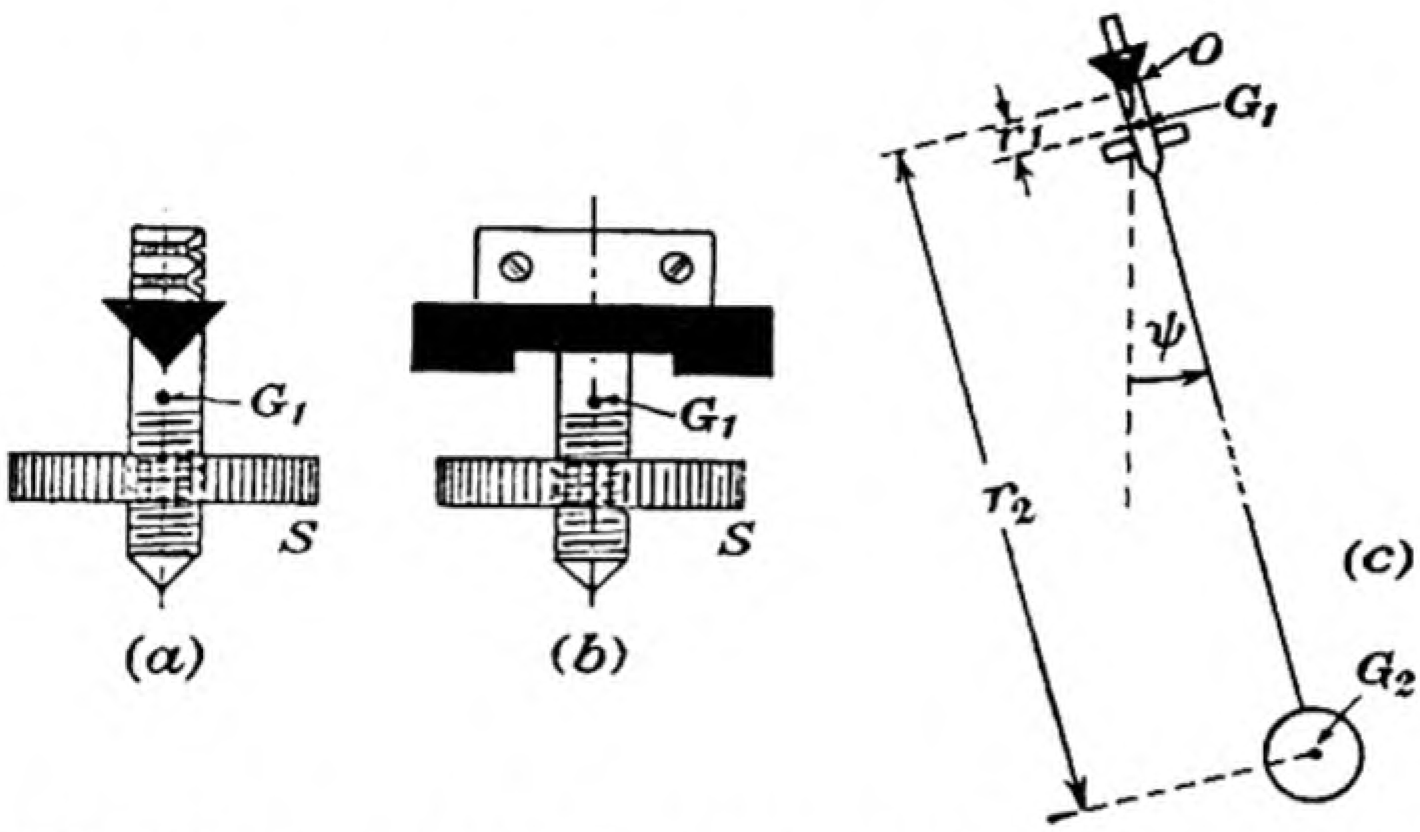


FIG. 5·03.—The period of the knife-edge and its effect on that of a simple pendulum.

Let G_2, Fig. 5·03(c), be the centre of gravity of the bob, of mass m_2, at a distance r_2 below the axis of suspension. Let $r_1 = OG_1$. If T_1 is the period of oscillation of the head alone, then

$$T_1 = 2\pi\sqrt{\frac{m_1\kappa_1^2}{m_1 r_1 g}} = 2\pi\sqrt{\frac{\kappa_1^2}{r_1 g}},$$

where m_1 is the mass of the head and κ_1 its radius of gyration about a horizontal axis through O.

Now T, the period of the complete pendulum, is given by

$$T = 2\pi\sqrt{\frac{m_1\kappa_1^2 + m_2\kappa_2^2}{(m_1 + m_2)rg}},$$

where r is the distance between O and the centre of gravity of the whole pendulum.

If the position of the screw S has been adjusted so that $T = T_1$, then

$$\frac{\kappa_1^2}{r_1} = \frac{m_1\kappa_1^2 + m_2\kappa_2^2}{(m_1 + m_2)r}$$

$$= \frac{m_1\kappa_1^2 + m_2\kappa_2^2}{m_1 r_1 + m_2 r_2}, \qquad [\because\ (m_1 + m_2)r = m_1 r_1 + m_2 r_2].$$

Hence
$$r_1\kappa_2^2 = r_2\kappa_1^2.$$

But T_2, the period of the pendulum in the absence of the head, is given by

$$T_2 = 2\pi\sqrt{\frac{\kappa_2^2}{r_2 g}} = 2\pi\sqrt{\frac{\kappa_1^2}{r_1 g}}$$

in virtue of the condition just established. Hence

$$T_2 = T = T_1,$$

i.e. the period is unaffected by the head of the pendulum if its period is adjusted to be equal to that of the pendulum as a whole.

The work of Borda and Cassini on the length of a seconds pendulum.—The apparatus used by Borda and Cassini (1792) to determine the length of a seconds pendulum at Paris is shown diagrammatically in Fig. 5·04. The bob of the pendulum consisted of a platinum sphere 3·48 cm. in diameter. A brass cap having a concave base with a radius of curvature equal to that of the sphere was attached to a thin iron wire whose upper end was fixed to a knife-edge in the manner indicated. The knife-edge rested on a steel plate rigidly attached to a stone slab projecting from a massive wall. The bob was attached to the cap with a thin layer of molten wax—this provided a means of rotating the sphere and then making observations on the period of the pendulum. The mean value for the period thus obtained was independent of the fact that the centre of gravity of the sphere may not coincide with its geometric centre. Borda and Cassini used a metallic wire for the suspension instead of one made of silk or from a material having a vegetable origin, because they found that threads of these offered a much larger area to the medium in which they moved than did metallic wires of equal strength. An iron wire was finally selected since, although it was thin, it was not stretched by the platinum bob. Platinum was chosen for the material of the bob since it could readily be obtained in homogeneous form.

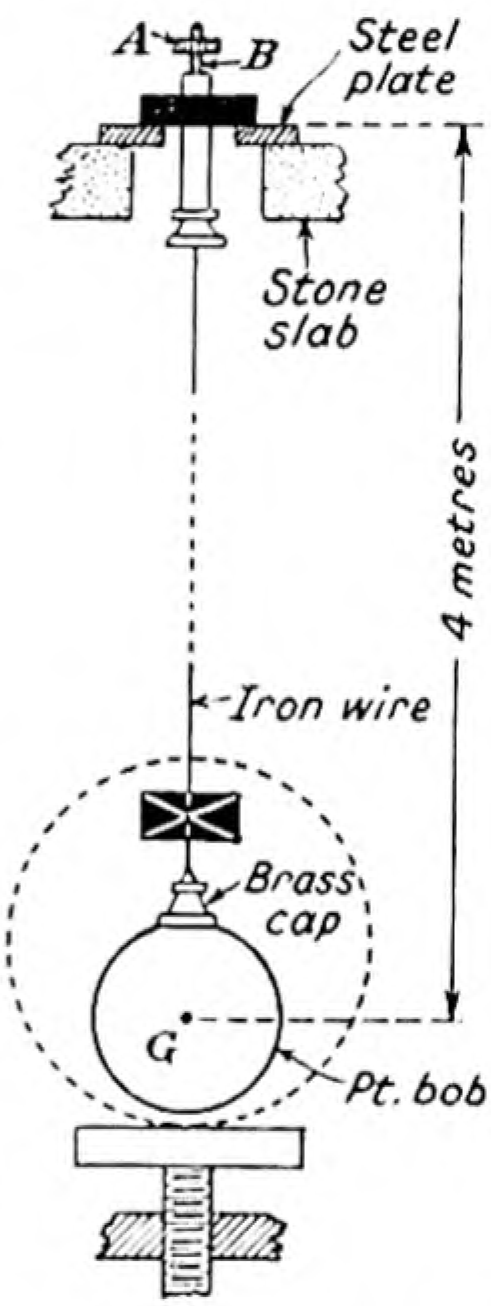


FIG. 5·04.—Borda and Cassini's pendulum. [The lower half of this diagram is drawn from a view-point in a direction at right angles to that for the upper, in order to show both the mounting for the knife-edge and the fiducial mark.]

The effective length of the pendulum, i.e. the length between the knife-edge and the centre of gravity of the bob was approximately 4 metres. These investigators preferred such a pendulum to one of length 1 metre since the relative error in measuring the length was

much less. Moreover its period was about four seconds so that it could easily be accurately determined by means of the method of coincidences using a standard clock beating seconds. The period of the head of the pendulum was first adjusted to be the same as that of the complete pendulum by altering the position of the nut A on the screw B. The pendulum of the clock used was one of the grid-iron variety so that its period was unaffected by small temperature changes.

To determine the period of a pendulum, the time interval between two occasions when the pendulum passes through its position of static equilibrium must be measured; this position must be definitely marked, for it is only possible to determine a time interval between two definite events in the world's history. Thus, if n complete swings are made in t seconds, $T = \frac{t}{n}$. If t is known to within one second, the period is known to within n^{-1} second. [It is always more expedient to consider the passage of the pendulum through its static equilibrium position rather than when it is at one extremity of its swing, because it passes through the former position with maximum velocity, a fact which helps to make the estimation of the instant when the particular event occurs, viz. the passage of the pendulum across a fiducial mark, more precise.]

In order to measure the period more accurately, Borda developed the so-called method of coincidences, a form of which had already been used by Bouguer in his experiments in the Andes in 1737 on the gravitational constant [cf. p. 227]. In this method it is necessary to use in conjunction with the pendulum whose period is required, the pendulum of a standard clock, i.e. one which beats seconds and therefore whose period is approximately two seconds, and this period may be determined accurately from observations on the successive transits of the sun across the meridian at the place of observation. Borda suspended the simple pendulum immediately in front of that of the standard clock and both were observed through a telescope. The bob of the clock's pendulum was marked with a white cross on a black background. The instant when the image of the wire of the experimental pendulum appeared to coincide with the centre of that of the white cross was noted; the instant when this next occurred was likewise observed. If the time interval between these events was t seconds the standard clock will have made $\frac{t}{2}$ complete oscillations, while the number made by the simple pendulum will be $\left(\frac{t}{2} \pm 1\right)$. Hence

$$T = \frac{t}{\frac{1}{2}t \pm 1} = \frac{2t}{t \pm 2} \simeq 2\left[1 \mp \frac{2}{t} + \frac{4}{t^2}\right].$$

The correct sign to be used follows at once if a note is made of the pendulum which appears to be gaining on the other.

In this method, if t is large, one is never quite sure when the coincidence is exact. Let Δt be the time interval between the instant when the coincidence appears to begin and when it appears to end. Then

$$T = \frac{t + \Delta t}{\left(\frac{t + \Delta t}{2} \pm 1\right)} = \frac{2(t + \Delta t)}{(t + \Delta t) \pm 2} = 2\left[\frac{1}{1 \pm \frac{2}{(t + \Delta t)}}\right]$$

$$= 2\left[1 \mp \frac{2}{t + \Delta t} + \frac{4}{(t + \Delta t)^2}\right], \qquad \text{[neglecting higher terms]},$$

$$= 2\left[1 \mp \frac{2}{t}\cdot\frac{1}{\left(1 + \frac{\Delta t}{t}\right)} + \frac{4}{t^2}\cdot\frac{1}{\left(1 + \frac{\Delta t}{t}\right)^2}\right]$$

$$= 2\left[1 \mp \frac{2}{t}\left(1 - \frac{\Delta t}{t} + \frac{(\Delta t)^2}{t^2}\right) + \frac{4}{t^2}\left(1 - \frac{2\Delta t}{t}\right)\right]$$

$$= 2\left[1 \mp \frac{2}{t} \pm \frac{2\Delta t}{t^2} + \frac{4}{t^2}\right].$$

$$\therefore \text{ Fractional error is } \pm \frac{2\Delta t}{t^2}.$$

Thus if $\Delta t = 5$ sec. and $t = 1000$ sec., the error is about 1 in 10^5.

The method of coincidences is most accurate when the period of the experimental pendulum is 2 seconds or $2n$ seconds, where n is a positive integer. It is for this reason that Borda used a pendulum in which $(\lambda - r) = 400$ cm. He actually measured the distance from the knife-edge to the lowest point in the sphere, and subtracted from this the radius of the sphere.

Borda displaced the pendulum 2° and found that even after five hours the amplitude was still sufficiently large for accurate observations to be made. The interval between successive coincidences was of the order 73 minutes with a possible error of 30 seconds owing to the difficulty of estimating an exact coincidence. But since five such periods were observed in succession the error on this account was much reduced. His final value for 'g' at Paris was 980·882 cm.sec.$^{-2}$, the length of the seconds pendulum being 99·353 cm. In connexion with this result it must be remembered that Borda and Cassini assumed that the bob and thread moved together as a rigid body, cf. p. 158. This is not so and hence the value for 'g' obtained in this way cannot be accurate to within the limits which the figures quoted suggest.

Other determinations of the intensity of gravity by means of simple pendulums.—Since $g = 4\pi^2 l T^{-2}$, it follows that when T is known it only remains to measure l accurately in order to obtain a reliable value for the intensity of gravity. This is somewhat difficult to do in practice, so that WHITEHURST (1787) and BESSEL (1826) used pendulums of lengths l_1 and l_2 and determined their periods T_1 and T_2. In Bessel's experiments, which will be briefly described, a bob was suspended by a wire first from one point and then from another, the distance between the points of suspension being accurately known. The position of static equilibrium for the bob was identical in each instance. Then

$$T_1 = 2\pi\sqrt{\frac{l_1}{g}}, \qquad \text{and} \qquad T_2 = 2\pi\sqrt{\frac{l_2}{g}},$$

so that

$$\frac{T_1^{\,2} - T_2^{\,2}}{4\pi^2}.g = (l_1 - l_2).$$

Thus, if it is legitimate to regard the pendulum as an ideal simple pendulum, the difference of the squares of the periodic times is the square of the periodic time of an ideal pendulum whose length is $(l_1 - l_2)$. But we have seen [cf. p. 153] that, more accurately,

$$T_1 = 2\pi\sqrt{\frac{l_1^{\,2} + \frac{2}{5}a^2}{l_1 g}}, \qquad \text{and} \qquad T_2 = 2\pi\sqrt{\frac{l_2^{\,2} + \frac{2}{5}a^2}{l_2 g}}.$$

$$\therefore \; \frac{T_1^{\,2} - T_2^{\,2}}{4\pi^2}.g = (l_1 - l_2) + \tfrac{2}{5}a^2\left[\frac{1}{l_1} - \frac{1}{l_2}\right].$$

It is therefore necessary to know l_1 and l_2, but since $\frac{2}{5}a^2$ is small, they need not be known with the same accuracy as their difference $(l_1 - l_2)$.

Instead of a knife-edge or a pair of jaws, Bessel suspended the wire from a horizontal cylinder on which the wire wrapped and unwrapped, having showed theoretically that for small oscillations this did not affect the period of the motion. He also made corrections for the stiffness of the wire for the want of rigidity between the wire and the bob. LAPLACE had first pointed out that the wire and the bob could not move as one—in practice, the bob turns through a slightly greater angle than does the wire, but the correction on this account is small if the radius of the bob is small compared with the length of the wire.

Not content with all these refinements in the theory and use of a simple pendulum, Bessel reinvestigated the effect of the medium

(air) in which the pendulum moved during its motion. The nature of the effect is threefold, being due to

(a) the buoyancy of the air,
(b) the fact that some of the air is carried along with the pendulum and thereby virtually increasing its mass,
(c) that air is a medium possessing a small viscosity, a fact which accounts in part at least for the motion of the pendulum being slightly damped.

Let us first consider the effects of (a) and (b). Let m be the mass of the pendulum, r the distance of the centre of gravity of the bob from the point of support, m_1 the mass of air displaced by the pendulum, and m_2 the mass of air carried along by the pendulum. The moment of inertia of the system about its axis of rotation will be $mk^2 + m_2d^2$, where κ is the radius of gyration for the pendulum about a horizontal axis through its point of support and m_2d^2 is a term representing the increase in the moment of inertia of the pendulum about the above axis due to the air carried along with the pendulum. The equation of motion is therefore

$$(m\kappa^2 + m_2d^2)\ddot{\theta} + (m - m_1)rg\theta = 0.$$

It should be noted that m_2 does not appear in the last term of this equation since the air carried along with the pendulum will be buoyed up by the atmosphere, its weight therefore not contributing to the restoring couple. The length of the simple equivalent pendulum is therefore

$$\frac{m\kappa^2 + m_2d^2}{r(m - m_1)} = \frac{\kappa^2 + \frac{m_2}{m}.d^2}{r\left(1 - \frac{m_1}{m}\right)}.$$

The value of m_1 is determined from the known volume of the pendulum bob and the density of the air under the conditions of the experiment. The value of m_2d^2 was determined by using bobs of the same size but of different materials, the periodic time being measured in each instance.

The above theory had already been given by NEWTON and by DU BUAT, the latter estimating the magnitude of the second effect due to the air in the way now used by Bessel, the work of Du Buat being unknown to him, however. Bessel showed that the above analysis is not quite accurate but his work is too difficult for discussion here.

The effect of the viscosity of air on the motion of the pendulum was examined by BESSEL, but the theory was not really adequate until it was developed by STOKES (1847). He showed that the effect could be represented by introducing into the equation of

motion a term proportional to the velocity of the pendulum, i.e. the equation assumes the form

$$\ddot{\theta} + \alpha\dot{\theta} + \beta\theta = 0, \qquad \text{[cf. p. 33]}.$$

The solution is

$$\theta = \text{C} \exp\left(-\tfrac{1}{2}\alpha t\right) \cos\left(\tfrac{1}{2}\sqrt{4\beta - \alpha^2}\,t + \phi\right),$$

where C and ϕ are constants. This assumes that $\alpha^2 < 4\beta$, but the observation that the motion is slightly damped shows that this is justified. The period is given by

$$\text{T} = \frac{4\pi}{\sqrt{4\beta - \alpha^2}} = \frac{2\pi}{\beta^{\frac{1}{2}}}\cdot\frac{1}{\sqrt{1 - \dfrac{\alpha^2}{4\beta}}} \doteq \frac{2\pi}{\beta^{\frac{1}{2}}}\left[1 + \frac{\alpha^2}{8\beta}\right].$$

In the absence of viscosity the period would be T_0, where $\text{T}_0 = 2\pi\beta^{-\frac{1}{2}}$. Hence the period is increased $\left[1 + \dfrac{\alpha^2}{8\beta}\right]$ times. Since

$$\frac{1}{\beta} = \frac{\text{T}_0{}^2}{4\pi^2}, \qquad \text{T} = \text{T}_0\left(1 + \frac{\alpha^2\text{T}_0{}^2}{32\pi^2}\right).$$

To examine the magnitude of this effect let us consider one of Borda's experiments in which $\text{T}_0 = 4$ sec. and the amplitude was reduced to half its value after about 4000 sec. Then

$$2 = \frac{\exp\left(-\tfrac{1}{2}\alpha t\right)}{\exp\left\{-\tfrac{1}{2}\alpha(t + 4000)\right\}} = \exp(2000\alpha).$$

$$\therefore\ \alpha = \frac{\ln 2}{2000}.$$

$$\therefore\ \frac{\alpha^2\text{T}_0{}^2}{32\pi^2} \doteq \frac{0{\cdot}5 \times 16}{4 \times 10^6 \times 320},$$

$$= 6 \times 10^{-9}.$$

Hence the effect of damping is negligible.

Kater's reversible pendulum.—A determination of 'g' accurate to one or two parts in a million is very difficult—perhaps impossible—with a simple pendulum because, although this may be constructed true to figure, slight variations in the density of the material of the bob displace the centre of gravity of the latter by an amount which cannot be determined with the necessary precision. In 1790 PRONY suggested using a compound pendulum provided with three knife-edges and developed a theory which would enable the length of the simple equivalent pendulum for a given period to be calculated from the times of swing and the positions of the knife-edges relative to the centre of gravity of the pendulum. Now although Prony was acquainted with Huygens' theorem concerning the reciprocal

nature of the centres of suspension and of oscillation for a compound pendulum, it does not appear that he realized the importance of it as providing a means for determining the length of a seconds pendulum. BOHNENBERGER (1811) pointed out this important deduction, but it was left to KATER† to realize again this important property of a compound pendulum and to put it to practical use. The pendulum he used is shown diagrammatically in Fig. 5·05(a). It consisted of a brass bar somewhat more than a metre long, 1·5 in. wide, and $\frac{1}{8}$ in. thick. It was provided with two knife-edges K_1 and K_2 turned towards each other, but on opposite sides of the centre of gravity of the complete pendulum. Near to the knife-edge K_2 there was rigidly fixed a cylindrical brass weight, B, of mass 2 lb. 7 oz., its diameter being 3·5 in., while it was 1·25 in. thick. The knife-edges were made in India of a special variety of steel called 'wootz.' They were made as hard as possible and tempered by immersing them in boiling water. The straightness of their edges was tested by holding them against a flat surface; they were considered straight if no streak of light could be seen coming through between the edge and the flat surface.

The ends of the pendulum were provided with brass plates about 6 in. long and 0·75 in. thick—a side view of the upper portion of the pendulum is shown in Fig. 5·05(b). They were screwed to the brass bar and extended beyond it. The gaps thus formed served to hold pieces of deal wood, each of which carried at its extremity a thin piece of whalebone which was used in estimating the amplitude of the pendulum, so that a correction could be applied to its period

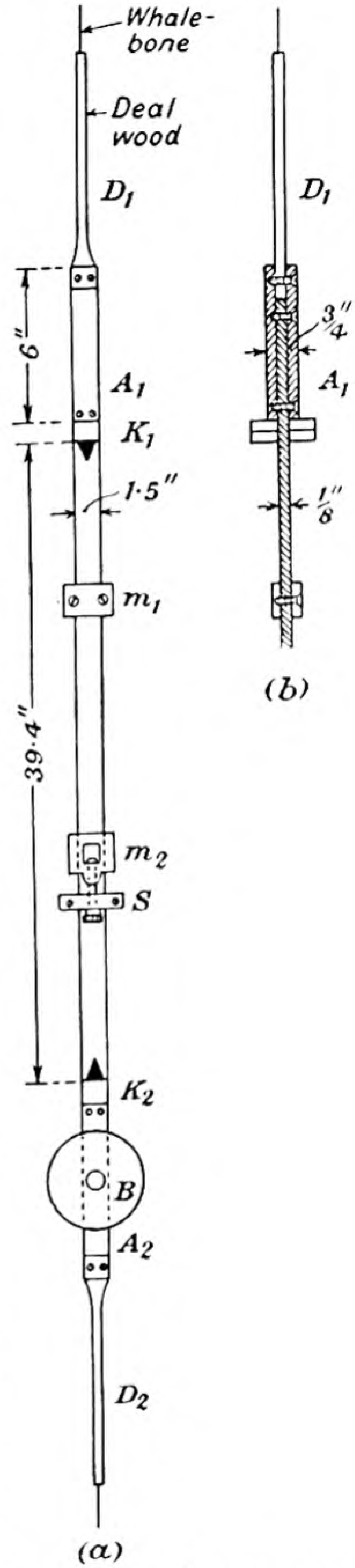


FIG. 5·05.—Kater's reversible pendulum.

† Kater, *Phil. Trans.*, p. 33 (1818).

owing to the fact that the arc of swing was finite. The pendulum was supported on agate plates fixed horizontally to a slab projecting from a stone wall.

Now the centres of suspension and oscillation are reciprocal, i.e. if the body be suspended on a horizontal axis passing through its centre of oscillation, its former point of suspension becomes the new centre of oscillation, and the vibrations in each position will be performed in equal times. If the times of vibration in each position are not equal, they may be made so by shifting a movable mass attached to the bar of the pendulum. In Kater's pendulum, m_1 was such a mass, about 4 oz., and m_2 a smaller mass whose position was capable of fine adjustment by means of the screw S.

Kater determined the time of vibration by the method of coincidences. The reversible pendulum was suspended in front of a standard clock to whose bob there was fixed a white circle drawn on black paper. When both pendulums were at rest the tail of the experimental pendulum just covered the white circle: both were viewed through a telescope a few feet distant. A slit was placed in the focal plane of the eye-piece of the telescope, the width of the slit being equal to that of the image of the tail, or of the white circle. A coincidence occurred when, as the two pendulums swung past their positions of rest, no image of any portion of the white circle was visible.

Kater used the knife-edges at the already mentioned fixed distance apart of 39·4 in. and adjusted the positions of m_1 and m_2 until the number of vibrations per day was independent of the knife-edge used. The mean time of swing was then corrected for amplitude, and another correction applied for the air displaced on the assumption that gravity was thereby diminished in the ratio of the mass of the pendulum in air to its mass in a vacuum. It only then remained to determine the distance between the two knife-edges in order to determine 'g,' the intensity of gravity. The value thus obtained was uninfluenced by any irregularity in the density of the material of the pendulum or by any departure of its form from that of some simple geometrical figure, for the pendulum had no particular shape.

In order to measure the distance between the two knife-edges Kater supported the pendulum in a horizontal position on a rigid piece of mahogany and stretched the pendulum with a force, applied by means of a common spring steelyard and slightly greater than the weight of the pendulum. Thus there was little error due to the fact that the pendulum was used in a vertical position while the distance between the edges was measured when the pendulum was in a horizontal position.

The final value for the length of the seconds pendulum at sea-level in the latitude of London was given by these experiments as 39·13929 in.

Bessel's theory of the symmetrical pendulum.—BESSEL'S main contribution in this field of investigation was his theory of a reversible pendulum whose external form was symmetrical. He proved that if such a pendulum is loaded at one end to lower the centre of gravity, and provided with two knife-edges, as in Kater's pendulum, one very nearly at the centre of oscillation for the other, then the length of the seconds pendulum could be derived without reference to the air effect.

LAPLACE had already shown that the knife-edges must be regarded as cylinders and Bessel proved that by inverting the pendulum the effect due to this curvature was eliminated if the knife-edges were the same; further, if they were different, the effect due to this was eliminated by interchanging the knife-edges and again determining the period of oscillation when the pendulum was swung from each. To realize such a pendulum Bessel suggested that two cylinders should be attached to the bar of the pendulum, the axes of the cylinders being normal to the axis of the bar and parallel to each other. The cylinders were placed to be symmetrical with respect to the centre of the bar, but one was to be solid while the other was to be hollow. The theory of this pendulum is as follows.

Let κ be the radius of gyration for the pendulum about an axis through its centre of gravity and parallel to the knife-edges. If G, the centre of gravity of the pendulum, is at distances r_1 and r_2 from the knife-edges, and T_1 and T_2 the periods appropriate to them, then

$$\frac{g}{4\pi^2}.T_1^2 = \frac{r_1^2 + \kappa^2}{r_1}, \quad \text{and} \quad \frac{g}{4\pi^2}.T_2^2 = \frac{r_2^2 + \kappa^2}{r_2}.$$

$$\therefore \frac{g}{4\pi^2}(r_1T_1^2 - r_2T_2^2) = r_1^2 - r_2^2,$$

or

$$\frac{4\pi^2}{g} = \frac{r_1T_1^2 - r_2T_2^2}{r_1^2 - r_2^2} \quad . \quad . \quad . \quad \text{(i)}$$

Let

$$\frac{r_1T_1^2 - r_2T_2^2}{r_1^2 - r_2^2} = \frac{A}{r_1 + r_2} + \frac{B}{r_1 - r_2},$$

where A and B are independent of r_1 and r_2. Then,

$$A = \frac{T_1^2 + T_2^2}{2} \quad \text{and} \quad B = \frac{T_1^2 - T_2^2}{2}.$$

$$\therefore \frac{4\pi^2}{g} = \frac{r_1T_1^2 - r_2T_2^2}{r_1^2 - r_2^2} = \frac{1}{2}.\frac{T_1^2 + T_2^2}{r_1 + r_2} + \frac{1}{2}.\frac{T_1^2 - T_2^2}{r_1 - r_2}.$$

Now $(r_1 + r_2)$ is the distance between the knife-edges, so that the first term in the above expression may be calculated from the observations. Moreover, $(T_1^2 - T_2^2) \rightarrow 0$, since the knife-edges

are so placed that the periods are approximately equal; hence the second term in the above expression is small compared with the first, and the quantity $(r_1 - r_2)$ need not be known very accurately. It is sufficient to determine the position of the centre of gravity by balancing the pendulum on a knife-edge, and measuring r_1 and r_2 in the usual manner. In this way the series of trials made by Kater before obtaining exact equality for the periods when the pendulum is swung in turn from each knife-edge is avoided.

Let T be the period of a simple pendulum of length $(r_1 + r_2)$. Then

$$T^2 = \frac{4\pi^2}{g}(r_1 + r_2) = \tfrac{1}{2}(T_1^2 + T_2^2) + \frac{1}{2}\cdot\frac{T_1^2 - T_2^2}{r_1 - r_2}(r_1 + r_2)$$

$$= \frac{r_1 T_1^2 - r_2 T_2^2}{r_1 - r_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

We shall call T the computed time. This relation is useless for calculating T from the observations, since it assumes r_1 and r_2 to be known accurately, but it is needed in developing the theory given in the following paragraphs.

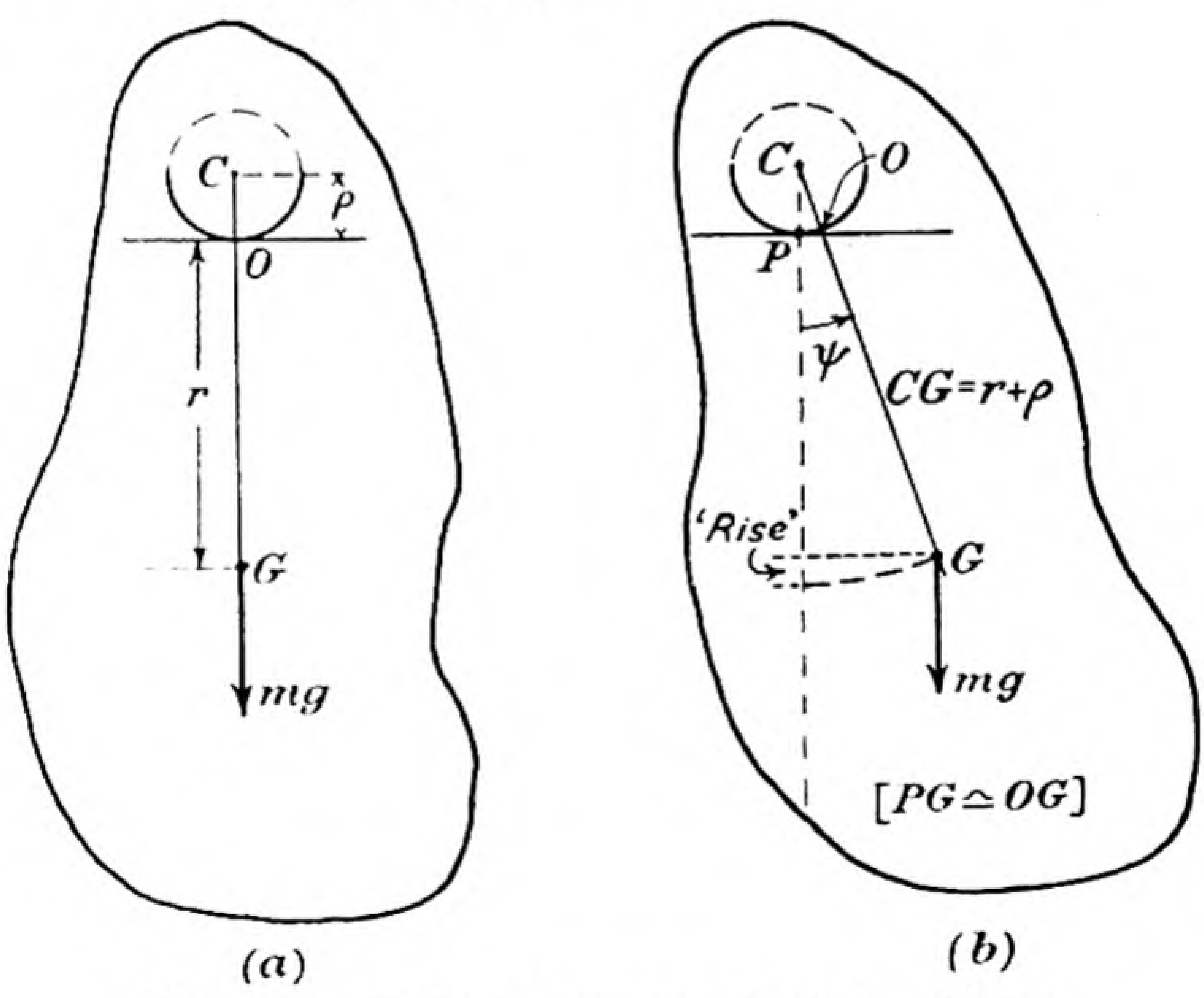


FIG. 5·06.—Effect of curvature of a knife-edge on the period of a pendulum.

To investigate the effect due to the curvature of a knife-edge, let us assume that this has a small but constant radius of curvature ρ. Fig. 5·06(a) shows such a pendulum in its position of static equilibrium. Thus if G is the centre of gravity of the pendulum it will

lie below O, the point of contact of the knife-edge with the horizontal plate on which it rests. When the pendulum undergoes a small angular displacement ψ, the instantaneous axis of rotation will pass through the point P, Fig. 5·06(b), where P is directly below C in its new position. During this displacement the centre of gravity G will have risen a distance

$$\mathrm{CG}(1 - \cos\psi) = (r + \rho)(1 - \cos\psi),$$

if $\mathrm{OG} = r$. Hence, if m is the mass of the pendulum and g the intensity of gravity, the potential energy of the pendulum in its displaced position is

$$mg(r + \rho)(1 - \cos\psi),$$

if this energy is considered zero when the pendulum is in its position of static equilibrium.

Now in so far as it is correct to assume that $\mathrm{PG} = r$, since P and O are close together, the kinetic energy of the pendulum at the instant considered will be $\frac{1}{2}m(r^2 + \kappa^2)\dot{\psi}^2$, where κ is the radius of gyration of the pendulum about a horizontal axis through G. Since the total energy of the pendulum is constant, we have

$$\tfrac{1}{2}(r^2 + \kappa^2)\dot{\psi}^2 + g(r + \rho)(1 - \cos\psi) = \text{constant}.$$

Differentiating with respect to time, we obtain

$$(r^2 + \kappa^2)\ddot{\psi} + g(r + \rho)\sin\psi = 0.$$

When ψ is small this corresponds to a simple harmonic motion of period T, where

$$\mathrm{T} = 2\pi\sqrt{\frac{r^2 + \kappa^2}{g(r + \rho)}}\,.$$

Now when a reversible compound pendulum is used, let O_1 and O_2 be its centre of suspension and its centre of oscillation respectively; then $\mathrm{O}_1\mathrm{G} = r_1$ and $\mathrm{O}_2\mathrm{G} = r_2$, while ρ_1 and ρ_2 may be taken as the radii of curvature of the knife-edges. Let T_1 be the period when the pendulum is supported on its first knife-edge; it may conveniently be called the 'erect time'. Then

$$\begin{aligned}\frac{g}{4\pi^2}\mathrm{T}_1^{\,2} &= \frac{\kappa^2 + r_1^{\,2}}{r_1 + \rho_1}\\ &= \frac{\kappa^2 + r_1^{\,2}}{r_1}\left[1 - \frac{\rho_1}{r_1}\right]. \qquad \left[\because \frac{\rho_1}{r_1} \to 0\right].\end{aligned}$$

Similarly, for the other knife-edge, T_2, the so-called 'inverted' time, is given by

$$\frac{g}{4\pi^2}\mathrm{T}_2^{\,2} = \frac{\kappa^2 + r_2^{\,2}}{r_2}\left[1 - \frac{\rho_2}{r_2}\right].$$

In the expression for T, the 'computed' time, when this is expressed in terms of T_1, T_2, etc., and hence in terms of κ, r_1, r_2, ρ_1 and ρ_2, κ^2 may be put equal to r_1r_2, cf. p. 164, in terms which are small compared with the remaining terms. Hence

$$\frac{g}{4\pi^2}T^2 = \frac{g}{4\pi^2}\left[\frac{r_1T_1^2 - r_2T_2^2}{r_1 - r_2}\right]$$

$$= \frac{(\kappa^2 + r_1^2)\left[1 - \frac{\rho_1}{r_1}\right] - (\kappa^2 + r_2^2)\left[1 - \frac{\rho_2}{r_2}\right]}{r_1 - r_2}$$

$$= \frac{(r_1^2 - r_2^2) - (\rho_1 - \rho_2)(r_1 + r_2)}{r_1 - r_2} \qquad [\because \kappa^2 \doteqdot r_1r_2]$$

$$= (r_1 + r_2) - \frac{\rho_1 - \rho_2}{r_1 - r_2}(r_1 + r_2).$$

Suppose now that the knife-edges are interchanged, no other alteration being made. If T′ is the computed time in this instance, we have

$$\frac{g}{4\pi^2}.T'^2 = (r_1 + r_2) - \frac{\rho_2 - \rho_1}{r_1 - r_2}(r_1 + r_2).$$

Hence, we have

$$\frac{g}{4\pi^2}.\frac{(T^2 + T'^2)}{2} = (r_1 + r_2),$$

so that the terms involving the radii of curvature of the knife-edges have disappeared.

The effect of the curvature of the knife-edges may also be eliminated by having plane bearings on the pendulum, and a fixed knife-edge. Under such circumstances, $\rho_1 = \rho_2$, so that the correction term disappears. The one disadvantage arises from the fact that it is difficult to be certain that the same part of the flat plate is being used on all occasions; against this, however, must be set the facts that it is not necessary to interchange the edges, and it is claimed that the distance between the plates, i.e. $(r_1 + r_2)$, can be more accurately measured. Moreover, if a knife-edge should be damaged it may be repaired without affecting the pendulum; in the pattern as ordinarily used, the pendulum becomes a different one if the knife-edges have to be reground.

It now remains for us to investigate how the effect of the air which is pushed along by the pendulum may be eliminated without having resort to suspending the pendulum in a vacuum. Consider the pendulum in its erect position. Let μ be the mass of air displaced by the pendulum. Then the moment of the forces tending to restore the pendulum to its equilibrium position when displaced, will be

reduced by the moment of a vertical force of magnitude μg acting vertically upwards through the centre of gravity of the displaced air. Let this point be at a distance s_1 from the axis of rotation. The mass of air flowing with the pendulum will increase the moment of inertia of the latter about its axis of rotation. Let this be represented by a term $\Delta \mathrm{I}_1$. The mass of this air will not affect the moment of the restoring forces since it will be buoyed up by the surrounding air and have no effective weight. The motion of the pendulum is given by the equation

$$[m(r_1^2 + \kappa^2) + \Delta \mathrm{I}_1]\ddot{\psi} + (mr_1 - \mu s_1)g\psi = 0,$$

if the angular displacement ψ is small. Hence the period T_1 is such that

$$\begin{aligned}\frac{g}{4\pi^2}\mathrm{T}_1^2 &= \frac{m(r_1^2 + \kappa^2) + \Delta \mathrm{I}_1}{mr_1 - \mu s_1} \\ &= \frac{m(r_1^2 + \kappa^2) + \Delta \mathrm{I}_1}{mr_1}\left[1 + \frac{\mu s_1}{mr_1}\right] \\ &= \frac{r_1^2 + \kappa^2}{r_1} + \frac{\Delta \mathrm{I}_1}{mr_1} + \frac{r_1^2 + \kappa^2}{r_1}\left(\frac{\mu s_1}{mr_1}\right),\end{aligned}$$

neglecting the term $\dfrac{\Delta \mathrm{I}_1}{mr_1}\cdot\dfrac{\mu s_1}{mr_1}$, which is very small in practice.

In the inverted position, the air set in motion may have a different effect on the moment of inertia about the axis of rotation—let it be denoted by $\delta \mathrm{I}_2$. The mass of air displaced by the pendulum will still be μ: let its centre of gravity be at a distance s_2 from the axis of rotation. Then the period T_2 is given by

$$\frac{g}{4\pi^2}.\mathrm{T}_2^2 = \frac{r_2^2 + \kappa^2}{r_2} + \frac{\Delta \mathrm{I}_2}{mr_2} + \frac{r_2^2 + \kappa^2}{r_2}\left(\frac{\mu s_2}{mr_2}\right),$$

to the same degree of approximation as before. The computed time T is therefore expressed by

$$\begin{aligned}\frac{g}{4\pi^2}\mathrm{T}^2 &= \frac{g}{4\pi^2}\left[\frac{r_1\mathrm{T}_1^2 - r_2\mathrm{T}_2^2}{r_1 - r_2}\right] \\ &= r_1 + r_2 + \frac{r_1 + r_2}{r_1 - r_2}.\frac{\mu(s_1 - s_2)}{m} + \frac{\Delta \mathrm{I}_1 - \Delta \mathrm{I}_2}{m(r_1 - r_2)},\end{aligned}$$

if, in the terms which are small we put $\kappa^2 = r_1 r_2$ as before.

If the external form of the pendulum is symmetrical about a horizontal axis through its geometric centre, then $s_1 = s_2$, and $\Delta \mathrm{I}_1 = \Delta \mathrm{I}_2$, so that

$$\frac{g}{4\pi^2}\mathrm{T}^2 = (r_1 + r_2).$$

Thus the air effect is eliminated provided that the temperature and pressure of the air remain constant while the experiment is in progress. Nowadays the pendulum is caused to swing in a vacuum so that the correction for the air effect is negligible except in work of the highest precision.

Repsold's pendulum.—Although Bessel is responsible for the theory of a reversible pendulum symmetrical in its external form, yet he did not construct one. The first such pendulum appears to have been made in 1860, i.e. some years after the death of Bessel, by Repsold, who was an instrument maker. One, as used in the Indian Survey, is shown in Fig. 5·07. It consists of a symmetrical brass frame AB, to which the knife-edges K_1 and K_2 are fixed. The masses attached to this frame were in the form of solid and hollow brass cylinders which could be screwed to it. In this way, G, the centre of gravity of the whole pendulum, was caused to lie nearer to K_1 than to K_2. According to Helmert, for whom Repsold made a reversible pendulum,† the knife-edges were not interchanged, but by interchanging the positions of C and D, the effect was the same as if the knife-edges had been interchanged provided, of course, that the frame had been made truly symmetrical about a horizontal axis through its geometrical centre. In this way the distance between the knife-edges remained constant and, moreover, the same points on them were always in contact with the flat plate which carried the pendulum. The pendulum used in the Indian Survey was a half-seconds pendulum, i.e. it was easily transportable since its overall dimensions were necessarily smaller than those of a seconds pendulum.

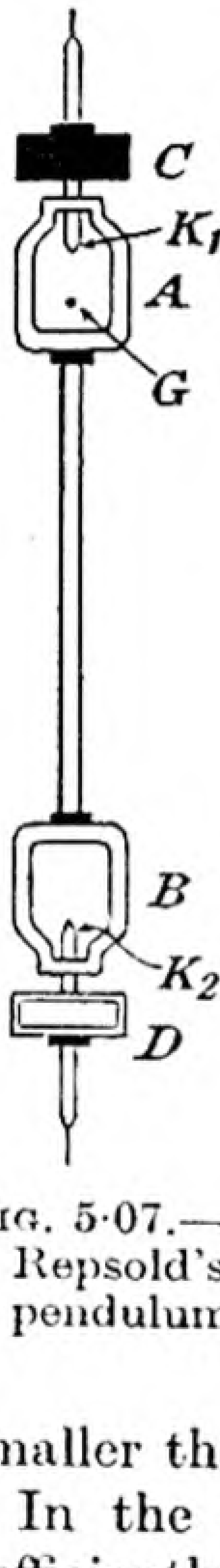


Fig. 5·07.—Repsold's pendulum.

In the earliest forms of this pendulum the support was not sufficiently rigid, so that reliable results were not possible. Perhaps this was not unfortunate, for it directed attention to the fact that corrections were necessary for the yielding of the support.

The yielding of the pendulum support.—Let G, Fig. 5·08(a), be the centre of gravity of a compound pendulum of mass m, and let O be the point of suspension. Let X and Y be the horizontal and vertical components of the force which the support exerts on the pendulum. Now the motion of the pendulum may be obtained by considering the external forces acting on it to be concentrated at its centre of gravity, the directions of these forces being unchanged—cf. Fig. 5·08(b). Call $OG = r_1$.

† *Beiträge zum Theorie des Reversionspendels*, Potsdam, 1898.

Now $r_1\dot{\psi}^2$ and $r_1\ddot{\psi}$ are the components of the acceleration of G along GO and normal to it respectively.

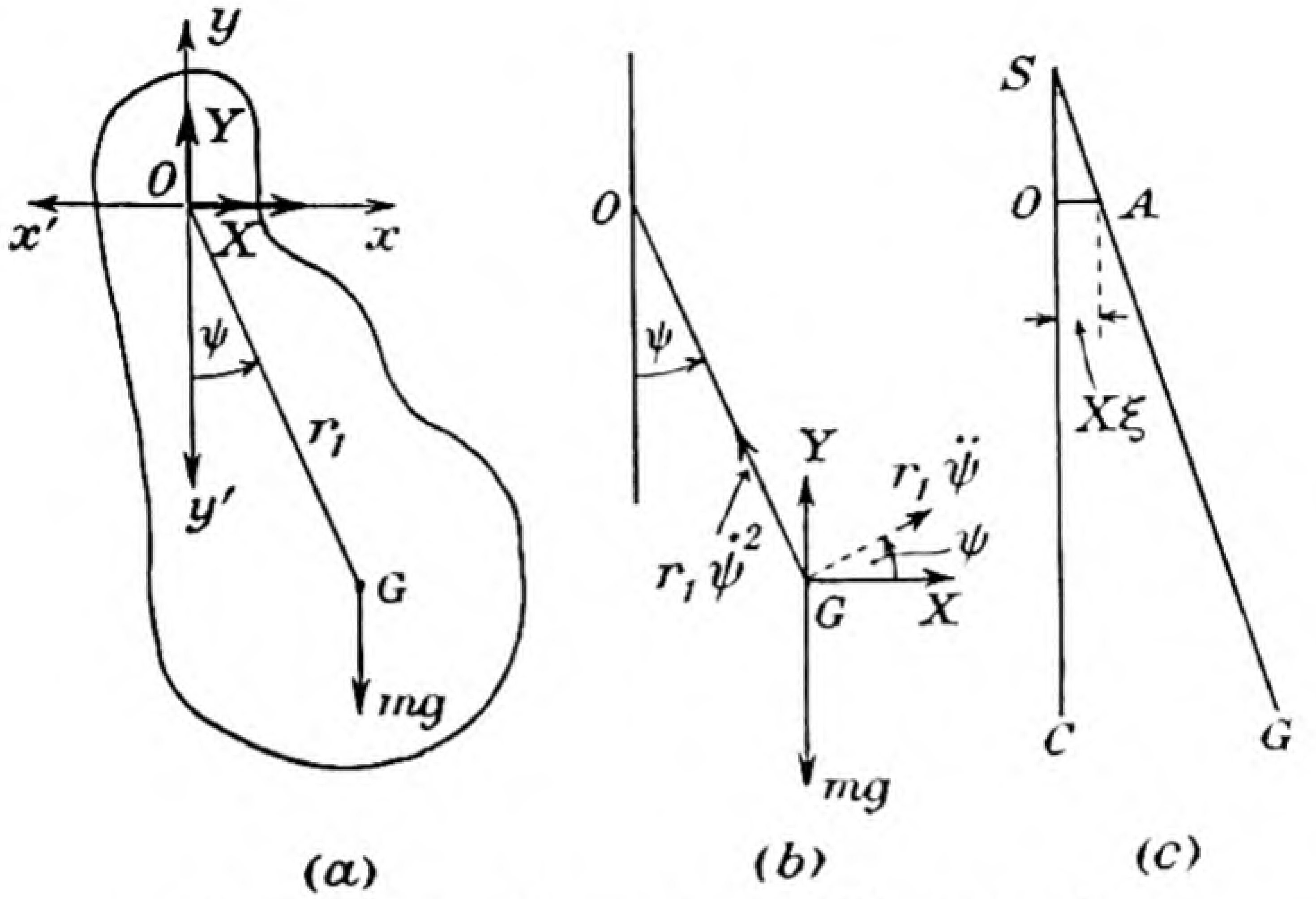


FIG. 5·08.—Pendulum with a yielding support.

$$\therefore \text{ Horizontal acceleration of G} = r_1\ddot{\psi}\cos\psi - r_1\dot{\psi}^2\sin\psi$$
$$= r_1\ddot{\psi}, \qquad \text{if } \psi \to 0.$$

Similarly, vertical acceleration of G $= r_1\dot{\psi}^2\cos\psi + r_1\ddot{\psi}\sin\psi$
$$= r_1\dot{\psi}^2, \qquad \text{if } \psi \to 0.$$

$$\therefore \text{ Force in horizontal direction} = mr_1\ddot{\psi}$$
$$= -mg\frac{r_1^{\,2}}{r_1^{\,2}+\kappa^2}\psi, \qquad \left[\because\ \ddot{\psi} = -\frac{gr_1\psi}{r_1^{\,2}+\kappa^2}\right],$$

if κ is the radius of gyration of the pendulum about a horizontal axis through G, normal to the plane of the diagram. The force is X and the minus sign indicates that it acts in the direction Ox'. Hence by Newton's third law of motion, the pendulum will exert a force of magnitude X in the direction Ox on the support. If the support is not rigid, it will be deflected from its zero position.

Let ξ be the displacement of the support in the direction Ox due to unit force acting in that direction. Let OC, Fig. 5·08(c), define the zero position of the pendulum, while AG defines its position when the amount of yielding of the support in a horizontal direction is OA, where

$$\text{OA} = |\,\text{X}\,|\,\xi \doteqdot mg\left[\frac{r_1\psi}{r_1+r_2}\right]\xi, \qquad [\because \text{ cf. p. 123 } \kappa^2 \doteqdot r_1r_2].$$

Now S, the instantaneous centre of motion, is given by the intersection of GA produced with the vertical through O. Hence

$$AS \fallingdotseq OS \fallingdotseq \frac{OA}{\psi} = mg.\frac{r_1}{r_1 + r_2}\xi,$$

or the effective distance of G from the axis of rotation is

$$r_1\left[1 + \frac{mg\xi}{r_1 + r_2}\right] = r_1 + \sigma_1 \quad \text{(say)}.$$

If T_1 is the period for the pendulum in its erect position,

$$\frac{g}{4\pi^2}T_1^2 = \frac{(r_1 + \sigma_1)^2 + \kappa^2}{r_1 + \sigma_1} = (r_1 + \sigma_1) + \frac{\kappa^2}{r_1 + \sigma_1}.$$

Similarly, in the inverted position,

$$\frac{g}{4\pi^2}T_2^2 = (r_2 + \sigma_2) + \frac{\kappa^2}{r_2 + \sigma_2}.$$

$\therefore$ T, the computed line, is given by

$$\frac{g}{4\pi^2}T^2 = \frac{g}{4\pi^2}\left[\frac{r_1T_1^2 - r_2T_2^2}{r_1 - r_2}\right]$$
$$= \frac{1}{r_1 - r_2}\left[r_1^2 + r_1\sigma_1 + \frac{\kappa^2 r_1}{r_1 + \sigma_1} - r_2^2 - r_2\sigma_2 - \frac{\kappa^2 r_2}{r_2 + \sigma_2}\right].$$

If in the terms in this expression which contain κ^2 we write, as an approximation, $\kappa^2 = r_1r_2$, we get

$$\frac{gT^2}{4\pi^2} = r_1 + r_2 + \frac{1}{r_1 - r_2}\left[r_1\sigma_1 - r_2\sigma_2 + r_1r_2\left(1 - \frac{\sigma_1}{r_1}\right) - r_1r_2\left(1 - \frac{\sigma_2}{r_2}\right)\right],$$

in so far as $\dfrac{1}{1 + \dfrac{\sigma_1}{r_1}}$ may be written $1 - \dfrac{\sigma_1}{r_1}$, etc. Hence

$$\frac{gT^2}{4\pi^2} = r_1 + r_2 + \frac{1}{r_1 - r_2}[(r_1 - r_2)\sigma_1 + (r_1 - r_2)\sigma_2]$$
$$= r_1 + r_2 + mg\xi,$$

since $\qquad \sigma_1 = \dfrac{mgr_1\xi}{r_1 + r_2}$, etc.

This expression shows that the effective length of the simple equivalent pendulum is increased, due to the yielding of the support, by an amount equal to the distance through which the weight of the pendulum, applied horizontally, would deflect the support.

It should be noted that in the above theory we have neglected the inertia of the support, and also the fact that there is a varying vertical force acting on the support. It may be shown that the effect of the varying portion of this vertical force which is of the second order in ψ is negligible in comparison with that due to the yielding in the horizontal direction.

Experimental investigation of the yielding of the support for a simple pendulum.—First let us regard a pendulum consisting of a light string and a bob as a compound pendulum. If l is the distance from the support to the centre of gravity of the bob, which point is also the centre of gravity of the pendulum, then, with the usual notation,

$$(r_1 + \sigma_1) + (r_2 + \sigma_2) = l,$$

and

$$(r_1 + \sigma_1)(r_2 + \sigma_2) = \kappa^2 \to 0.$$

Since r_1 is finite, it follows that $(r_2 + \sigma_2) \to 0$, and since each quantity in this expression is positive, r_2 and σ_2 must each tend to zero. For this pendulum, σ_1, will equal the deflexion of the support in a horizontal direction due to a force, equal to the weight of the pendulum, acting in that direction.

The above argument shows quite clearly that although σ_1 and σ_2 must be small for the theory we have developed to apply to a compound pendulum vibrating on a yielding support, yet for a simple pendulum σ_1 may be finite since r_2 is necessarily very small.

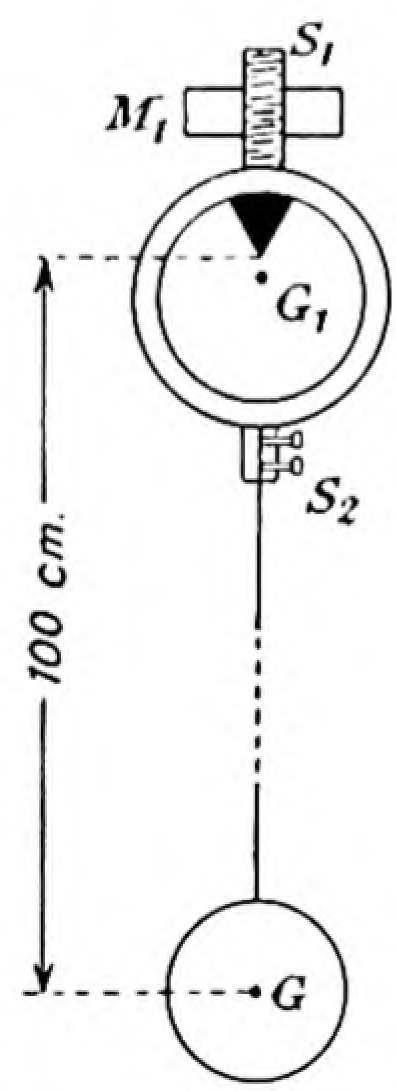


FIG. 5·09.—simple pendulum on a yielding support.

To determine σ_1 experimentally, a simple pendulum is constructed in the following manner. Fig. 5·09 shows the pendulum, in which the frame carrying the knife-edge consists of a circular ring about 10 cm. in diameter, the metal rim being 1 cm. thick and 1 cm. wide. To this rim there is fixed the knife-edge and vertically above this a screw S_1 carrying a movable mass M_1. Below this there is a pair of metal jaws with clamping screws S_2 to carry the fine wire and the bob of the pendulum. This bob should have a mass of about 5 kgm.

First remove the wire and bob and adjust the position of M_1 until the period of the frame carrying the knife-edge is approximately equal to the period of the pendulum to be constructed. Let this be a seconds pendulum. When the period of the frame is large its centre of gravity G_1 will be just below the knife-edge but the position for any given period must be determined experimentally.

The wire and bob are then placed in position, the distance from the knife-edge to the centre of gravity of the bob being made about 100 cm. Then the period of the pendulum will be about one second —and under the conditions stipulated this period will not be affected by the presence of the frame carrying the knife-edge.

Now let the period of the pendulum be determined, using the method indicated on p. 128 when it is mounted, (*a*) on a rigid support, (*b*) on a yielding support. This latter may consist of a brass bar, 50 cm. by 3 cm. by 0·5 cm., clamped horizontally in a vice so that the frame carrying the pendulum may be supported on the narrow edge of the bar.

Let T_1 and T_2 be the periods of the pendulum in these two instances. If r is the length of the pendulum, and σ the increase in this length due to the presence of a non-rigid support, then

$$T_1^2 = \frac{4\pi^2}{g} r, \qquad \text{and} \qquad T_2^2 = \frac{4\pi^2}{g}(r + \sigma),$$

so that σ may be determined. The value so obtained should be compared with the deflexion of the support due to a horizontal force equal to the weight of the pendulum.

MODERN WORK ON THE ABSOLUTE MEASUREMENT OF GRAVITY

Introduction.—One of the main reasons for determining, in absolute units, the intensity of gravity is that the earth's gravitational attraction on a body of known mass provides a convenient standard of force. Again, in using a current balance to measure a current in absolute units, the force of attraction between two coils carrying a current is balanced against the gravitational pull on a known mass. Also, in an experiment to determine the gyromagnetic ratio of the proton, it is necessary to balance the force on a coil carrying a current against a similar gravitational force. For these and similar applications the value of g at the site should be known to 1 or 2 parts in 10^5, but in the calculation of pressure from a barometric height the error in g should not exceed 1 part in 10^6, i.e. 1 p.p.m. This very high accuracy is needed to improve the precision with which the international scale of temperature can be reproduced, for one of the chief limitations at present is the measurement of the pressure at which water is boiling in the apparatus used to establish the 100° C. point. If this temperature is to be known to 10^{-4} deg.C., the pressure must be correct to 3 p.p.m. Unfortunately, at the present time, the difference in the value of 'g' as measured at two standardizing laboratories does not agree with the difference as recorded by a gravimeter standardized at two other laboratories.

Viewed in the light of modern achievements the results obtained by Kater are valueless, for in his day the effects of the surrounding air and the elasticity of the pendulum and its supports were not well understood.

The Clark reversible pendulum.—A value for the intensity of gravity at Potsdam was made by KÜHNEN and FURTWÄNGLER in 1906. The reversible pendulums used by these investigators had knife-edges attached to the pendulum rods so that much difficulty was experienced in measuring the precise distances between these edges. Subsequent experiments have confirmed the view that the length measurements could be carried out with much higher precision if the knife-edges formed a part of the pendulum support and the reversible pendulum itself carried two flat and parallel surfaces which rested in turn on the knife-edges. Since the distance between such surfaces can now be measured interferometrically with high precision, CLARK (1938) designed and used a suitable pendulum. It is now known that the earth's magnetic field has an appreciable effect on a pendulum made of invar steel; Clark therefore used a pendulum rod made of a non-magnetic light alloy. It is shown diagrammatically in Fig. 5·10(*a*). The rod was machined out of a solid block of forged aluminium alloy so that it had an I-section. The ends of the pendulum were reinforced so that rectangular blocks could be attached; these reinforcements were cut away at A_1 and A_2 so as to permit the insertion of the knife-edge on which the pendulum swings. The heavier block was made of three pieces, C, D and E. C is identical with the lighter block B; each of these has its opposite faces *aa* and *bb* ground and lapped flat, after being chromium plated, and then made parallel. One of these two plated surfaces *aa* on B serves as the plane which rests on the knife-edge K and the other serves as a plane mirror by means of which the amplitude of the pendulum is observed. The block D is reduced in length by repeated trial until the pendulum is found to have very nearly the same period whether it is swinging with the heavy end or the light end above the knife-edge; these arrangements are shown in Fig. 5·10(*b*) and (*c*). The lower face of E was plated on one surface only, viz. *cc*, so that it provided a plane mirror for determining the amplitude when the pendulum is in the inverted position. Each of the blocks B and C is bored transversely so that the pendulum may rest on a suitable support until it is ready to be lowered on to the knife-edge.

The support for the pendulum is a truncated pyramidal iron casting. The pendulum itself was enclosed in a high vacuum; the residual pressure was measured with the aid of a McLeod and a Pirani gauge, cf. 705 et seq. Three platinum thermometers were used to determine the mean temperature of the pendulum rod. A precision chronograph

was available for measuring time correct to 0·0002 second and a suitable electrical device, which can be brought into action when required for a short time at the beginning and end of any period of

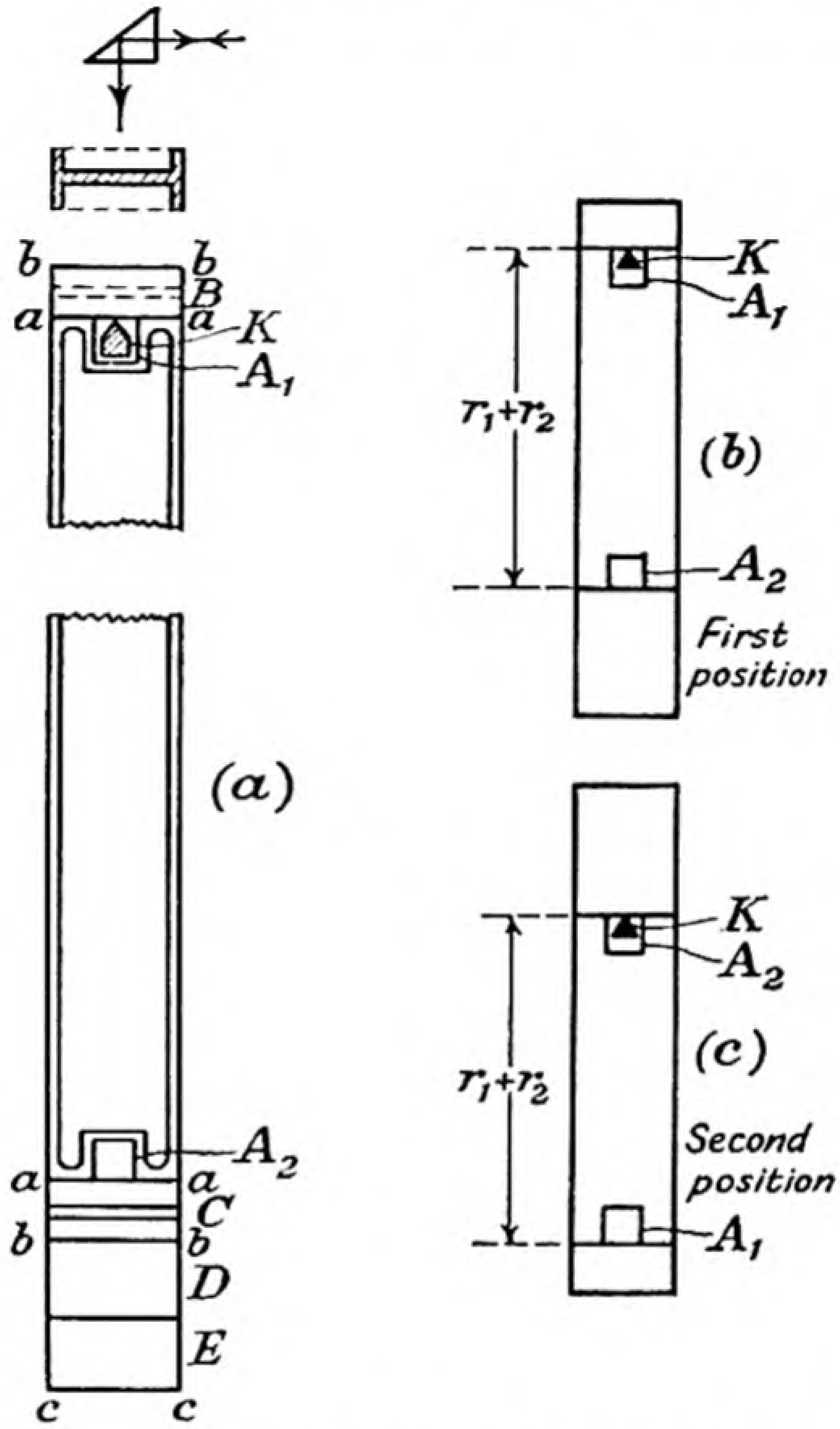


FIG. 5·10.—Clark's reversible pendulum. [N.P.L.]

time which it is desired to measure, enabled a value for the mean period to be determined with very high precision in a much shorter time than had hitherto been possible.

The knife-edge was made of hardened steel; many such edges were used, the most 'sharp' having a radius less than 20 μ (0·0008 in.). Clark found that the apparent values of g, viz. g_ρ, were related to the value of ρ by the equation

$$g_\rho = g_{\rho=0}[1 + 0{\cdot}65\rho \times 10^{-7}],$$

where ρ is the radius of curvature of the knife-edge in microns. A value for $g_{\rho=0}$ was determined by using the above equation. Clark attributes the increase of g with ρ to an irreversible loss of energy caused by the friction between the knife-edge and plane.

Corrections were made for the change in length of the rod due to temperature variations, the departure from zero amplitude, the effect of residual air pressure on the period and non-uniformity in the rate of the standard clock.

The length of the pendulum, i.e. $(r_1 + r_2)$ was measured by Barrell using a wavelength comparator, cf. Vol. III, p. 556, and the values of r_1 and r_2 were determined by balancing the pendulum horizontally on a length of steel wire 0·36 cm. in diameter; a steel metre scale enabled these distances to be measured with sufficient accuracy since the last term in the equation

$$\frac{8\pi^2}{g} = \frac{T_1^{\,2} + T_2^{\,2}}{r_1 + r_2} + \frac{T_1^{\,2} - T_2^{\,2}}{r_1 - r_2},$$

which was used in these calculations, cf. p. 163, is very small compared with the term involving $r_1 + r_2$.

Clark found

$$g_{\text{N.P.L.}} = 981{\cdot}1815 \text{ cm.sec.}^{-2}$$

with a possible error of $\pm 1{\cdot}6$ mgal., cf. p. 185. Jeffreys has recalculated the elasticity correction which Clark had applied with the wrong sign and gives

$$\begin{aligned} g_{\text{N.P.L.}} &= 981{\cdot}1832 \text{ cm.sec.}^{-2} \\ &= 981183{\cdot}2 \text{ mgal.} \end{aligned}$$

Since 1946 AGELETSKI and EGOROV have used three fused silica pendulums and their results are in good agreement with those obtained by Clark.

The long pendulum.—The idea of the long pendulum method is that the work done by forces at the support is much reduced in comparison with the total energy of the system. IVANOFF, 1936, used pendulums 20 and 30 metres long. To correct for the mass m of the wire let us consider Fig. 5·11 in which a sphere of diameter 2R and mass M is shown supported at the end of the wire of length L; G_1 and G_2 are the centroids of the sphere and wire respectively. If O is the point of suspension, the moment of inertia of the system about a horizontal axis through O and normal to the plane of the diagram is

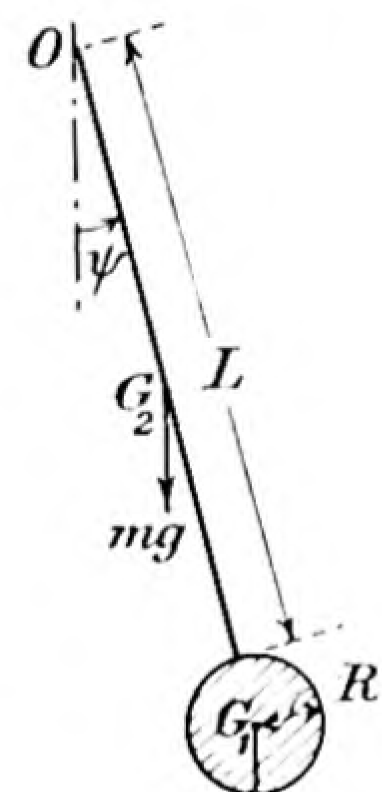


FIG. 5·11.—Ivanoff's long pendulum.

$$M\left[\frac{2}{5}R^2 + (L + R)^2\right] + m\left[\frac{1}{3}\left(\frac{L}{2}\right)^2 + \left(\frac{L}{2}\right)^2\right] = I, \quad \text{(say)}.$$

If ψ is the small angular displacement of the pendulum from its position of static equilibrium the equation of motion is

$$\left[M\left\{\frac{2}{5}R^2 + (L+R)^2\right\} + m\frac{1}{3}L^2\right]\ddot{\psi} + \left\{m\frac{L}{2} + M(L+R)\right\}g\psi = 0,$$

so that the motion is simple harmonic with a period

$$T = 2\pi\sqrt{\frac{(L+R) + \frac{2}{5}\frac{R^2}{L+R} + \frac{m}{M}.\frac{L^2}{3}.\frac{1}{L+R}}{\left(1 + \frac{m}{2M}.\frac{L}{L+R}\right)g}}.$$

Hence l, the length of the simple equivalent pendulum, is given by

$$l = \frac{L + R + \frac{2}{5}\frac{R^2}{L+R} + \frac{m}{M}.\frac{L^2}{3}.\frac{1}{L+R}}{1 + \frac{m}{2M}.\frac{L}{L+R}}$$

$$\simeq \left[(L+R) + \frac{2}{5}.\frac{R^2}{L+R} + \frac{m}{M}.\frac{L^2}{3}.\frac{1}{L+R}\right]\left[1 - \frac{m}{2M}.\frac{L}{L+R}\right]$$

$$= L + R + \frac{2}{5}.\frac{R^2}{L+R} + \frac{m}{M}\left[\frac{L^2}{3(L+R)} - \frac{L}{2} - \frac{1}{5}.\frac{R^2L}{(L+R)^2}\right]$$

$$= \quad \dots \quad + \frac{m}{M}\left[\frac{1}{3}(L-R) + \frac{R^2}{3(L+R)} - \frac{1}{2}L - \frac{1}{5}\frac{R^2L}{(L+R)^2}\right]$$

$$= \quad \dots \quad - \frac{m}{M}\left[\frac{3L - 2L + 2R}{6} - R^2\left\{\frac{5(L+R) - 3L}{15(L+R)^2}\right\}\right]$$

$$= \quad \dots \quad - \frac{m}{M}\left[\frac{L+2R}{6} - \frac{2}{15}\frac{R^2(L+\frac{1}{2}R)}{(L+R)^2}\right]$$

$$= L + R + \frac{2}{5}.\frac{R^2}{L+R} - \frac{m}{M}\left[\frac{L+2R}{6} - \frac{2}{15}.\frac{R^2}{L+R}\right].$$

In these experiments the wire was clamped rigidly to the support so that it was necessary to apply a correction for the elasticity of the wire. This correction could have been much reduced if advantage had been taken of a device due to Biot who employed a knife-edge with a free period equal to that of the pendulum itself. Now there is

no reason why the distance L could not be measured to 1 in 10^6 by a tape hanging beside the wire but this requires that the mean temperature of the wire should be known to within 0·1 deg.C. The pendulum used by Ivanoff was mounted in a double-walled tube through which water circulated. The spheres were non-magnetic but unfortunately a steel wire was used so that small forces arising from its presence in a magnetic field may have caused errors which are by no means negligible. Ivanoff found

$$g_{\text{Leningrad}} = 981{\cdot}928 \text{ gal.}$$

The rotating liquid method.—In Chap. XI p. 611 it is shown that if a liquid rotates about a vertical axis with uniform angular velocity ω, then the free surface of the liquid assumes the form of a paraboloid of revolution with a focal length f, where

$$f = \frac{g}{2\omega^2}.$$

Now it may be shown that the disturbing effect of surface tension on the shape of the above surface is very small and confined to the annulus which is close to the boundary. With these known facts to hand MEDI has proposed to determine the value of g from measurements on the focal length of the paraboloidal surface formed when mercury is rotating as described above. The focal length will probably be of the order 10 cm. and although such a length may be measured to 1 p.p.m. yet the focus may not be sufficiently well defined for this to be possible. In this proposed experiment, contamination of the mercury surface may destroy its regular outline and so render meaningless the use of the term 'point focus' in connexion with it. A Michelson interferometer is to be used for this purpose while the angular velocity will be measured by means of a pulse generator giving 10,000 pulses per revolution and fitted to the axis of the disc.

No accurate results for this method have yet been reported and this is not surprising when other sources of trouble, as first reported by WOOD (1909), are recalled. Wood attempted to use the surface of mercury in a rotating dish for astronomical purposes but found that there was a periodic change of focus due to periodic variations in the speed of rotation and this caused ripples to spread out from any irregularity in the sides of the dish. Wood also found that the axis of rotation must be truly vertical and the base of the dish flat and perfectly horizontal, for otherwise a rotating wave will develop in the mercury.

Methods using the free fall of a body.—In 1946 VOLET suggested that the value for g might be determined by high speed photography of a line standard falling freely. His method was improved by RIECKMANN who used a plain bar coated with a

photographic emulsion and on this images of a fine slit illuminated by timed flashes of light were formed. MARTSINYAK has made a similar determination using a quartz rod similarly coated.

The definition of the time interval in these experiments is quite precise since lamps with flashes lasting for less than 0·2 μsec. are available; they are controlled by means of a quartz oscillator. It is doubtful, however, whether or not the distance can be measured so precisely. In addition, the temperature of the bar must be known and forces other than that of gravity may be exerted on the bar; they will arise from its motion in magnetic and electric fields and from the air itself.

The falling bar experiments suffer from the following weaknesses; the two images on the photographic plate of the illuminated slit are not equally well defined and distances are not measured interferometrically. COOK, at the National Physical Laboratory, is therefore carrying out experiments on the following lines.

Let a ball be thrown vertically upwards and let the times at which it crosses two planes separated by a distance s be measured. Let Δt_1 be the time between successive passages (up and down) across the lower plane and Δt_2 the corresponding interval for the upper plane. If s_0 is the height above the upper plane when the ball is momentarily at rest, we have

$$s + s_0 = \tfrac{1}{2}g[\tfrac{1}{2}(\Delta t_1)]^2,$$

and

$$s_0 = \tfrac{1}{2}g[\tfrac{1}{2}(\Delta t_2)]^2.$$

$$\therefore\ g = \frac{8s}{(\Delta t_1)^2 - (\Delta t_2)^2}.$$

The method, in principle, has two main advantages:—

(*a*) The up and down passages across a plane occur with the same velocity so that Δt is the difference in time between two events occurring with equal sharpness.

(*b*) The planes defining the length are stationary and their separation can be measured interferometrically.

To investigate any effect on the sphere due to its motion in a viscous medium, let distances measured downwards be denoted by x. If the force due to viscosity is directly proportional to the velocity, the equation of motion for the downward flight is

$$m\ddot{x} = mg - \kappa\dot{x},$$

or

$$\ddot{x} + \alpha\dot{x} = g,$$

where κ and α are constants. Let the origin of time and of height be taken at the highest point of the trajectory, i.e. at $t = 0$, $x = 0$ and $\dot{x} = 0$.

Writing $\dot{x} = v$, we have

$$\dot{v} + \alpha v = g,$$

so that using $\exp \int \alpha\, dt = \exp \alpha t$ as an integrating factor we get

$$v = \exp(-\alpha t)\left[\frac{g}{\alpha} \exp \alpha t - \frac{g}{\alpha}\right].$$

since the constant of integration is $-\dfrac{g}{\alpha} \exp(-\alpha t)$.

Integrating again we get

$$x = \frac{g}{\alpha^2} \exp(-\alpha t) + \frac{gt}{\alpha} + \mathrm{B},$$

where $\mathrm{B} = -\dfrac{g}{\alpha^2}$, since at $t = 0$, $x = 0$.

$$\begin{aligned}\therefore\ x &= \frac{g}{\alpha^2}\{\exp(-\alpha t) - 1\} + \frac{gt}{\alpha} \\ &= \frac{g}{\alpha^2}\left\{-\alpha t + \frac{\alpha^2 t^2}{2!} - \frac{\alpha^3 t^3}{3!} + \ldots\right\} + \frac{gt}{\alpha} \\ &= g\left\{\frac{t^2}{2!} - \frac{\alpha t^3}{3!} + \frac{\alpha^2 t^5}{5!} - \ldots\right\}. \qquad \text{(i)}\end{aligned}$$

Let t_0 be the time required for the body to descend a distance x in vacuo, i.e. $t_0 = \sqrt{\dfrac{2x}{g}}$, and $t_0 + \tau$ be the time to descend the same distance in a slightly viscous medium. If α is small $\tau \ll t_0$. Substituting in (i) we get

$$\tfrac{1}{2}gt_0^2 = g\{\tfrac{1}{2}(t_0 + \tau)^2 - \tfrac{1}{6}\alpha(t_0 + \tau)^3 + \ldots\},$$

i.e. neglecting terms in the second order in α and τ,

$$\tfrac{1}{2}gt_0^2 = g\{\tfrac{1}{2}(t_0^2 + 2t_0\tau) - \tfrac{1}{6}\alpha t_0^3\}$$

or

$$\tau = \frac{\alpha t_0^2}{6}.$$

Thus

$$t_0 + \tau \simeq t_0\left(1 + \frac{\alpha t_0}{6}\right).$$

Similarly, for an upward flight, the time is approximately

$$t_0\left(1 - \frac{\alpha t_0}{6}\right),$$

since the equation of motion is

$$\ddot{z} = -g - \alpha\dot{z},$$

if z is measured vertically upwards, i.e. $x = -z + \mathrm{A}$, where A is a constant, so that

$$\ddot{x} - \alpha\dot{x} = g.$$

Hence, to the first order in α, the total time of flight is

$$2\left(\frac{2x}{g}\right)^{\frac{1}{2}},$$

which is independent of α, i.e. of the small damping effect.

To investigate the effect of variation of gravity with height, we take our origin for t and x as before, and assume that if gravity has a value g at $x = 0$, then at a distance x below its value is

$$g + \beta x,$$

where β is 0·309 mgal.metre^{-1}, or $0{\cdot}31 \times 10^{-3}$ cm.sec.$^{-2}$metre^{-1}, i.e. $0{\cdot}31 \times 10^{-5}$ sec.$^{-2}$. We therefore have to solve the equation

$$\ddot{x} = g + \beta x, \quad \text{or} \quad \ddot{x} - \beta x = g.$$

The primitive is

$$x = \text{A} \cosh (\beta^{\frac{1}{2}}t + \phi),$$

where A and ϕ are constants, while the particular integral is

$$\frac{g}{\text{D}^2 - \beta} = -\frac{g}{\beta}. \qquad \text{[cf. p. 35]}$$

Since $\dot{x} = 0$ at $t = 0$, $\phi = 0$; again $x = 0$ at $t = 0$ so that $\text{A} = \dfrac{g}{\beta}$ and the appropriate solution is therefore

$$x = \frac{g}{\beta}.\cosh \beta^{\frac{1}{2}}t - \frac{g}{\beta}. \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(ii)}$$

In this equation let $t_0 + t'$ be the time to descend when variation in gravity is taken into account. Substituting $t = t_0 + t'$ in (ii) we have, since $t' \ll t_0$ and $\beta \ll g$,

$$\tfrac{1}{2}gt_0^{\,2} = \frac{g}{\beta}\left\{\frac{\beta(t_0 + t')^2}{2!} + \frac{\beta^2(t_0 + t')^4}{4!} + \ldots\right\}$$

$$\simeq g\{\tfrac{1}{2}(t_0 + 2t_0t') + \tfrac{1}{24}\beta t_0^{\,4}\},$$

if second order terms in β and t' are neglected.

$$\therefore \; t_0t' \simeq -\tfrac{1}{24}\beta t_0^{\,4}, \quad \text{or} \quad t' \simeq -\frac{\beta t_0^{\,3}}{3!}.$$

$$\therefore \; t_0 + t' = t_0\left(1 - \frac{\beta t_0^{\,2}}{4!}\right).$$

Similarly, for the upward flight, the time required is also $t_0 + t'$. Hence to the first order in β, the time of flight is

$$\Delta t = 2t_0\left(1 - \frac{\beta t_0^{\,2}}{24}\right) = 2\sqrt{\frac{2x}{g}}\left(1 - \frac{\beta x}{12g}\right).$$

$$\therefore \; (\Delta t_1)^2 - (\Delta t_2)^2 = \frac{8(x_1 - x_2)}{g}\left\{1 - \frac{\beta}{g}.\frac{x_1 + x_2}{6}\right\},$$

i.e. the value of gravity determined is that corresponding to a point $\frac{1}{6}(x_1 + x_2)$ below the highest point of the trajectory.

A ball seems the only practical form of body to use for any rotation given to it when it is projected upwards will not affect the detection of its passage across the reference planes. A ball of highest quality optical glass is therefore used so that forces due to magnetic and electric fields are negligible and to detect the position of the centre of the ball relative to a plane to better than 1 μ the following procedure is adopted.

The essential features of the apparatus are shown in Fig. 5·12(*a*); further details are shown in the frontispiece. G_1 and G_2 are glass blocks with their opposing faces truly parallel and horizontal; the distance between these faces is determined interferometrically. One of these blocks is shown in greater detail in Fig. 5·12(*b*). S and S′ are fine slits and as a glass sphere passes across the plane containing them an image of S is focussed on S′ when the ball is symmetrically between the slits. Thus a flash of light falls on a photomultiplier placed behind the second slit and the time interval between successive signals from the photomultiplier are measured. To do this the current from the photomultiplier is amplified and the time interval measured on a counter in terms of the period of a 100 kilocycle.sec.$^{-1}$ standard frequency signal. This standard oscillator is started and stopped by the ball in flight. To measure the fractional part of a period both at the beginning and end of a flight by the sphere a differentiator is used in conjunction with a cathode ray tube; in such a way the positions of maxima on the main curve become zero on the differential curve and these can be located much more accurately than any maxima.

By means of a crossbow-type catapult, the ball is put into flight in a high vacuum. To eliminate all corrections for change in temperature the separation of the opposing faces of G_1 and G_2 is measured during the time the ball is in motion. For this purpose the separation is expressed in terms of the optical path in a 20 cm. étalon. To understand the principle involved suppose that two pairs of plates with semi-reflecting surfaces are set up as shown in Fig. 5·12(*c*) so that the separation S of one pair of plates is a small multiple n of the separation s of the other pair. If a collimated beam of light is incident normally on the plates, light which undergoes two reflexions in the first pair will be retarded by the same amount as light which is reflected $2n$ times in the second pair. The two beams give rise to interference fringes and since the path difference is nominally zero, the fringes will be seen in white light. Such are Brewster's fringes, cf. Vol. III, p. 533, and when they are observed visually they are sometimes indistinct since the two interfering beams have different intensities and the fringes have to be observed over a background

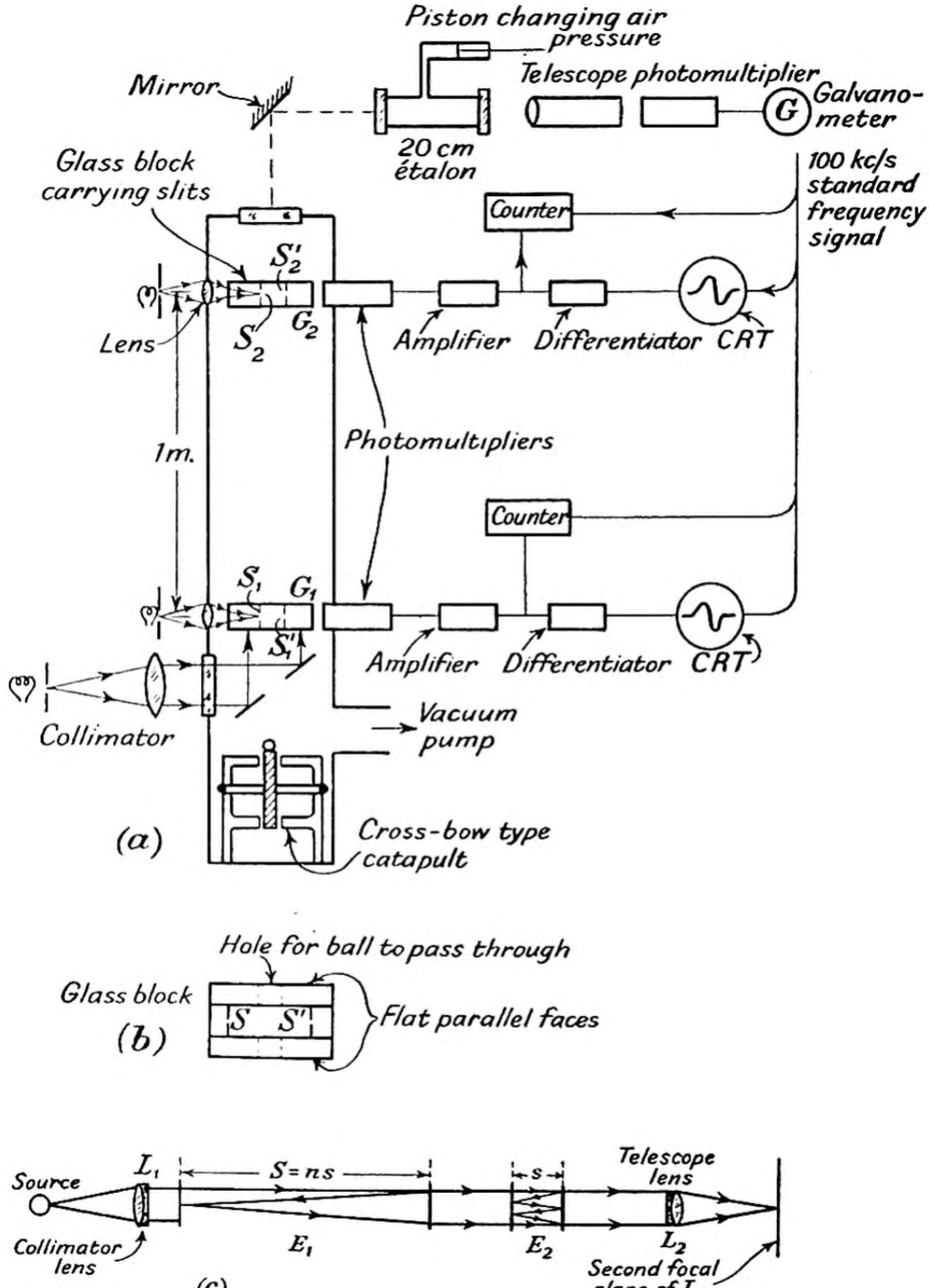
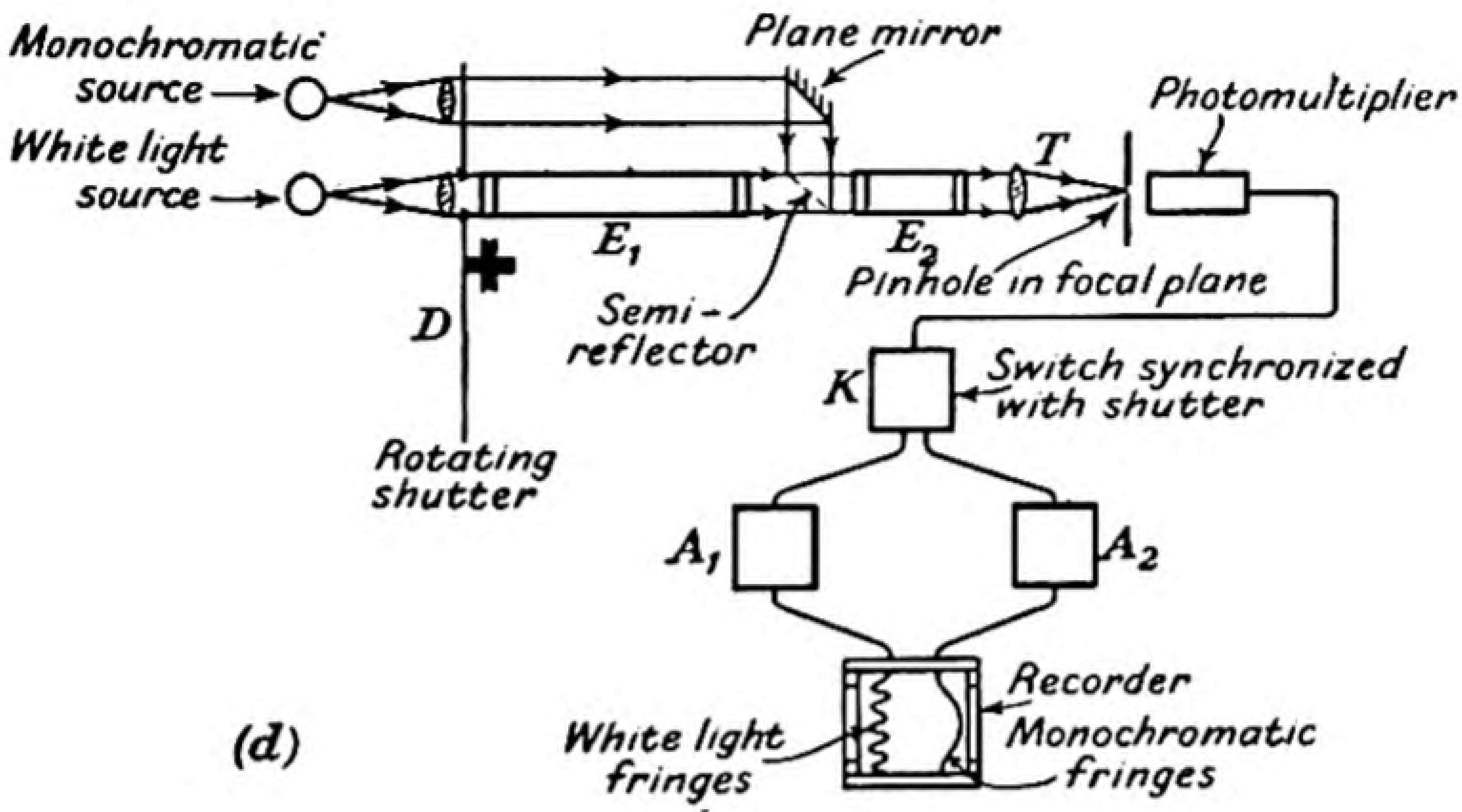


FIG. 5·12.—Cook's precision method for the determination of gravity [N.P.L.]

which is not dark. COOK and RICHARDSON (1959) examined the fringes photoelectrically and at the same time brought the path differences concerned to the correct ratio by altering the pressure, and therefore the refractive index, of the air in the shorter étalon.



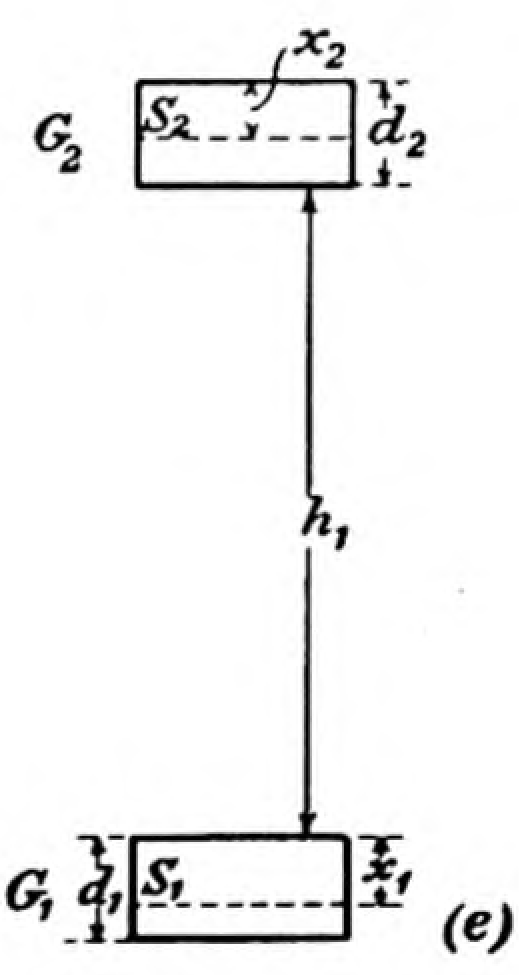


FIG. 5·12.

The essential principles of the optical system are shown in Fig. 5·12(d). The longer étalon had a length of one metre; the axis of the 20 cm. étalon was parallel to that of an observing telescope as defined by a pinhole, diameter 0·8 mm., in its focal plane. The incident light was interrupted at 200 cycle.sec.$^{-1}$ by rotating a sector disc shutter D. The light passing through the pinhole fell on a photomultiplier and the output from this was amplified and rectified to give a direct current proportional to the intensity of the light received by the pinhole. If this system is to be used to measure the

longer étalon in terms of the standard wavelength, the optical path of the shorter étalon must be known in terms of this standard at the pressure for which the central white light fringe falls on the pinhole; this necessitates measuring the pressure with an error not exceeding a few thousandths of a millibar. Such a difficult operation is avoided by measuring the length of the shorter étalon in terms of the standard wavelength simultaneously with the white light comparison, for then the error in measuring the pressure may be as large as one millibar since the value of the pressure is only required to obtain the integral part of the order of interference in the short étalon.

The output of the multiplier is connected alternately to two amplifiers A_1 and A_2 by a switch K synchronized with the shutter obscuring the sources so that the output of one amplifier is proportional to the intensity of the white light and that of the other to the intensity of the monochromatic light. The two outputs are recorded simultaneously and from the combined record it is possible to get the order of interference in monochromatic light corresponding to the central white fringe without needing to know the air pressure very precisely.

The distance between the horizontal planes containing the slits cannot be measured directly and so the following procedure is adopted. Let h_1 be the distance between the nearer surfaces of the two blocks G_1 and G_2; this is the distance measured interferometrically when they are arranged as in Fig. 5·12(e). Then if d_1 and d_2 are the thicknesses of the blocks and x_1 and x_2 the distances of the slits from the upper surfaces of the respective blocks,

$$s_1 = h_1 + x_1 + (d_2 - x_2),$$

where s_1 is the vertical distance between the slits.

When G_1 is inverted, the distance between the slits becomes s_2, where

$$s_2 = h_2 + (d_1 - x_1) + (d_2 - x_2),$$

if h_2 is the separation of the nearer surfaces of the two blocks under these conditions.

When G_2 is also inverted, we have

$$s_3 = h_3 + (d_1 - x_1) + x_2,$$

while when G_1 is restored to its initial position and G_2 alone is inverted

$$s_4 = h_4 + x_1 + x_2$$

$$\therefore\ s_1 + s_2 + s_3 + s_4 = 4\bar{s} \text{ (say)}$$

$$= h_1 + h_2 + h_3 + h_4 + 2d_1 + 2d_2,$$

i.e.

$$\bar{s} = \bar{h} + \tfrac{1}{2}(d_1 + d_2),$$

where $$4\bar{h} = h_1 + h_2 + h_3 + h_4.$$

Now the expression for g in terms of s, Δt_1 and Δt_2 may be written

$$g[(\Delta t_1)^2 - (\Delta t_2)^2] = 8s,$$

i.e. $$g[\text{Mean value of } \{(\Delta t_1)^2 - (\Delta t_2)^2\}] = 8[\text{Mean value of } s] = 8\bar{s}.$$

This is the formula from which g is finally evaluated. The final result of these experiments is not yet (July, 1960) available.

GRAVITY SURVEYS

Introduction and early work.—A gravity survey has for its main object a determination of gravity at various points on a sea-level surface which is defined as that surface bounding the ocean in the absence of all tides. Nearly all instruments measure the vertical component of the earth's gravitational field at a station and it is this vertical component which is denoted by 'g' and termed ***gravity***. A subsidiary object is to ascertain from the results obtained how matter is distributed below the earth's surface for it is now a well-established fact that small local variations in gravity, after small corrections, cf. p. 197, have been applied, are intimately connected with definite geological structures almost immediately below the area surveyed.

The earlier workers in this field used reversible pendulums and made absolute determinations of gravity. At the present time, instruments, known as ***gravimeters***, enable small changes in gravity to be carried out with much smaller expenditure of time and trouble than is incidental to all experiments with reversible pendulums.

The gal and milligal.—In recent years an acceleration of 1 cm.sec.$^{-2}$ has been termed a ***gal***. In geophysics it is customary to express small differences in the intensity of gravity in terms of a unit known as a milligal. Thus

$$1 \text{ mgal.} = 10^{-3} \text{ gal} = 10^{-3} \text{ cm.sec.}^{-2}$$

Gravimeters.—There are to-day, three basic types of instrument for determining small variations in gravity; they depend upon dynamic, gas-pressure or static principles. Later on it will be convenient to divide the static type of gravimeter into two classes, called respectively ***astatized*** and ***unastatized***; formerly they were called unstable and stable but since it is only when an instrument is about to become unstable and not when actual instability has set in

that it can be used, the more precise terms stated above have been introduced.

Dynamic gravimeters.—All pendulums may be used as dynamic gravimeters but they are not very sensitive to small variations in gravity. Many modern dynamic gravimeters make use of the HOLWECK–LEJAY inverted pendulum, the essential theory of which is as follows.

Let G, Fig. 5·13(a), be the centre of gravity of a rigid and uniform

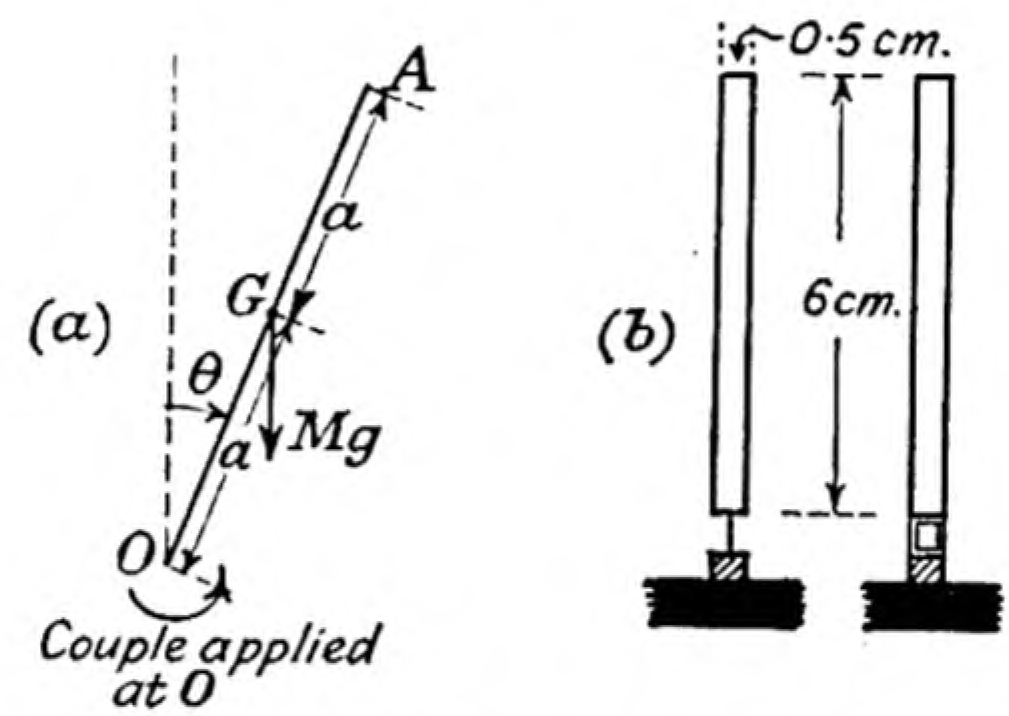


FIG. 5·13.—A dynamic gravimeter.

rod OA of length $2a$ and mass M, and let its axis make an angle θ with the vertical when a couple expressed by $C\theta + D$, where C and D are constants, acts upon the rod; its sense is such that the couple tends to decrease θ.

When the rod is in equilibrium let the couple be $C\theta_0 + D$; then

$$Mga \sin \theta_0 = C\theta_0 + D,$$

where θ_0 is the equilibrium value of θ and g is gravity.

$$\therefore \; D = Mga \sin \theta_0 - C\theta_0$$
$$= (Mga - C)\theta_0,$$

if θ_0 is small.

The equation of motion, for small angular oscillations, is

$$I\ddot{\theta} = Mga\theta - (C\theta + D),$$

or

$$I\ddot{\theta} + (C - Mga)(\theta - \theta_0) = 0.$$

Since $\ddot{\theta} = \dfrac{d^2}{dt^2}(\theta - \theta_0)$, a solution to the above equation is

$$\theta - \theta_0 = A \sin \sqrt{\frac{C - Mga}{I}}\, t,$$

if $\theta = \theta_0$ at time $t = 0$ and A is an integration constant. Thus the oscillation is about $\theta = \theta_0$ with a period given by

$$T = 2\pi\sqrt{\frac{I}{C - Mga}}.$$

Differentiating logarithmically, we obtain

$$\frac{\delta T}{T} = \frac{1}{2}\frac{Ma}{C - Mga}.\delta g$$

$$= \frac{1}{2}\frac{Mga}{C - Mga}.\frac{\delta g}{g}.$$

For a simple pendulum

$$\frac{\delta T}{T} = -\frac{1}{2}\frac{\delta g}{g}.$$

Thus the sensitivity of the inverted pendulum exceeds that of a simple pendulum of the same period by a factor of $\left|\frac{Mga}{C - Mga}\right|$. This ratio can be made very large by chosing the elastic constants of the lamina, which supports the rod and exerts a couple on it when the rod is displaced, so that $C - Mga \rightarrow 0$. In practice, however, $C - Mga$ cannot be made too small for then the lamina becomes so thin that a twist in it may develop easily; under such conditions the motion of the rod is no longer simple harmonic.

Lejay found that the above fraction should not exceed a value of 200; with such a value, to detect a change in gravity of 1 milligal, i.e. $\frac{\delta g}{g} \simeq 10^{-6}$, the corresponding change in a time interval, T_0, will be $10^{-4}\,T_0$. Thus, if the time interval is 1,000 seconds this need only be measured to within 0·1 second to give the required accuracy.

The pendulum designed by Holweck–Lejay consists of a rod of quartz, 6 cm. long and 0·5 cm. diameter, supported at its lower end by a thin lamina made of elinvar. Its width is equal to the diameter of the quartz rod but its thickness is only a few thousandths of a centimetre. Fig. 5·13(*b*) shows two side views of the pendulum one of which reveals the fact that part of the elinvar spring lamina is removed in order to reduce the couple per unit angular displacement which this lamina exerts, when bent, on the quartz rod. This consists of two parts; one forms the main part of the pendulum while the other, which is much lighter enables the period to be adjusted to a suitable value before the pendulum is finally enclosed in a glass bulb, which is highly exhausted. This allows the pendulum to oscillate for a sufficiently long time interval for accurate observations on the periodic time to be made.

To eliminate electrostatic effects the surface of the quartz rod is silvered and the pendulum itself lies within a Faraday cylinder.

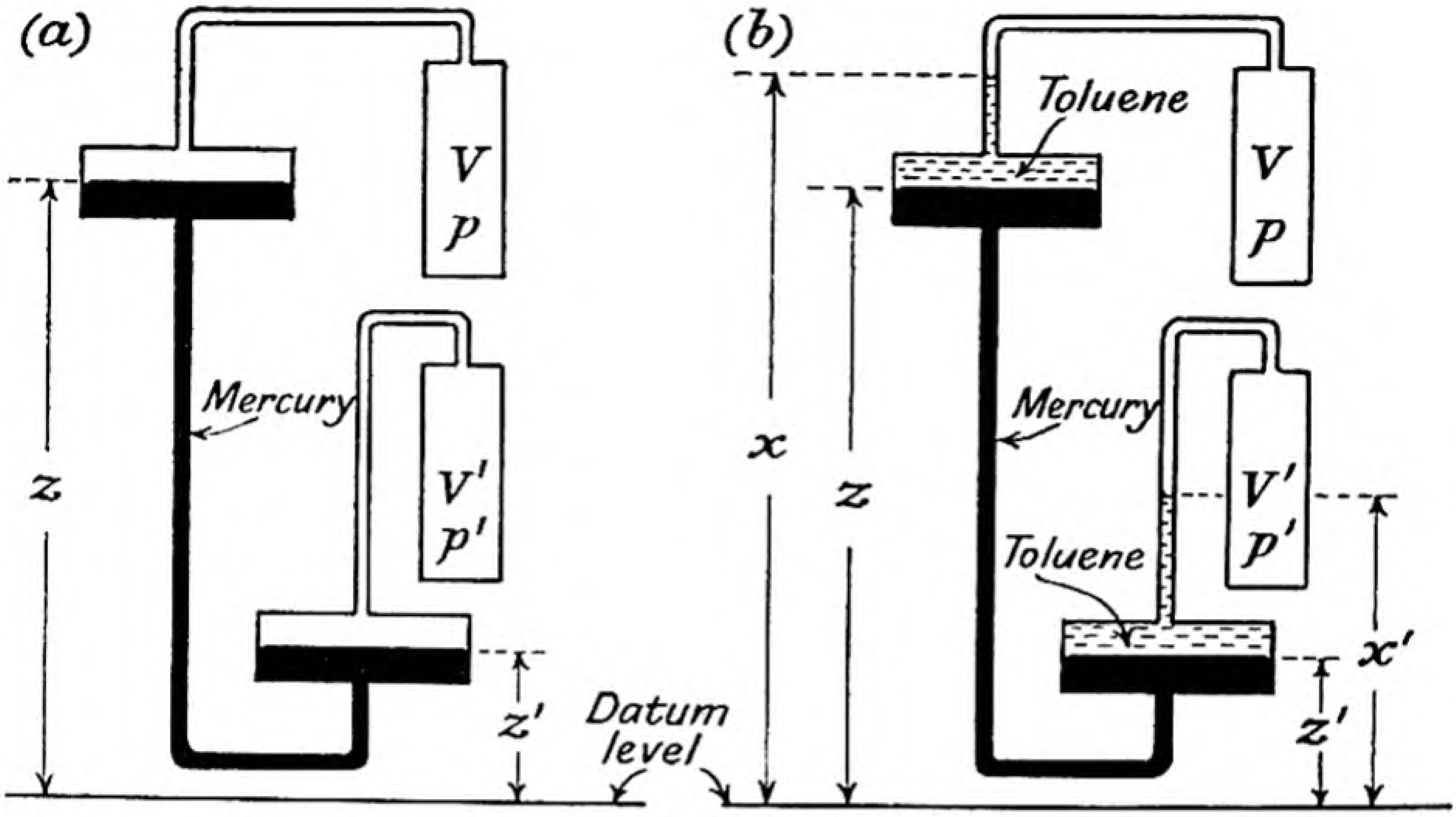


FIG. 5·14.—Principle of Haalck's gas-pressure gravimeter.

The instrument is calibrated by erecting it at two stations where the values for gravity are accurately known and observing the time period at each. The formula

$$T = 2\pi\sqrt{\frac{I}{C - Mga}}$$

may be written

$$g = g_0 - \frac{\alpha}{T^2},$$

where g_0 and α are constants. Thus while observations at two stations serve to standardize the instrument, it is always advisable to check the calibration from observations made at other stations where absolute values for gravity are known.

Gas-pressure gravimeters.—In the gas-pressure gravimeter designed by HAALCK in 1935, use is made of the elastic properties of a gas to exert a force which will oppose one due to gravity. To appreciate its principles let us consider, cf. Fig. 5·14(a), two glass vessels of volumes V and V′ respectively, filled with a gas at pressures p and p' and connected by a system of tubes filled with mercury. Let z and z' be the heights of the mercury surfaces above a datum level and suppose that the temperature is invariable. Then if ρ is the density of mercury, we have

$$p + g\rho(z - z') = p',$$

which, on differentiating, gives

$$\delta p + g\rho(\delta z - \delta z') + \rho(z - z')\,\delta g = \delta p'.$$

Since pV is constant

$$\delta p = -\frac{p}{V}\,\delta V,$$

with a similar expression for $\delta p'$. If S is the cross-sectional area of each tube at the upper and lower levels of the mercury column

$$\delta V = -S\,\delta z \quad \text{and} \quad \delta V' = -S\,\delta z'.$$

$$\therefore\ \frac{pS}{V}\,\delta z + g\rho(\delta z - \delta z') + \rho(z - z')\,\delta g = \frac{p'S}{V'}\,\delta z'.$$

$$\therefore\ \delta g = \frac{1}{\rho(z - z')}\left[\left(\frac{p'S}{V'} + g\rho\right)\delta z' - \left(\frac{pS}{V} + g\rho\right)\delta z\right]$$

$$= \frac{g}{p' - p}\left[\left(\frac{p'S}{V'} + g\rho\right)\delta z' - \left(\frac{pS}{V} + g\rho\right)\delta z\right].$$

Haalck then increased the sensitivity of this gravimeter by placing toluene, density σ, on top of each mercury surface and arranging for the toluene surfaces in contact with the gas to be in capillary tubes as shown in Fig. 5·14(b). Let x and x' be the heights of these surfaces above the datum level and s the cross-sectional area of each capillary. Then

$$p + g\sigma(x - z) + g\rho(z - z') = p' + g\sigma(x' - z').$$

$$\therefore\ \left\{\begin{aligned}&\delta p + \delta g\,.\,\sigma(x - z) + g\sigma(\delta x - \delta z)\\ &\quad + \delta g\,.\,\rho(z - z') + g\rho(\delta z - \delta z')\end{aligned}\right\} = \left\{\begin{aligned}&\delta p' + \delta g\,.\,\sigma(x' - z')\\ &\quad + g\sigma(\delta x' - \delta z')\end{aligned}\right\}.$$

As before

$$\delta p = -\frac{p}{V}\,\delta V \quad \text{and} \quad \delta p' = -\frac{p'}{V'}\,\delta V',$$

and also

$$\delta V = -s\,\delta x = -S\,\delta z,$$

and

$$\delta V' = -s\,\delta x' = -S\,\delta z'.$$

$$\therefore\ \left\{\begin{aligned}&-\frac{p}{V}\,\delta V + \delta g\,.\,\sigma(x - z) + g\sigma\left(1 - \frac{s}{S}\right)\delta x\\ &\qquad + \delta g\,.\,\rho(z - z') + g\rho\,.\,\frac{s}{S}\,(\delta x - \delta x')\\ &\qquad - \delta g\,.\,\sigma(x' - z') - g\sigma\left(1 - \frac{s}{S}\right)\delta x'\end{aligned}\right\} = -\frac{p'}{V'}\,\delta V'.$$

$$\therefore \frac{p}{V} s\,\delta x - \frac{p'}{V'} s\,\delta x' + \delta g\{\sigma(x - z) + \rho(z - z') + \sigma(x' - z')\}$$

$$= g\left[\sigma\left(\frac{s}{S} - 1\right)\delta x + \rho \,.\, \frac{s}{S}(\delta x' - \delta x) - \sigma\left(\frac{s}{S} - 1\right)\delta x'\right]$$

$$= g\left[\sigma\left(\frac{s}{S} - 1\right) - \rho\frac{s}{S}\right](\delta x - \delta x').$$

$$\therefore\ \delta g = C\,\delta x - C'\,\delta x',$$

where C and C′ are instrumental constants. By tilting the capillary tubes the instrument becomes more sensitive but even so the error is seldom less than 3 or 4 mgal. It possesses, however, one great advantage and that is its ability to record variations in gravity at sea.

Static gravimeters.—This type of gravimeter is based either on the spring-balance (unastatized) or on the astatic-balance (astatized) principle.

(*a*) ***Unastatized gravimeters.*** When a mass is suspended from a helical spring the length of the spring will change with variations in g; the greater gravity becomes, the longer is the spring. Actual changes in the length of the spring are very small and since any satisfactory gravimeter must be capable of detecting changes in g of the order of 0·1 mgal, this means that changes in length of the spring amounting to no more than one fiftieth of the wavelength of light must be measured. Optical, mechanical or electrical means must be employed to get the necessary magnification and because of these practical difficulties many years had to elapse before a marketable instrument of this type appeared.

The Hartley gravimeter.—This instrument, which is shown diagrammatically in Fig. 5·15, was designed in 1932 and was one of the first successful gravimeters. In each previous design an attempt had been made to obtain extreme sensitivity by some mechanical arrangement approaching unstable equilibrium, and one reason for inevitable failure which followed is that the unstable condition exaggerates all disturbing factors. In Hartley's instrument the design is such that there is maximum stability and the small displacements caused by changes in 'g' are amplified by optical methods.

The fundamental principle is as follows. If we can apply to a suspended mass an upward force almost, but not quite, equal to the gravitational attraction on it, then it will be relatively easy to measure changes in the small additional force required to hold the system in equilibrium. The main spring is made from an alloy of tungsten and tantalum and carries about 99·9 per cent of the

weight of the load. The wire used in constructing this spring has a fairly heavy gauge and the spring is not stressed to more than 20 per cent of its apparent elastic limit so as to give the maximum

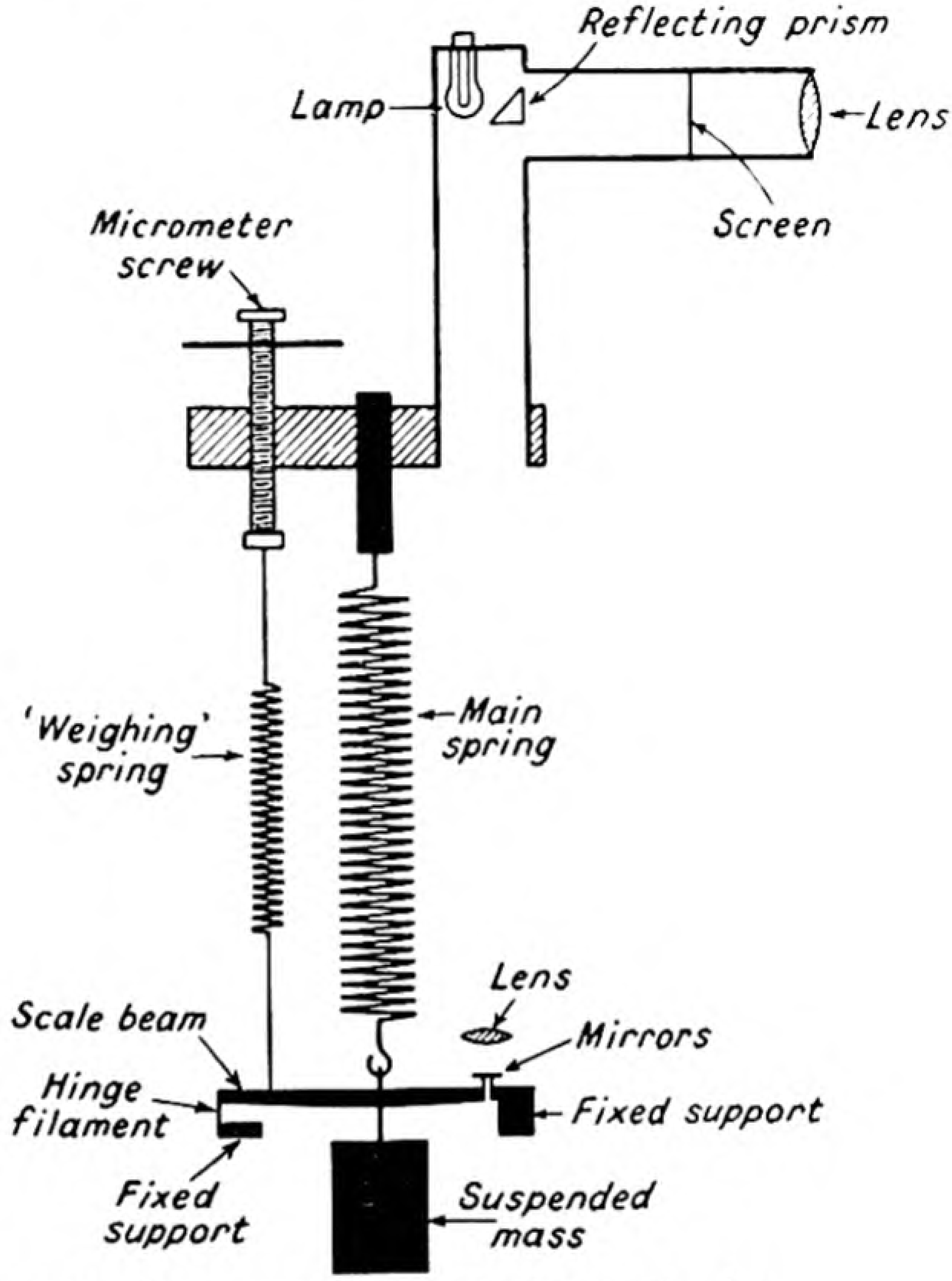


FIG. 5·15.—The principle of a Hartley gravimeter.

dependability. A very light beam is attached by suitable flexible ligaments between the spring and load. This beam is hinged, on the left, to a fixed support, and at the other end carries two small mirrors.

Although very few details concerning the method for mounting the mirrors have been published, it is probable that it operates somewhat as follows.

The beam is hinged by a ligature type of suspension at its left hand end and, except for the mounting of the mirrors is completely free at its opposite end. The mirrors, one on either side of the beam, are mounted by fine ligaments between the free end of the beam and a separate fixed support. The two ligaments supporting the mirrors are arranged differently for the mirrors on opposite sides of the beam as in Fig. 5·16(*a*) and (*b*). Hence for a downward movement of the beam, the right-hand mirror tilts anticlockwise and the left-hand

mirror clockwise. [It is probable that the ligament to the fixed support may in each instance be stronger than that to the beam and be attached to the mirror through a collar which allows a slight rotation. This may not be absolutely necessary, but would give

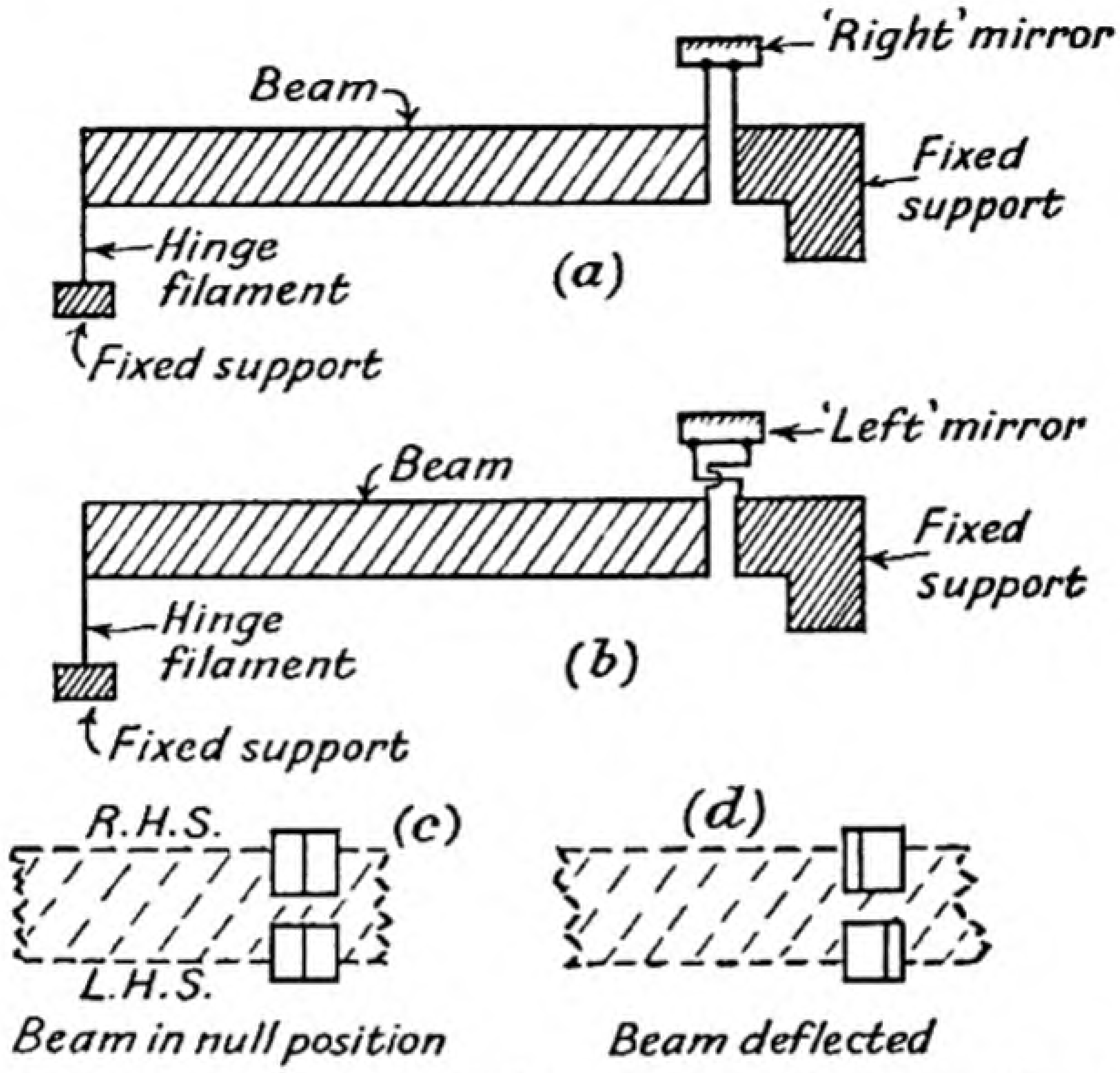


FIG. 5·16.—Some possible details of a Hartley gravimeter

greater smoothness perhaps. It may also be true that the ligament from the fixed support is attached to the centre of the mirror in each instance.]

These two mirrors form part of an optical lever and are so mounted that a vertical displacement of the beam causes them to rotate in opposite directions about horizontal axes. This not only doubles the magnification but avoids the necessity of a fiducial mark in the optical system. When the two mirrors are coplanar the two images of the lamp filament will be collinear irrespective of any small displacements in the remaining parts of the optical system. A very light 'weighing spring' is attached to the beam in the position indicated and the tension in this spring can be adjusted by means of a micrometer screw. The sum of the tensions in the two springs and the magnitude of the weight of the load and beam are such that there is tension in the hinge filament. This is found to be essential if the beam is to be stable.

Since the optical system is used only as a means for obtaining a null setting, actual measurements being made on the reading spring, slight defocusing when the beam is out of the null position is only a slight disadvantage. The changes in the field of view when g is changed are indicated in Fig. 5·16(c) and (d).

The instrument is adjusted at a certain station so that the two images of the filament are collinear. When it is taken to a second station the change in the value of the intensity of gravity will cause a slight displacement of the two images. By means of the micrometer screw this displacement is made zero; the difference between the two readings of the micrometer dial is a measure of the difference in the intensity of gravity at the two stations. If this difference is known the instrument is at once standardized.

Although the frame carrying the spring assembly is in part temperature-compensated by the use of aluminium and ingot steel rods, in practice it is found necessary to house the instrument in an air-tight cylindrical case kept at a constant temperature. The instrument will detect differences in 'g' equal to one milligal.

(*b*) ***Astatized gravimeters. Schematic design for such a gravimeter.*** Fig. 5·17 shows how a beam A and a mass M at one end are supported by two springs S and S_1 attached to a rigid framework; a frictionless pivot at P makes contact between the beam and frame. The tensions in the springs are adjusted so that

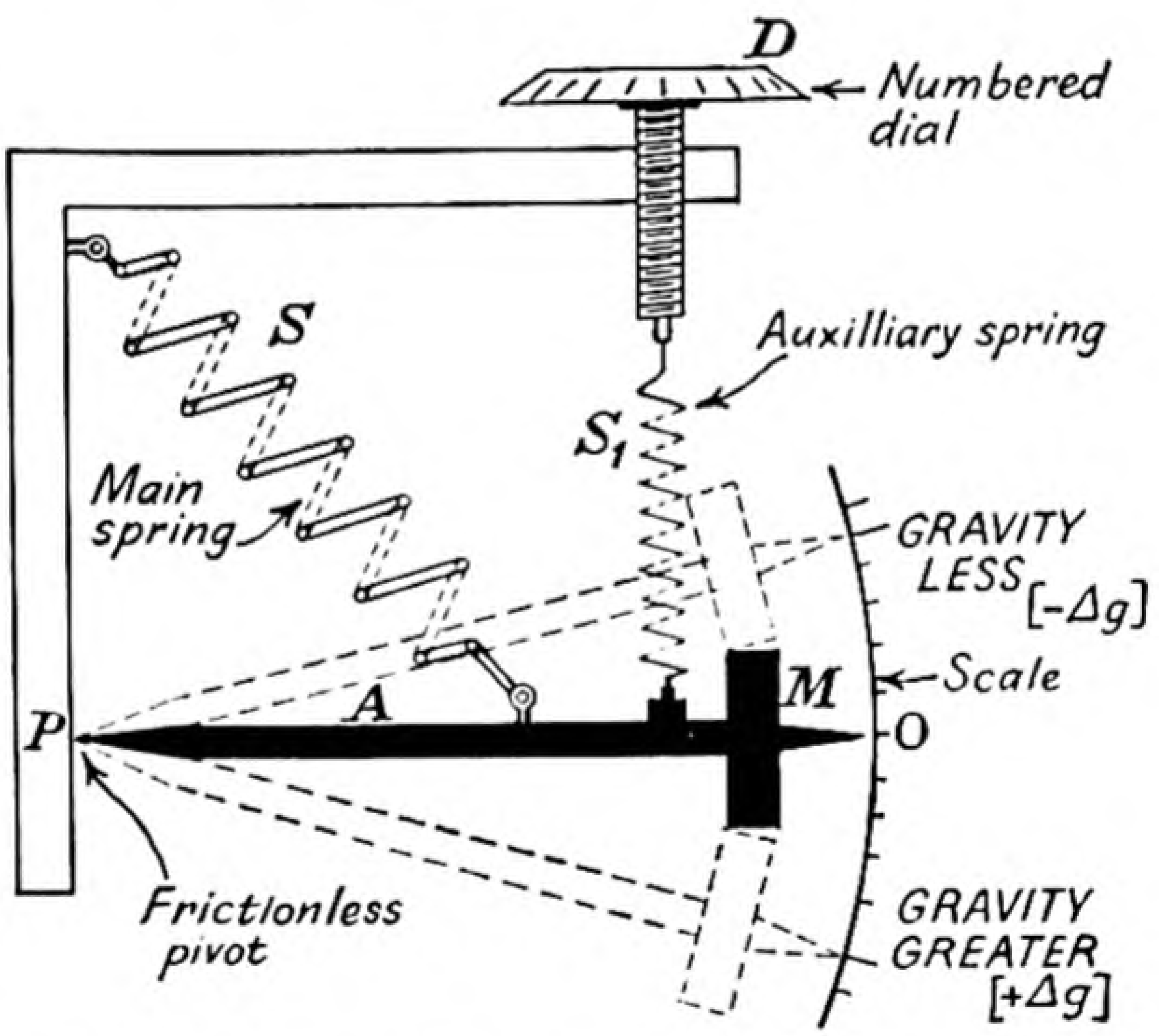


FIG. 5·17.—Schematic design of an astatized gravimeter.

a pointer attached to A reads zero when the gravimeter is at a station where an absolute value for gravity is known. When taken to another station where, in general, g will be different, the couple on the beam due to the weight of M will change and this will cause corresponding changes in the tensions in the springs. By rotating the numbered reading dial D, which controls the spring S_1, the mass

M may be restored to its original position. Provided the readings on D have been standardized, a value for Δg is easily obtained. Such a principle is the basis of a ***Worden gravimeter***, an instrument in the foremost class of its kind.

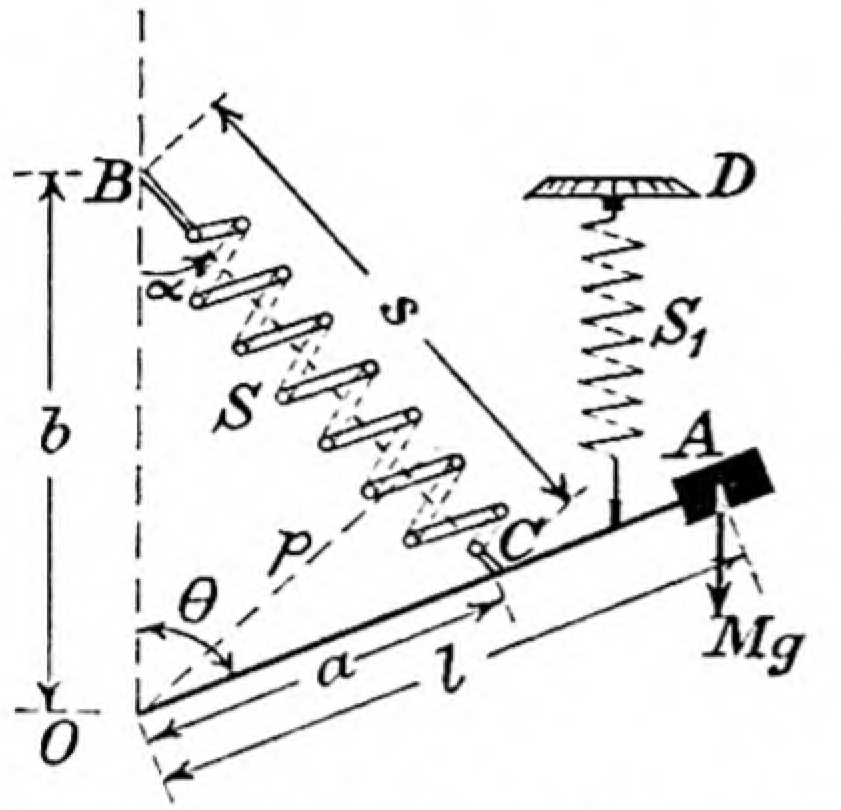


FIG. 5·18.—The principle of a Worden gravimeter. [Based on La Coste's seismograph.]

The essential theory of a Worden gravimeter.—Let the mass M be carried at the end A of a light but rigid rod or lever OA, Fig. 5·18, which is free to rotate about a horizontal axis through O; let B be a point vertically above O and to which one end of a spring of 'zero length', cf. p. 393, is fixed. The other end of this spring is attached to C, a point in OA. Then if a, b, p and s are the distances indicated, the moment of the weight of M about O is M$gl \sin\theta$, where g is gravity, l the length of the lever and $\theta = \widehat{\text{AOB}}$. The moment of the stretching force in the spring about the same point is ksp, where k is the spring constant. But from ΔBOC, we have

$$\frac{s}{\sin\theta} = \frac{a}{\sin\alpha}.$$

$$\therefore\ ksp = k.a.\frac{\sin\theta}{\sin\alpha}.b\sin\alpha = kab.\sin\theta.$$

When the system is in equilibrium let S_0 be the couple due to the auxiliary spring S_1, i.e. the total torque on the lever is

$$(kab - Mgl)\sin\theta + S_0,$$

and this is zero. In actual instruments $a = b$, so that

$$(ka^2 - Mgl)\sin\theta + S_0 = 0.$$

Now suppose that the beam is displaced slightly from its zero position, i.e. θ becomes $\theta + \beta$, where β is small. If the increase in the restoring couple due to S_1 is $Z\beta$, where Z is assumed constant, the equation of motion, if I is the moment of inertia of M about a horizontal axis through O, is

$$I\ddot{\beta} + (ka^2 - Mgl)\sin(\theta + \beta) + (S_0 + Z\beta) = 0,$$

since
$$\frac{d^2}{dt^2}(\theta + \beta) = \ddot{\beta}.$$

Expanding $\sin(\theta + \beta)$ and using the fact that

$$(ka^2 - Mgl)\sin\theta + S_0 = 0,$$

we have $$I\ddot{\beta} + \{(ka^2 - Mgl)\cos\theta + Z\}\beta = 0,$$

since $\sin\beta$ may be replaced by β when this is small. Thus the period T, of small oscillations of the system, is given by

$$T = 2\pi\left[\frac{Ml^2}{(ka^2 - Mgl)\cos\theta + Z}\right]^{\frac{1}{2}}. \quad [\because\ I = Ml^2.]$$

In passing it should be noticed that if $f(\theta)$ is the couple Γ on a system, then when θ becomes $\theta + \beta$, where β is small,

$$\begin{aligned} f(\theta + \beta) &= f(\theta) + \beta f'(\theta) \quad \text{[Taylor's theorem.]} \\ &= 0 + \beta f'(\theta), \end{aligned}$$

if the system is in equilibrium before being displaced. Since

$$\ddot{\beta} = \frac{d^2}{dt^2}(\theta + \beta),$$

the equation of motion is

$$I\ddot{\beta} + f'(\theta)\beta = 0,$$

i.e. $$T = 2\pi\sqrt{\frac{I}{\dfrac{\partial\Gamma}{\partial\theta}}},$$

which agrees with the result obtained above, when it is recalled that

$$S_0 + Z\beta \equiv S_0 + \frac{\partial S_0}{\partial\theta}\cdot\beta,$$

so that $$Z = \frac{\partial S_0}{\partial\theta}.$$

A Worden gravimeter.—The instrument itself, shown diagrammatically in Fig. 5·19, is made almost entirely of fused quartz; three such rods form part of a rectangular framework whose fourth arm consists of a small quartz rod R_1 attached to the rest of the frame by two torsion fibres also made of quartz. The arm of the rectangle opposite to that carrying R_1 is attached to a support so constructed that the inclination of the frame to the horizontal may be adjusted. This is a very desirable feature of this gravimeter for the structure of the elastic quartz system is such that its response to changes in gravity is directly proportional to the cosine of the angle of tilt and this affords an accurate means of calibration. R_2 is another quartz

rod supported by two torsion fibres and carries a long pointer P whose upper end is horizontal. This end is illuminated and it is the central fringe in its diffraction pattern, observed through the microscope M, which serves as a fiducial mark; the field of view is shown in Fig. 5·17(*b*), where the shorter lines are marks on a graticule in the eyepiece while the space between the two long thin lines represents the central white light fringe of the diffraction pattern.

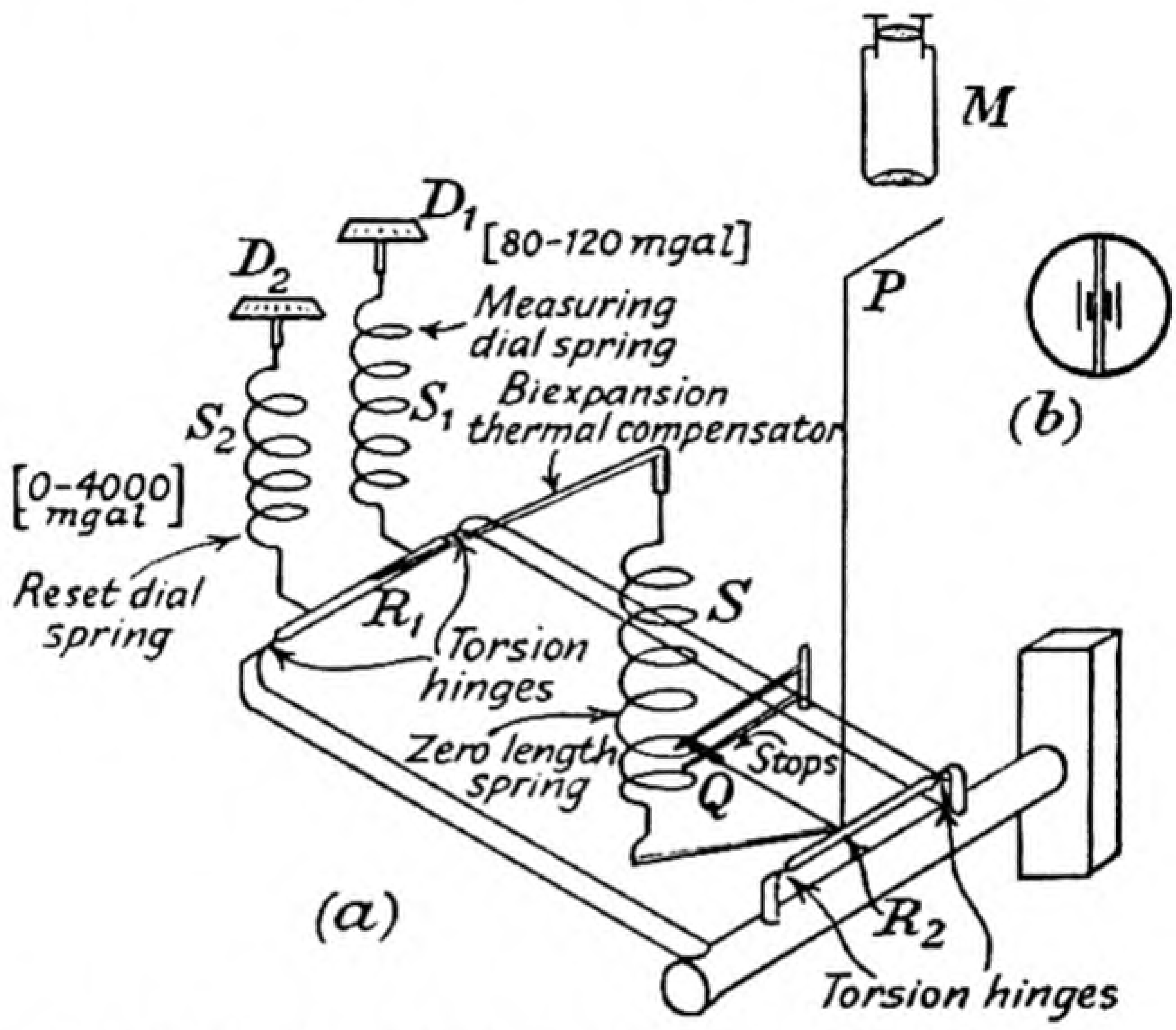


FIG. 5·19.—The structure of a Worden gravimeter; diagrammatic and not to scale.

The quartz load Q is carried by a lever rigidly fixed to R_2; the mass of this load is about 5 mgm. and stops prevent it from moving except between narrow limits. The low mass of Q and the almost ideally elastic properties of quartz eliminate the necessity of clamping the suspended system while the instrument is in transit.

S is the zero length spring with its lower end fixed to a lever attached to R_2; its upper end is carried by a composite biexpansion thermal compensator joined to R_1.

S_1 is the measuring dial spring and the tension in it can be altered by rotating the head D_1; the full range of this spring corresponds to a change in gravity of 80–120 mgal. To extend the range of the gravimeter to 4,000 mgal, the re-set spring S_2 is also attached to R_1 and the tension in it controlled by the knob D_2.

The gravimeter is set up in turn at two or more stations, at each of which g is known accurately, and this enables the readings on D to be interpreted. When the instrument is at a station for which $\Delta g = 120$ mgal., the spring S_2 is used to bring P to its zero position after setting D to read zero. When the gravimeter is taken to other

stations further values of Δg are obtained from the readings on D when P is restored to its zero position.

Since the elastic system of a Worden gravimeter is constructed of fused quartz, it is highly resistant to shock and fatigue and entirely free from all magnetic effects. The complete instrument is sealed in an enclosure when a low vacuum has been produced and the whole then fitted inside a thermos flask. The scale readings on the dial D are linear to within one part in a thousand. The complete gravimeter and its carrying case have a mass only just exceeding 5 kgm.; the small size of the outfit and the elimination of an external power source for temperature control are the chief reasons why this gravimeter is used extensively in geological surveys.

The instrument is null reading, balance being obtained by rotating the reading dial until the pointer, viewed through a reading microscope is central with respect to the fiducial marks on a graticule.

The small reading dial D_1 has 100 divisions each of which corresponds to a change in gravity of 0·1 mgal. or 0·001 cm.sec.$^{-2}$; a vernier enables observations to be made to $\pm$0·01 mgal. A Worden gravimeter is capable of measuring changes of between $10^{-8}g$ and $10^{-7}g$, where g denotes gravity, i.e. the vertical component of the earth's gravitational field.

The reduction of field data obtained with a Worden gravimeter.—The observed changes in gravity must be corrected for several effects; the more important are as follows.

(*a*) ***Instrumental drift.*** The existence of instrumental drift in apparatus used in gravity surveys is perhaps the greatest disadvantage of this method of carrying out a geological survey. Such drift in the Worden gravimeter, which is much less than that in many other instruments, is caused by departures from the ideal elastic properties which are assumed to be associated with the suspension and the net effect of this drift is to create an apparent discrepancy in the value of Δg at a given station as determined on a subsequent occasion. The effect is eliminated by using the gravimeter at $(t + 1)$ hours at a station where it was first used at t hours. The drift is small and is assumed to be linear; it seldom exceeds 0·1 mgal.hr.$^{-1}$.

(*b*) ***Altitude effect.*** The altitude of a station causes the observed value of Δg to differ in two ways from the variation due to changes in the distribution and nature of the local rocks etc. In the first place gravity varies inversely as the square of the distance of the station from the centre of the earth, the normal vertical gradient of gravity being +0·3085 mgal.metre^{-1}. This is the so-called ***'free air' correction***. Secondly, the material between the reference plane and the horizontal plane through the gravity station tends to

increase the value of g; the necessary correction is $-0{\cdot}04187\rho$ mgal.metre^{-1}, where ρ is the mean density of the rocks lying between the above planes. This is called the ***Bougner correction.***

(c) ***The latitude effect.*** All values of g must finally be reduced to a standard latitude, usually that of the station at the centre of the north-south axis of the area being surveyed. This variation is due to the rotation of the earth, cf. p. 150, and its departure in shape from that of a sphere.

(d) ***Tidal and temperature effects.*** These are very small and are incorporated in the ordinary drift correction.

EXAMPLES V

5·01. A uniform lamina in the form of a circular sector makes small oscillations under gravity about a horizontal axis through its vertex and normal to its plane. Find the angle of the sector when the length of the equivalent simple pendulum is equal to one half the length of the arc. [$2 \sin^{-1} \frac{3}{4}$.]

5·02. Investigate the angle of a uniform lamina in the form of an isosceles triangle, so that the period of small oscillations in a vertical plane and about a horizontal axis may be the same when the axis passes through its vertex as when it passes through the mid-point of its base. $\left[\text{The vertical angle must be } \frac{\pi}{2}.\right]$

5·03. A light rod, pivoted about a horizontal axis normal to its length, carries at its lower end a particle of mass m whose distance from the pivot is r_1. At the upper end is a mass $\theta m (0 < \theta < 1)$ and the distance of this mass from the pivot is r_2. Find an expression for the period of small oscillations under gravity. $\left[2\pi\sqrt{\dfrac{r_1^2 + \theta r_2^2}{r_1 - \theta r_2}}.\right]$

5·04. A uniform square lamina oscillates about a horizontal axis normal to its plane. Find where the axis must pierce the square in order that the period may be a minimum.

[If a is the side of the square, the locus of the required point is a circle, radius $\dfrac{a}{\sqrt{6}}$ and centre at the centre of the plate.]

5·05. A uniform beam of length $2a$ and mass $3m$ has a light arm of length a attached to it perpendicularly at its mid-point, and hangs in a horizontal position from a pivot at the end of the arm. At each end of the beam is a simple pendulum of length a and mass m. The whole system executes small oscillations in a vertical plane. Show that the lengths of the equivalent simple pendulums are a, $2a$, $\frac{4}{5}a$, and describe the corresponding modes.

5·06. Show how the moment of inertia of a uniform horizontal bar about a vertical axis through its centre of gravity may be found from observations of its period when supported by a bifilar suspension and making small oscillations in a horizontal plane.

5·07. Show that a compound pendulum will oscillate with the same period about four points in a straight line through its centre of gravity.

Explain the application of this result to the determination of the acceleration due to gravity.

5·08. By writing

$$\frac{4\pi^2}{g} = \frac{r_1 T_1^2 - r_2 T_2^2}{r_1^2 - r_2^2} = \frac{A}{r_1 + r_2} + \frac{B}{r_1 - r_2}$$

show that
$$\frac{4\pi^2}{g} = \frac{T_1^2 + T_2^2}{2(r_1 + r_2)} + \frac{T_1^2 - T_2^2}{2(r_1 - r_2)}.$$

A Kater pendulum gains one whole swing every 60 seconds on a clock whose period is 2·00 seconds. When inverted so that the more massive end is at the top it gains one swing in every 58 seconds. If the distances of the knife-edges from the centre of gravity of the pendulum are 65·62 cm. and 27·61 cm. respectively, find a value for the intensity of gravity. [980 cm. sec.$^{-2}$]

5·09. Explain the method of using Kater's pendulum to determine the acceleration of gravity in any locality. Why are the determinations with such a pendulum much more accurate than those made by timing the pendular oscillations of a small body at the end of a thread?

Indicate the nature of the variations of g over the earth's surface.

5·10. Give the theory and practical details of the determination of the acceleration due to gravity by means of Kater's pendulum. The period of small oscillations in a vertical plane of a uniform bar about a smooth horizontal axis 10 cm. from the centre of gravity is 1·25 seconds. Find the length of the bar and determine the other position of the axis about which the period has the same value. In making the calculation, the lateral dimension of the bar may be neglected. (G)

[58·8 cm., 28·8 cm. from the centre of the bar.]

5·11. Describe an experiment to determine the time period of a pendulum by the 'method of coincidences'. What is the advantage of the method?

A pendulum consists of a flat vertical disc of diameter 6·0 cm. and mass 220 gm. fixed at a point on its rim to the end of a rod 22 cm. long and of mass 30 gm. It oscillates about an axis through the upper end of the rod perpendicular to the plane of the disc. Calculate its time period for oscillations of small amplitude. [0·994 sec.]

5·12. A flywheel having a moment of inertia I about its axis of revolution is mounted on frictionless bearings with its axis horizontal. A mass m of negligible size is attached to the wheel at a distance r from the axis and the wheel is released from rest in a position for which the radius through m makes an angle θ with the vertical. Derive an expression for the maximum angular velocity attained by the wheel.

The above system is then allowed to perform small oscillations and the length l of a simple pendulum adjusted to have an equal time period. This is repeated for different values of r. Explain how the relation between r and l may be represented by a straight line graph and show how, if m is known, a value for I may be determined from the graph.

Give the theory of any other method for determining the moment of inertia of a flywheel about its axis of revolution. (L.Sch.)

$\left[\left\{\frac{2mgr(1 - \cos\theta)}{I + mr^2}\right\}^{0\cdot5}\right.$; plot r^2 against rl when $I = m \times$ (intercept).$\left.\right]$

5·13. A thin uniform rod of length l swings in a vertical plane about a horizontal knife-edge passing through one extremity of the rod. At

what point in the rod must a concentrated load be placed if the period of small oscillations under gravity is to remain unchanged?

$$\left[\frac{2l}{3} \text{ from the upper end.}\right]$$

5·14. (*a*) If the intensity of gravity be increased by $\frac{1}{n}$th of its value, prove that a pendulum will gain one complete oscillation in every $2n$.

(*b*) The period of a simple pendulum at the top of a mountain is observed to be less than its period at the foot by $\frac{1}{60}$ per cent and the decrease in length due to the change in temperature $\frac{1}{80}$ per cent. Show that the percentage increase in the intensity of gravity is $\frac{1}{48}$.

(*c*) A pendulum clock is observed to gain 24 seconds a day when taken to a different locality. Assuming that there is no change in temperature, find a value for the difference in the intensity of gravity in the two localities. [$g/1800$]

5·15. The balance wheel of a watch may be regarded as a heavy rim of radius 0·80 cm. Its mass is 0·96 gm. Find the restoring couple per unit angular displacement which the spring must exert in order that the period of one oscillation may be one second. [0·0156 erg.radian.$^{-1}$]

5·16. A quadrant of a uniform circular disc of radius a oscillates under gravity in its own plane about a horizontal axis through its apex. Find its period.

$$\left[\pi\sqrt{\frac{3\pi}{8}\cdot\frac{a}{g}}.\right]$$

5·17. A piece of wire is bent to form two arms of a rectangle with lengths a and b respectively. It swings in its own plane under gravity about a horizontal axis passing through the bend in the wire. Show that the length of the simple equivalent pendulum is

$$\tfrac{2}{3}(a^3 + b^3)(a^4 + b^4)^{-\frac{1}{2}}.$$

5·18. A thin circular loop of radius a swings in its own plane about a horizontal axis passing through a point in its rim. If the oscillations are small show that the length of the simple equivalent pendulum is $2a$.

5·19. A light rod of length $2a$ has two equally massive particles fixed at its ends. If the rod is pivoted at a point distant r from its centre, show that the length of the simple equivalent pendulum is $(a^2 + r^2)r^{-1}$.

5·20. The amplitude of oscillations of a long pendulum is 10·0 cm. After 4 minutes the amplitude is 8·0 cm. What time elapses before the amplitude is 6·0 cm.? [10 min.]

5·21. A uniform elliptic plate makes small oscillations in a vertical plane under the influence of gravity. If the axis of rotation passes through one end of the major axis, prove that the length of the simple equivalent pendulum is $(5a^2 + b^2)(4a)^{-1}$, where a, b are the semi-axes with $a > b$.

5·22. Prove that the time of small oscillation under gravity of a rhombus in its own plane about a horizontal axis passing through one corner is

$$\pi\sqrt{\frac{7a}{3g}},$$

where a is the length of the diagonal through the point of suspension.

5·23. A non-uniform rod may oscillate in turn about two horizontal axes passing through points in the rod at a distance l apart. If t_1 and t_2 are the respective periods of small oscillations, prove that the mass centre of the rod divides the distance l in the ratio

$$\frac{gt_1^2 - 4\pi^2 l}{gt_2^2 - 4\pi^2 l}.$$

5·24. Give in outline the chief corrections which have to be made in making an accurate determination of the acceleration due to gravity by means of the compound pendulum and consider in detail the correction due to the curvatures of the knife-edges. (S)

5·25. Define ***mass*** and ***weight*** and describe the experiments you would carry out to test the proportionality of the mass and weight of a body at some given place. (G)

5.26. A rigid body is mounted so that it can turn without friction about a fixed horizontal axis. Deduce an expression for the period of its small oscillations.

Describe how you would use such a system for the purpose of measuring g, the intensity of gravity. (G)

5·27. If a body is let fall from a point above the earth's surface, prove that it undergoes a small easterly deviation from the vertical and also a very small deviation towards the equator.

Show that the easterly deviation varies as the cosine of the latitude and as the cube of the square root of the vertical distance fallen. Show further that in latitude $51\frac{1}{2}°$ this deviation is about $\frac{1}{11}$ inch for a fall of 100 feet.

5·28. A faulty seconds pendulum loses 20 'seconds' a day. Show that it will keep correct time at the same station if its length is shortened by about 0·046 cm.

5·29. Show that if the earth were not rotating about its axis the intensity of gravity at the equator would exceed its present value by 3·38 cm.sec.$^{-2}$.

5·30. Describe in detail how the intensities of gravity at two stations may be accurately compared. What knowledge can be obtained from the results of a gravitational survey?

5·31. A square metal plate of uniform thickness is arranged to execute small oscillations in its own plane, about a horizontal axis through one corner. What is the length of the simple equivalent pendulum if the side of the square is of length $2a$?

$$\left[\frac{4\sqrt{2}}{3}a.\right]$$

5·32. A straight uniform rod of length λ metres and mass m kgm., hinged at one end so that it may move in a vertical plane, is released from a horizontal position. Calculate the angular velocity when the rod passes through the vertical position. Would this velocity be doubled if the rod were released from its position of unstable equilibrium.

$$\left[(a)\ \frac{1}{10}\sqrt{\frac{3g}{\lambda}}\ \text{radian.sec.}^{-1};\quad (b)\ \frac{1}{10}\sqrt{\frac{6g}{\lambda}}\ \text{radian.sec.}^{-1}.\right]$$

5·33. A uniform metal rod of length 30 cm. is supported horizontally by a bifilar suspension consisting of two vertical strings each 100 cm. long. These strings are attached to the ends of the rod. If the mass of the bar is 1000 gm. determine the couple acting upon it when it is twisted through a small angle θ.

If the radius of gyration for the bar referred to a vertical axis through its centre of gravity is 10 cm., find the period for small oscillations of the bar in a horizontal plane. [1·34 sec.]

5·34. Derive an expression for the periodic time of a body oscillating with simple harmonic motion. Use the result to find the time of oscillation of a compound pendulum when the amplitude of its motion is small.

A rigid circular disc of radius r and mass m is supported so that it may perform oscillations of small amplitude about a horizontal axis through its rim and perpendicular to its plane. Find the periodic time, T, when a particle of mass M is fixed to the lowest point of the disc and show that, however great the value of M, the change in T cannot exceed about 15 per cent. (G)

5·35. Show that the kinetic energy of a rigid body rotating about a fixed axis is $\frac{1}{2}I\omega^2$, where the symbols have their usual meanings. Describe and explain how the moment of inertia of a flywheel about its axis, which is horizontal, may be determined by fixing a small body of known mass to the rim and observing the oscillations as a compound pendulum.

5·36. Explain the term ***moment of inertia*** and state ***the theorem of parallel axes.*** Obtain, from first principles, the equation of motion (damping negligible) of a pendulum consisting of a bob of finite radius suspended by a thin string. Show how the period of the pendulum may be derived from the equation of motion.

A pendulum consists of a spherical bob of radius 2·24 cm. suspended by a string of length 15·0 cm. Calculate values for (*a*) the period of the pendulum, (*b*) the length of the equivalent simple pendulum.

[(*a*) 0·836 sec., (*b*) 17·4 cm.]

5·37. Obtain an expression for the period of oscillation, T, of a uniform circular sheet of metal, radius a, about an horizontal axis perpendicular to the plane of the sheet and at a distance r from its centre.

If the diameter of the sheet is 36·0 cm., calculate

(*a*) the period of oscillation, T_r, when $r = a$.

(*b*) another value of r for the same period T_r.

(*c*) the minimum period of oscillation, T_{min}.

[(*a*) 1·04 sec., (*b*) 9·0 cm., (*c*) 1·01 sec.]

CHAPTER VI

GRAVITATION

Kepler's laws of planetary motion.—The motion of the planets has been a subject of much interest from very early times for even Babylonian astronomers were able to compute tables of the planetary motions with some success. PTOLEMY, assuming that the earth was the centre of the universe, discovered that the apparent motions of the planets as observed from the earth could be represented by a system of circles and epicycles. In 1542 COPERNICUS tried to simplify the above representation and in his theory the sun was considered to be the centre about which the planets revolved in circles. He died shortly before his results could be published in book form.

In 1609, KEPLER, having made a systematic study of the astronomical observations of TYCHO BRAHE, published two laws concerning planetary motion; these were followed in 1618 by a third law. Kepler's laws of planetary motion are as follows.—

(*a*) ***Each planet describes an ellipse with the sun in one focus.***

(*b*) ***The radius vector from sun to planet sweeps out equal areas in equal times.***

(*c*) ***The squares of the times of revolution of the planets are proportional to the cubes of their mean distances from the sun.***

[To-day the expression 'mean distance' is replaced by 'the ***major semi-axis*** of the ellipse'.]

The first and second laws, it should be noted, refer to the motion of a single planet; the third expresses a relation between the motions of the different planets.

Newton's law of gravitation.—Kepler's laws are a summary of observations on the motions of the planets and, as such, give a simple and accurate description of these motions without offering any explanation. In 1666 NEWTON, at the age of twenty four, gave us an explanation of planetary motion in terms of the force of gravitation which exists between sun and planet; this actual work was not published for about twenty years and it was largely due to the influence of HALLEY that the work was published as part of the *Principia.*

To gain an insight into the reasoning whereby Newton may have discovered his law of gravitation let us assume that a planet moves about the sun as centre in a circle of radius r, T is the period of revolution of the planet and ω its angular velocity. The acceleration of the planet towards the sun is $\omega^2 r$, so that if its mass is m the force of attraction on the planet is

$$m\omega^2 r = m\left(\frac{2\pi}{\mathrm{T}}\right)^2 r,$$

where T is the period. Now according to the third law $\mathrm{T}^2 = \kappa r^3$, where κ is a constant. Hence the force on the planet is proportional to $\frac{m}{r^2}$ i.e. the attraction is inversely proportional to the square of the distance between the sun and the planet.

By some such reasoning as that just given Newton was led to propose his law of gravitation, which states that ***any particle of matter attracts any other particle with a force which is directly proportional to the product of their masses and inversely proportional to the square of their distance apart, the line of action of the force being the line joining the particles.*** If m_1 and m_2 are the masses of the two particles and r their distance apart, the force F tending to increase their separation is given by

$$\mathrm{F} = -\gamma . \frac{m_1 m_2}{r^2},$$

where γ is a universal constant known as ***the constant of gravitation.*** The negative sign indicates that the force is one of attraction, which is in conformity with the usual practice in magnetism and electricity, although it is not always so in works on gravitation.

Now if, with Newton, it is assumed† that the attraction due to a sphere at an external point is the same as if the mass of the sphere were concentrated at its centre, the validity of the inverse square law in the case of the moon's motion may be tested. Let m_1 and m_2 be the masses of the earth and the moon, r the distance between their centres and a the radius of the earth. If T is the period of the moon (27 d. 7 hr. 43 min.) then

$$m_2\left(\frac{2\pi}{\mathrm{T}}\right)^2 r = \text{force of attraction on the moon and due to the earth}$$

$$= \gamma\,\frac{m_1 m_2}{r^2}.$$

† Cf. p. 212 for a proof of this statement.

If g is the intensity of gravity at a point on the earth's surface, by considering the attraction of the earth on a unit mass at this point, we have

$$g \times 1 = \gamma . \frac{m_1 \times 1}{a^2}.$$

These equations give

$$ga^2 = \left(\frac{2\pi}{T}\right)^2 r^3,$$

so that if we take as a close approximation $r = 60a$, we have

$$g = \frac{4\pi^2}{T^2}(60)^3 a.$$

Assuming $a = 4{,}000$ miles, we find

$$g = 32{\cdot}4 \text{ ft.sec.}^{-2}.$$

This deduction from observations on the moon's motion is in accord with experimental facts and thus points to the validity of gravitational theory.

In the above discussion the planetary orbits have been treated as circles, whereas Kepler asserts that they are ellipses. Let us therefore examine Kepler's laws more critically and try to discover what deductions can be made from them.

First let P and Q, Fig. 6·01(a), be the positions of a planet at times t and $(t + \delta t)$ in its motion round the sun. Then S is one of the foci of an ellipse and if v is the speed of the planet

$$PQ = v . \delta t.$$

Thus if A is the area swept out by SP and as measured from some fixed position of that vector,

$$\delta A = \Delta\, SPQ = \tfrac{1}{2} p . PQ,$$

where p is the perpendicular SM from S on to the chord PQ produced or, in the limit, on the tangent PT at P.

$$\therefore\ \frac{dA}{dt} = \tfrac{1}{2} pv.$$

Since equal areas are described in equal times (Kepler's second law) $\dfrac{dA}{dt}$ is constant, i.e. pv is constant, say h. If m is the mass of the planet it follows that

$$mpv = \text{constant},$$

i.e. the moment of momentum, or angular momentum, cf. p. 90, of the planet is constant.

In the second place it remains to show that the only law of force consistent with an elliptic orbit having the sun at one focus is an inverse square one. Thus let

$$\frac{dA}{dt} = \tfrac{1}{2}h,$$

and the force of attraction be $f(r) = \overline{F}$. Also let ϕ be the angle which the chord PQ makes with the radius vector SP. In the limit,

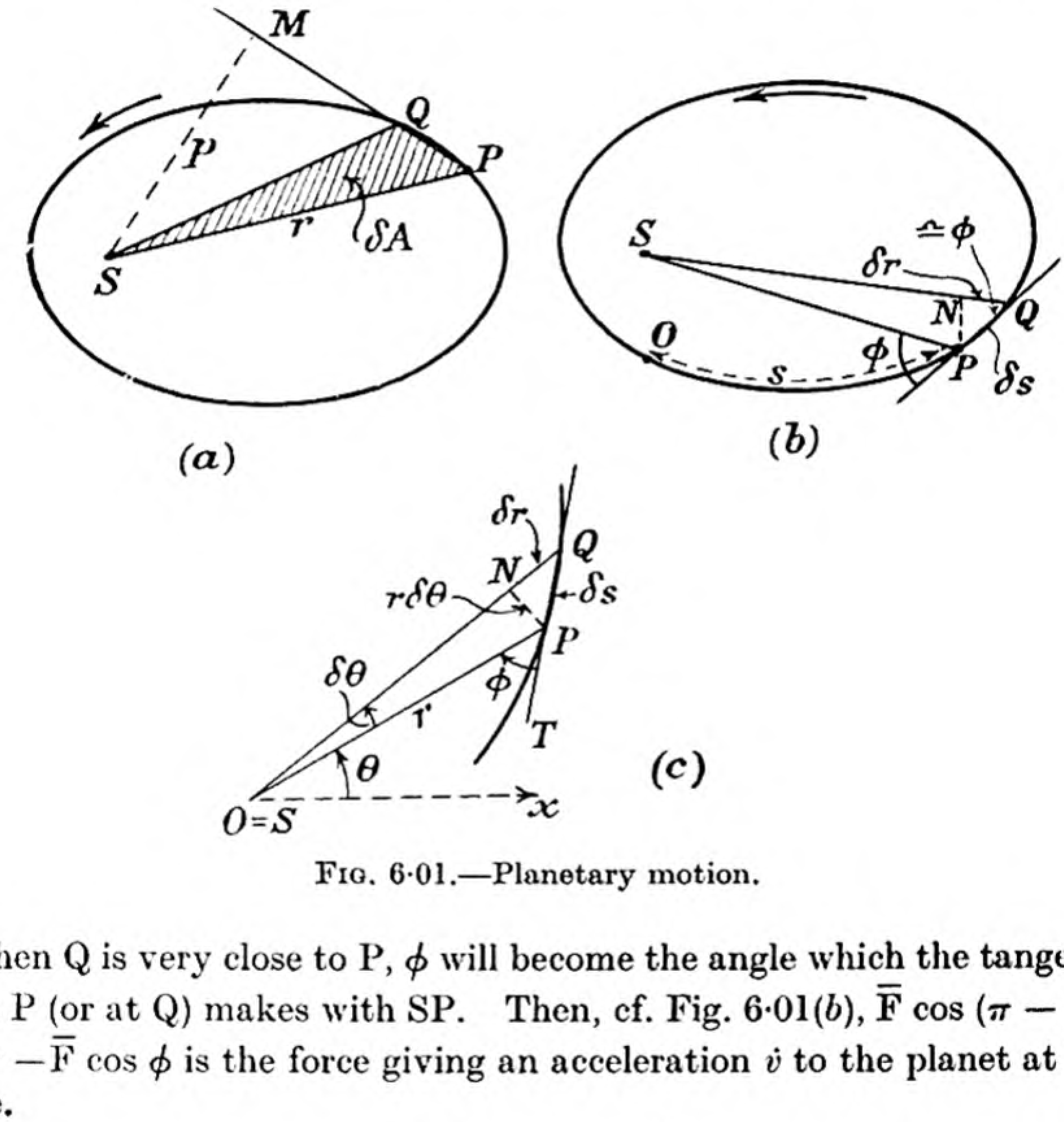


FIG. 6·01.—Planetary motion.

when Q is very close to P, ϕ will become the angle which the tangent at P (or at Q) makes with SP. Then, cf. Fig. 6·01(*b*), $\overline{F}\cos(\pi - \phi)$ or $-\overline{F}\cos\phi$ is the force giving an acceleration $\dot{v}$ to the planet at P, i.e.

$$-\overline{F}\cos\phi = m\frac{dv}{dt},$$

and since $pv = h$, a constant,

$$p\frac{dv}{dt} + v\frac{dp}{dt} = 0.$$

$$\therefore\ -\overline{F}\frac{dr}{ds} = -\frac{mv}{p}\cdot\frac{dp}{dt}, \qquad \left[\because\ \cos\phi = \frac{dr}{ds},\ \text{cf. Fig. 6·01}(c).\right]$$

$$\therefore\ \overline{F} = \frac{mv}{p}\cdot\frac{ds}{dt}\cdot\frac{dp}{dr} = \frac{mv^2}{p}\frac{dp}{dr}.$$

Now the (p, r) equation to an ellipse with regard to a focus is

$$\frac{b^2}{p^2} = \frac{2a}{r} - 1, \qquad \text{[cf. p. 14.]}$$

$$\therefore \frac{dp}{dr} = \frac{p^3 a}{r^2 b^2}.$$

$$\therefore \bar{F} = \frac{mh^2}{p^3} \cdot \frac{p^3 a}{r^2 b^2} = \frac{m}{r^2} \cdot \frac{h^2 a}{b^2}.$$

If we write $\bar{F} = \mu \frac{m}{r^2}$, where $\mu = \frac{h^2 a}{b^2}$, we have for the period of the planet,

$$\tfrac{1}{2}hT = \pi ab. \qquad [\because \delta A = \tfrac{1}{2} pv\, \delta t]$$

$$\therefore T = \frac{2\pi ab}{h} = \frac{2\pi a^{\frac{3}{2}}}{\mu^{\frac{1}{2}}},$$

i.e.

$$T^2 \propto a^3.$$

Hence Kepler's third law implies that μ has the same value for all planets, i.e. the force on a planet is proportional to its mass. Since the attraction is mutual, it will also be proportional to the mass of the other body (the sun). Then

$$\bar{F} = \gamma . \frac{mM}{r^2}, \quad \text{or} \quad F = -\gamma . \frac{mM}{r^2},$$

where M is the mass of the sun and γ is the universal constant of gravitation.

The above discussion is necessarily only an approximate one since the sun has been assumed to have a fixed position. Strictly speaking, both sun and planet describe ellipses about their common centre of mass as a focus, the sun's ellipse being very small on account of its large mass compared with that of a planet.

There is no *a priori* reason for Newton's law of gravitation, but astronomical calculations based upon it agreed so well with observation that its validity was established. As one very important discovery made as the result of an application of this law, we may cite the following. In 1781, Sir Wm. Herschel discovered the planet Uranus and for a time it appeared to behave in the same manner as did the other planets, but early in the nineteenth century certain discrepancies with regard to its motion were discerned. Some force was evidently at work on this distant planet, causing it to disagree with its motion as calculated according to the Newtonian law of gravitation. In 1845, J. C. Adams, at Cambridge, showed that the refractory nature of Uranus would be explained if there were an outer, and as yet unknown, planet in a position he specified.

He communicated his theory to the Astronomer Royal who took little notice of the news. In 1846, LE VERRIER announced a similar theoretical deduction, and the Astronomer Royal noted that each of these investigators predicted the existence of a new planet, now called Neptune, in practically the same position. Only a casual search was made for this new planet in England and it was left to GALLE to discover it on 23rd September 1846. As further support in evidence for the validity of Newton's law of gravitation, it may be mentioned that predictions of the times of return of certain comets, based on it, have been fulfilled. Thus, in general, Newton's law of gravitation has proved adequate as a basis for astronomical calculations; there are, however, a few instances where minute discrepancies between calculation and observation occur but these have been accounted for by Einstein's Theory of Relativity.

The strength of a gravitational field—Gravitational intensity.—If a small mass δm is placed in a gravitational field and it experiences a force $\overrightarrow{\delta F}$, then

$$\lim_{\delta m \to 0} \frac{\overrightarrow{\delta F}}{\delta m}$$

is called ***the strength of the gravitational field*** (or ***gravitational intensity***), $\overrightarrow{G}$, at the point occupied by δm. Thus

$$\overrightarrow{G} = \lim_{\delta m \to 0} \frac{\overrightarrow{\delta F}}{\delta m}.$$

Consider a small mass δm at P, Fig. 6·02(a), a point situated at

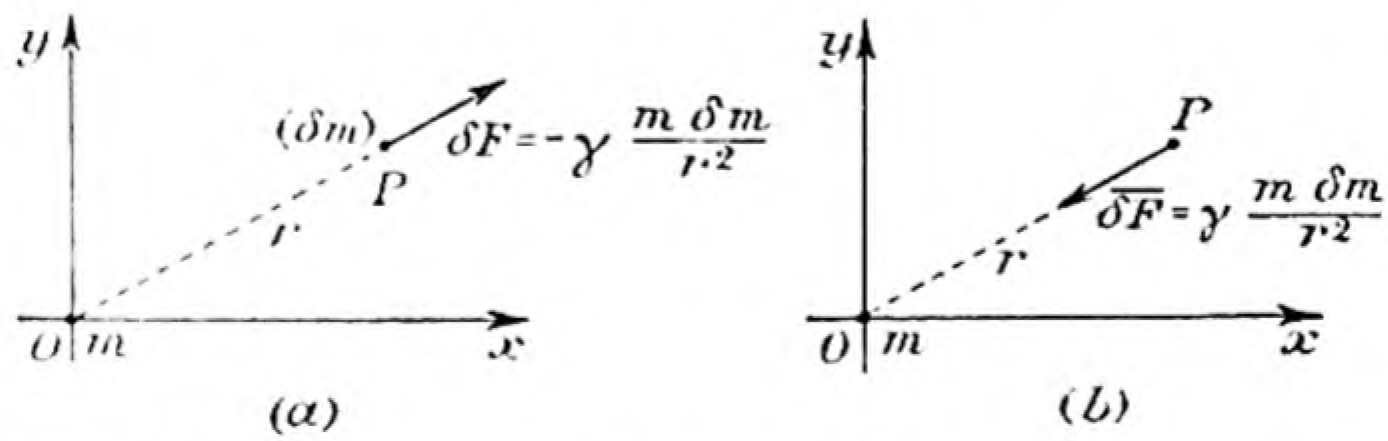


FIG. 6·02.—Intensity in a gravitational field due to a material particle.

a distance r from a particle of mass m. Then δF, the force on δm, is given by

$$\delta F = -\gamma . \frac{m\, \delta m}{r^2}.$$

Hence

$$\overrightarrow{G} = \lim_{\delta m \to 0} \frac{\overrightarrow{\delta F}}{\delta m} = -\gamma \frac{m}{r^2} \hat{r} = G\hat{r}.$$

This is the gravitational field strength at P due to a particle of mass m at O. The negative sign indicates that the intensity (as also the force $\overrightarrow{\delta F}$) is actually directed towards O; cf. Fig. 6·02(b), where $\delta \overline{F}$ denotes the force on δm when it is measured along the direction of r decreasing.

The gravitational field due to a uniform linear distribution of matter.—Let AB, Fig. 6·03(a), represent a uniform linear

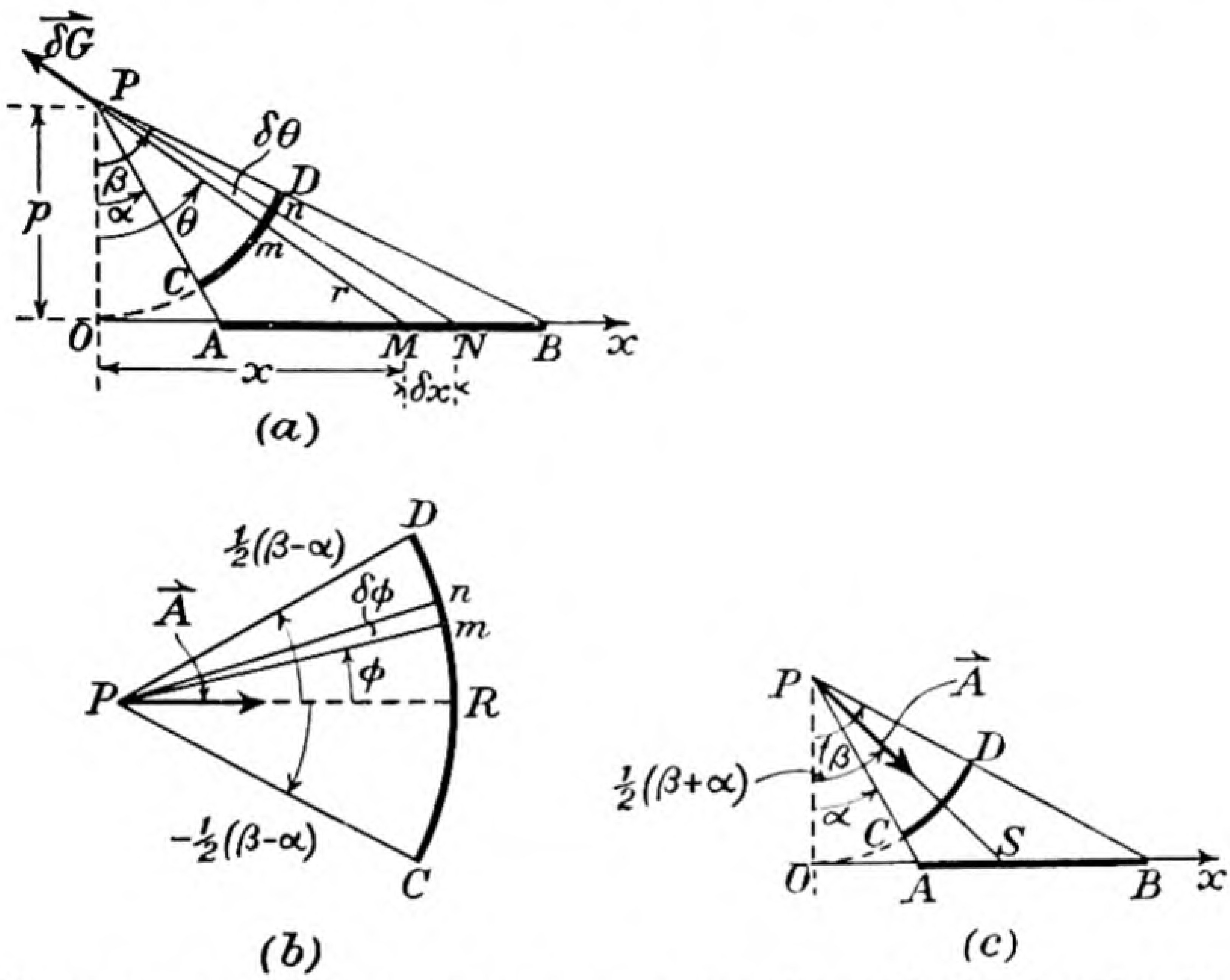


FIG. 6·03.—Gravitational field due to a uniform linear distribution of matter.

distribution of matter of density μ per unit length. Let P be the point at which the gravitational intensity is to be calculated. Consider the contribution to the gravitational field at P due to the element MN of the distribution of matter. Let OM $= x$, and MN $= \delta x$. Then the contribution we are considering is

$$\delta G = -\frac{\gamma\mu\,\delta x}{r^2},$$

if MP $= r$. If OP $= p$, $x = p\tan\theta$ and $r = p\sec\theta$.

$$\therefore\ \delta G = -\frac{\gamma\mu p\sec^2\theta\,\delta\theta}{p^2\sec^2\theta} = -\frac{\gamma\mu}{p}\,\delta\theta$$

$$= -\gamma\left(\frac{\mu . p\,\delta\theta}{p^2}\right).$$

Thus the field at P due to MN is the same as that of an element mn of a circular arc CD, radius p and centre P and linear density μ. This is true for all such elements of the rod and hence the field at P is equal to that of a circular arc CD, of radius p and linear density μ.

The component of the field at P due to mn along PR, the bisector of $\widehat{CPD}$, is, cf. Fig. 6·03(b),

$$-\frac{\gamma\mu}{p}\cos\phi\,\delta\phi,$$

where ϕ is the angle indicated. On account of symmetry only these components have to be considered and we have accordingly

$$G = -\frac{\gamma\mu}{p}\int_{-\frac{\beta-\alpha}{2}}^{\frac{\beta-\alpha}{2}}\cos\phi\,d\phi$$

$$= -\frac{2\gamma\mu}{p}\sin\tfrac{1}{2}(\beta-\alpha).$$

Hence if $\vec{A}$ is the attraction at P, i.e. $\vec{A} = -\vec{G}$, we see that $\vec{A}$ has a magnitude $\frac{2\gamma\mu}{p}\sin\frac{1}{2}(\beta-\alpha)$, and makes an angle $\frac{1}{2}(\alpha+\beta)$ with the vertical through P drawn downwards, i.e. cf. Fig. 6·03(c), along the bisector of the $\widehat{APB}$.

Gravitational field due to a uniform spherical shell of matter.—Let O, Fig. 6·04(a), be the centre of the spherical shell,

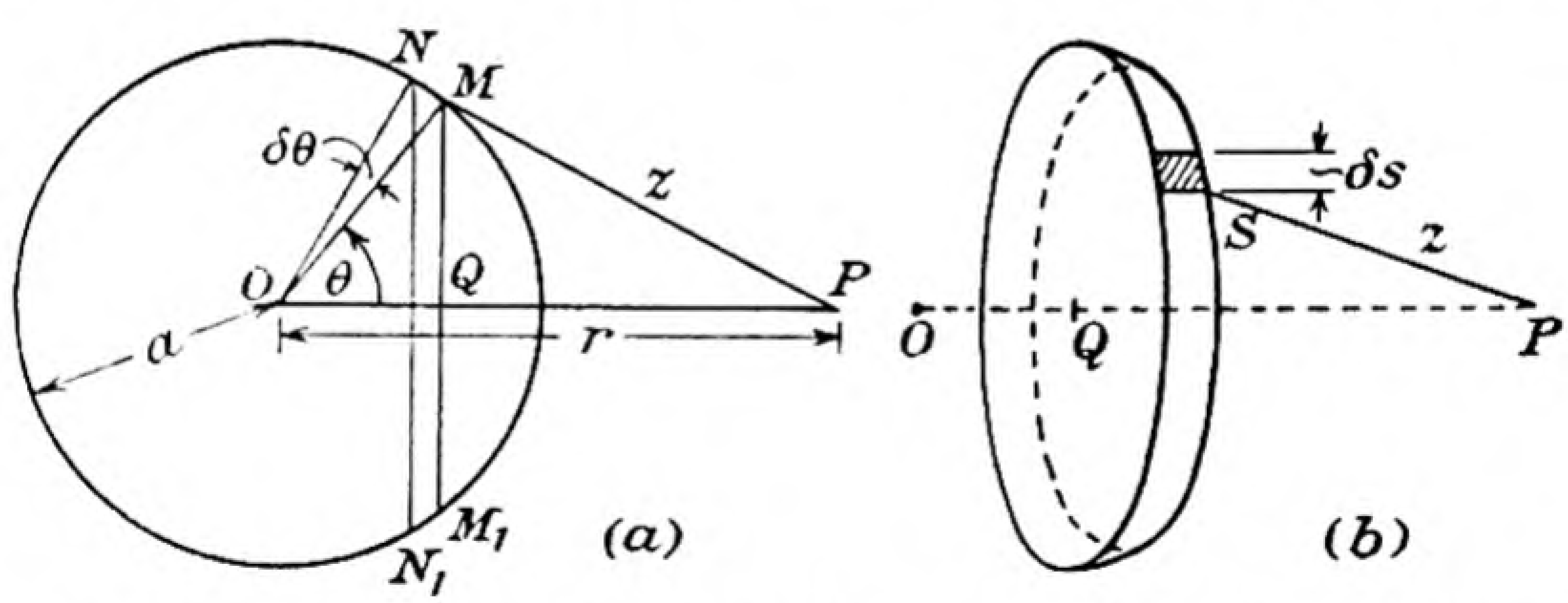


FIG. 6·04.—Gravitational intensity due to a uniform spherical shell.

of radius a, and P the point at distance r from O at which the gravitational intensity is required. Consider that portion of the shell which is cut off by two vertical neighbouring planes intersecting the surface of the shell in the plane of the diagram at points M, N, N_1 and M_1 respectively. Then the surface area of the elementary shell is $2\pi a\sin\theta . a\,\delta\theta$; its mass is therefore $2\pi a^2\sigma\sin\theta\,\delta\theta$, where σ is the surface density of the material of the shell, and every point in the shell is at a distance $PM = \sqrt{(a^2 + r^2 - 2ar\cos\theta)}$

from P. We cannot, however, proceed at once to evaluate the intensity at P due to this ring of matter for the intensity due to each element of it will have a different direction. If δs is the length of such an element at S, Fig. 6·04(b), then the intensity at P due to this element is

$$-\frac{\gamma.\sigma.a\,\delta\theta.\delta s}{\mathrm{PS}^2},$$

and this may be resolved into two components, one along OP and the other at right angles to it. Now diametrically opposite to the element considered there will be another element the intensity due to which may likewise be resolved. The component at right angles to OP will counterbalance the corresponding component due to the first element. Thus, on the whole, it will only be necessary to consider the components along OP. Each of these will be equal to

$$-\frac{\gamma.\sigma.a\;\delta\theta.\delta s}{\mathrm{PS}^2}\cos \widehat{\mathrm{SPQ}},$$

where Q is the point in which the plane MM_1 intersects OP.

The intensity at P, δG, due to the whole of the elementary ring will be found by integrating the above expression round the ring. It is given by

$$\delta \mathrm{G} = -\gamma\sigma\frac{2\pi.a^2\sin\theta.\delta\theta}{a^2+r^2-2ar\cos\theta}\cos\widehat{\mathrm{MPQ}}. \qquad [\because\ \widehat{\mathrm{MPQ}} = \widehat{\mathrm{SPQ}}.]$$

$$\therefore\ \mathrm{G} = -2\pi\gamma\sigma a^2\int_0^\pi \frac{\sin\theta(r-a\cos\theta)\,d\theta}{(a^2+r^2-2ar\cos\theta)^{\frac{3}{2}}}.$$

To integrate this expression, let $a^2 + r^2 - 2ar\cos\theta = z^2$. Then

$$ar\sin\theta.d\theta = z\,dz, \qquad \text{and} \qquad 2r(r - a\cos\theta) = z^2 - a^2 + r^2.$$

Two instances arise:—(a) If P is outside the shell, the limits of integration are $(r - a)$ and $(r + a)$.

$$\therefore\ \mathrm{G} = -\frac{2\pi\gamma\sigma a}{2r^2}\int_{r-a}^{r+a}\left(1+\frac{r^2-a^2}{z^2}\right)dz$$

$$= -\frac{\pi\gamma\sigma a}{r^2}\left[z - \frac{r^2-a^2}{z}\right]_{r-a}^{r+a} = -\frac{\pi\gamma\sigma a}{r^2}\ (4a)$$

$$= -\frac{\gamma(\text{mass of shell})}{r^2}.$$

Hence the shell considered attracts a particle at an external point as if its whole mass were concentrated at the centre of the shell.

(*b*) If P lies inside the shell, the limits of integration are $(a - r)$ and $(a + r)$. Hence

$$G = -\frac{\pi\gamma\sigma a}{r^2}\left[z - \frac{r^2 - a^2}{z}\right]_{a-r}^{a+r} = 0,$$

i.e. the gravitational field inside a spherical shell is zero. [This result is true for all closed shells.†]

The gravitational field due to a uniform sphere.—Let O be the centre of a sphere of radius a, and let its material have a density ρ. Two cases arise for discussion.

(*a*) Let P, Fig. 6·05(*a*), be a point outside the sphere and at a

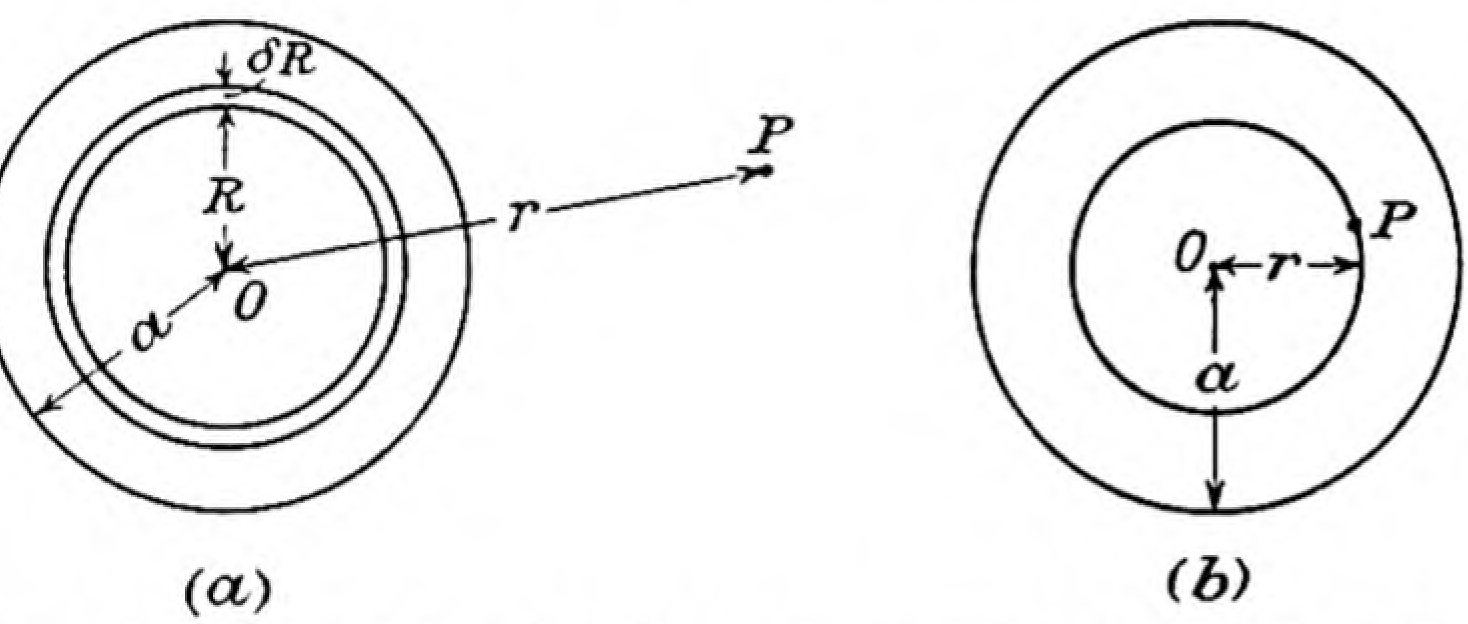


FIG. 6·05.—Gravitational field due to an isolated uniform sphere.

distance r from O. Consider the sphere to be made up of concentric shells of which one is shown; let R be the radius of this shell and δR its thickness. Then δG, the gravitational intensity at P due to this shell, is given by

$$\delta G = -\gamma\,\frac{4\pi R^2\rho\,\delta R}{r^2},$$

in virtue of the proposition proved above.

$$\therefore\ G = -\frac{4\pi\gamma\rho}{r^2}\int_0^a R^2\,dR = -\frac{4}{3}.\frac{\pi\gamma\rho a^3}{r^2},$$

i.e. the intensity is the same as if the whole mass of the sphere were concentrated at its centre.

(*b*) If P is inside, cf. Fig. 6·05(*b*), with O as centre and OP as radius, construct a sphere of radius r. Then the matter outside this sphere makes no contribution to the intensity at P, since P is inside every spherical concentric shell into which this outer portion of the solid sphere may be divided. The intensity at P will therefore be due entirely to the matter inside a sphere of radius r, i.e.

$$G = -\frac{4}{3}.\frac{\pi\gamma\rho r^3}{r^2} = -\frac{4}{3}\pi\gamma\rho r.$$

† Cf. Vol. V, p. 40.

[N.B.—Each of the above expressions gives the same value for the field strength at a point on the solid sphere of radius a. This equality is not true in all instances, cf. the internal and external fields at points near to the surface of a spherical shell.]

The flux of gravitational intensity.—Let P, Fig. 6·06(a), be a point in a gravitational field where the gravitational intensity is G,

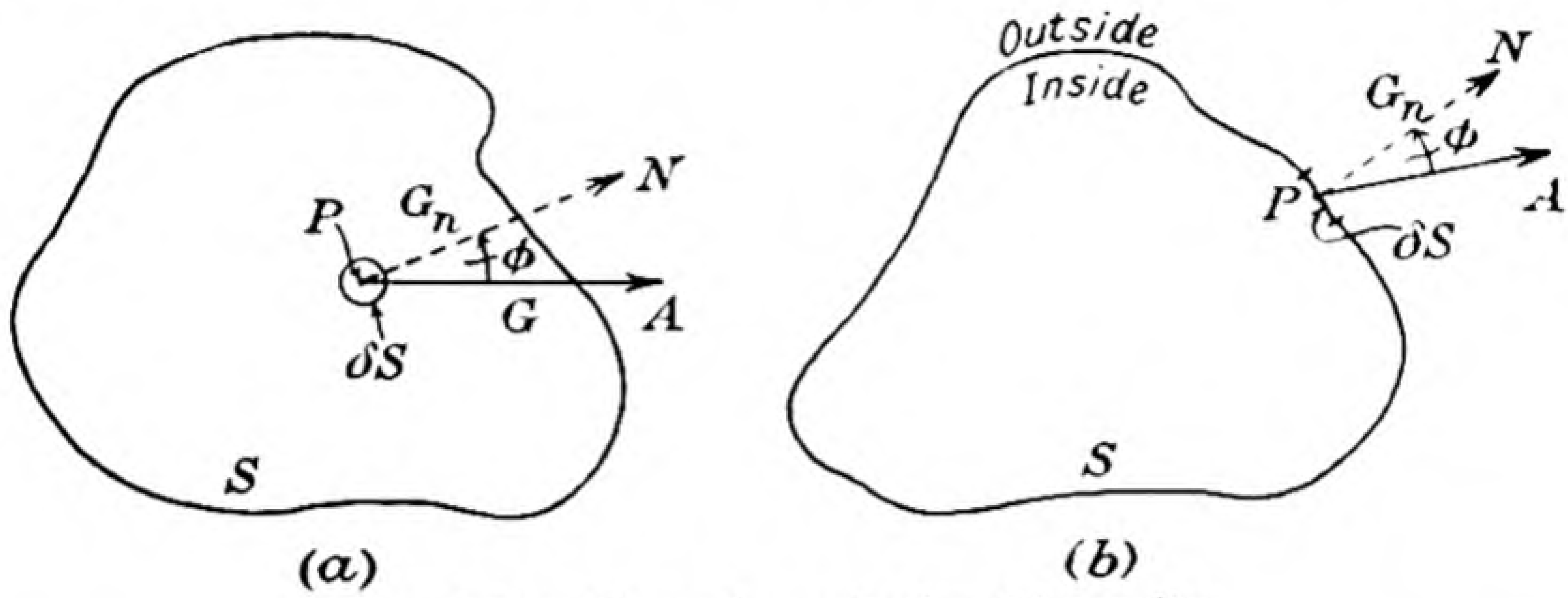


FIG. 6·06.—Flux of gravitational intensity.

acting along PA. Through P draw any† closed surface and consider that small element of this surface surrounding P. Through P draw PN normal to δS. Then G_n, the component of G along the outward direction of the normal, is G cos ϕ, where ϕ is $\widehat{APN}$—cf. Fig. 6·06(b).

The quantity $G_n \, \delta S$, or $(G \cos \phi)\, \delta S$, is termed the ***flux of gravitational intensity*** across the surface δS. Similarly,

$$\int G_n \, dS, \qquad \text{or} \qquad \int G \cos \phi \, dS,$$

where the integration extends over the whole of the surface S, is the flux of gravitational intensity across that surface. In vector notation, the flux is $\int \overrightarrow{G} \cdot \hat{n} \, dS$, where $\hat{n}$ is a unit vector along the outward drawn normal at δS.

Gauss' theorem.—***This states that the flux of gravitational intensity across a closed surface which has a unique outward drawn normal at every point is $-4\pi\gamma$ times the total mass enclosed by that surface, i.e.***

$$\int G_n \, dS = -4\pi\gamma \, \Sigma (m), \quad \textit{or} \quad \int \overrightarrow{G} \cdot \hat{n} \, dS = -4\pi\gamma \, \Sigma (m),$$

where $\Sigma (m)$ is the total mass inside the closed surface.

† The surface must be finite and have a unique outward drawn normal at every point; moreover, it must not be a skew surface.

To prove this,† let there be a particle of mass m at A, Fig. 6·07, and suppose S is any‡ closed surface surrounding A. Let BC be a

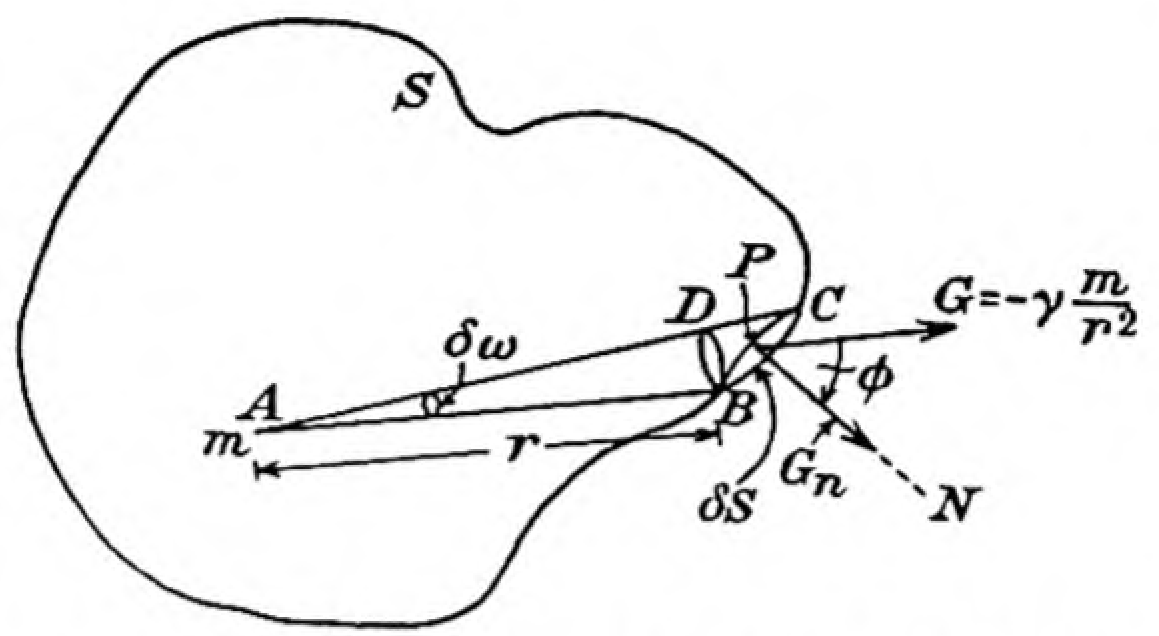


FIG. 6·07.—Gauss' theorem for a gravitational field.

small area δS surrounding a point P on the above surface, where the gravitational intensity is G. If r is the distance of P from A, then

$$G = -\gamma . \frac{m}{r^2},$$

and $$G_n = -\gamma . \frac{m}{r^2} \cos \phi,$$

where ϕ is the angle between the line of action of G and the normal to the surface at P. [The line of action of G is along PA.] With A as centre and AB as radius describe a sphere to cut the cone ABC along the section BD. Then

$$BC \cos \phi = BD = r^2 \, \delta\omega,$$

where $\delta\omega$ is the solid angle of the elementary cone ABC.

$$\therefore \text{ Flux of gravitational intensity across } \delta S = -\gamma . \frac{m}{r^2} . \cos \phi . BC$$

$$= -\gamma . \frac{m}{r^2} . BD$$

$$= -\gamma . m . \delta\omega.$$

$\therefore$ The flux of gravitational intensity across S

$$= -\gamma m \int d\omega = -4\pi\gamma m.$$

† Only a simple form of surface is considered here. The theorem and its applications are elaborated in connexion with electrical phenomena, cf. Vol. V, p. 35.

‡ The surface must be finite and have a unique outward drawn normal at every point; moreover, it must not be a skew surface.

Thus the theorem is established for one small particle of matter lying inside a closed surface. It is therefore true in general, for it applies to every such particle into which a material system may be divided, i.e.

$$\int G_n\, dS = -4\pi\gamma\, \Sigma\, (m), \qquad \text{or} \qquad \int \vec{G}\cdot\hat{n}\, dS = -4\pi\gamma\, \Sigma\, (m).$$

Simple applications of Gauss' theorem.—(*a*) As an illustration of the use of this theorem let us calculate the field due to an isolated uniform sphere at a point outside it. Let O, Fig. 6·08, be the centre of such a sphere of radius a and mass m. To find the gravitational intensity at P, a point at distance r from O, $(r > a)$, with O as centre construct a sphere of radius r. Then, by symmetry, the intensity at every point on the surface of this sphere will have the same value, G, say, and will be everywhere normal to the surface. The flux of gravitational intensity across the surface of this sphere is therefore $G(4\pi r^2)$, by definition. But by Gauss' theorem it is $-4\pi\gamma m$. Hence, equating these two expressions, we have

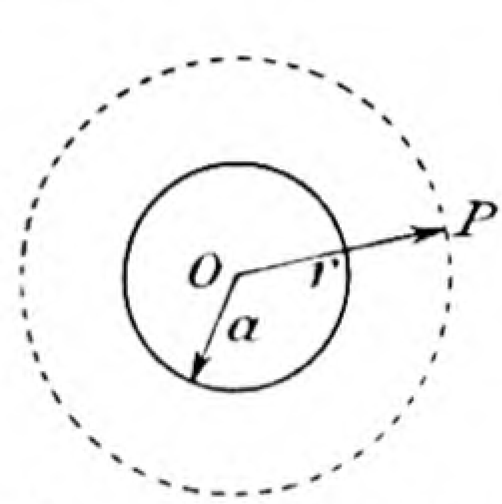


FIG. 6·08.—Gravitational intensity due to an isolated uniform sphere.

$$G = -\gamma\frac{m}{r^2},$$

as already proved. [It must be remembered that this is the value of G in the direction of r increasing; the negative sign indicates that the field is actually directed towards O.]

At a point inside the sphere and at a distance r from its centre, the strength of the field is determined solely by the matter enclosed within the sphere of radius r. Applying Gauss' theorem to the surface of this sphere, we have

$$4\pi r^2 G = -4\pi\gamma(\tfrac{4}{3}\pi\rho r^3),$$

where ρ is the density of the material in the sphere.

$$\therefore\ G = -\tfrac{4}{3}\pi\gamma\rho r, \qquad \text{or} \qquad \vec{G} = -\tfrac{4}{3}\pi\gamma\rho r\hat{r},$$

i.e. the field strength at a point inside a uniform solid sphere varies as the distance of the point from the centre of the sphere.

At the surface of the sphere $G_{r=a} = -\frac{4}{3}\pi\gamma\rho a$, and this is identical with the value of the field if the point at the surface is considered to be outside the sphere; this means that at the surface the field strength is continuous.

(*b*) For a spherical shell it is at once seen that at points outside the shell, the field is the same as if the mass of the shell were

concentrated at its centre; at a point inside the shell Gauss' theorem shows at once that the field is zero. Thus, at the shell, there is a finite discontinuity in the field strength.

Example.—Assume that a smooth straight tunnel exists through the earth which may be regarded as a sphere, at rest, radius 6400 kilometres and of uniform density. Obtain a value for the time required for a particle, initially at rest, to pass from one end of the tunnel to the other. Show that this time is independent of the direction of the tunnel.

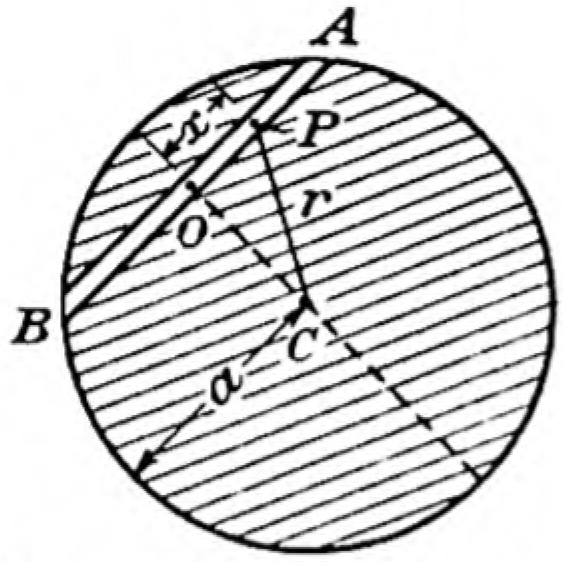


FIG. 6·09.—Motion of a particle in a 'tunnel' through an isolated sphere.

Let C, Fig. 6·09, be the centre of the earth and let AB be the tunnel. Let the diameter through C and perpendicular to AB cut this in O. At time t let the particle be at P, where CP $= r$ and OP $= x$. Then the force per unit mass on the particle is $-\frac{4}{3}\pi\gamma\rho r$, where γ and ρ have their usual meanings and this force is directed along CP. The force per unit mass along PO is $\frac{4}{3}\pi\gamma\rho r\left(\frac{x}{r}\right) = \frac{4}{3}\pi\gamma\rho x$. Thus the motion of the particle is represented by

$$\ddot{x} + \tfrac{4}{3}\pi\gamma\rho x = 0.$$

The motion is therefore simple harmonic with a period

$$T = 2\pi\sqrt{\frac{3}{4\pi\gamma\rho}} = \sqrt{\frac{3\pi}{\gamma\rho}}.$$

This is independent of the size of the sphere and the length of the tunnel. Since $g = \frac{4}{3}\pi\gamma\rho a$, we have $T = 2\pi\sqrt{\frac{a}{g}}$.

The time required is $\frac{1}{2}T = \pi\sqrt{\frac{64 \times 10^7}{980}}$ sec. $\fallingdotseq$ 42 min.

The gravitational field due to an infinite and uniform plate of matter.—Let X_1X_2, Fig. 6·10(a), be a portion of an infinite plate

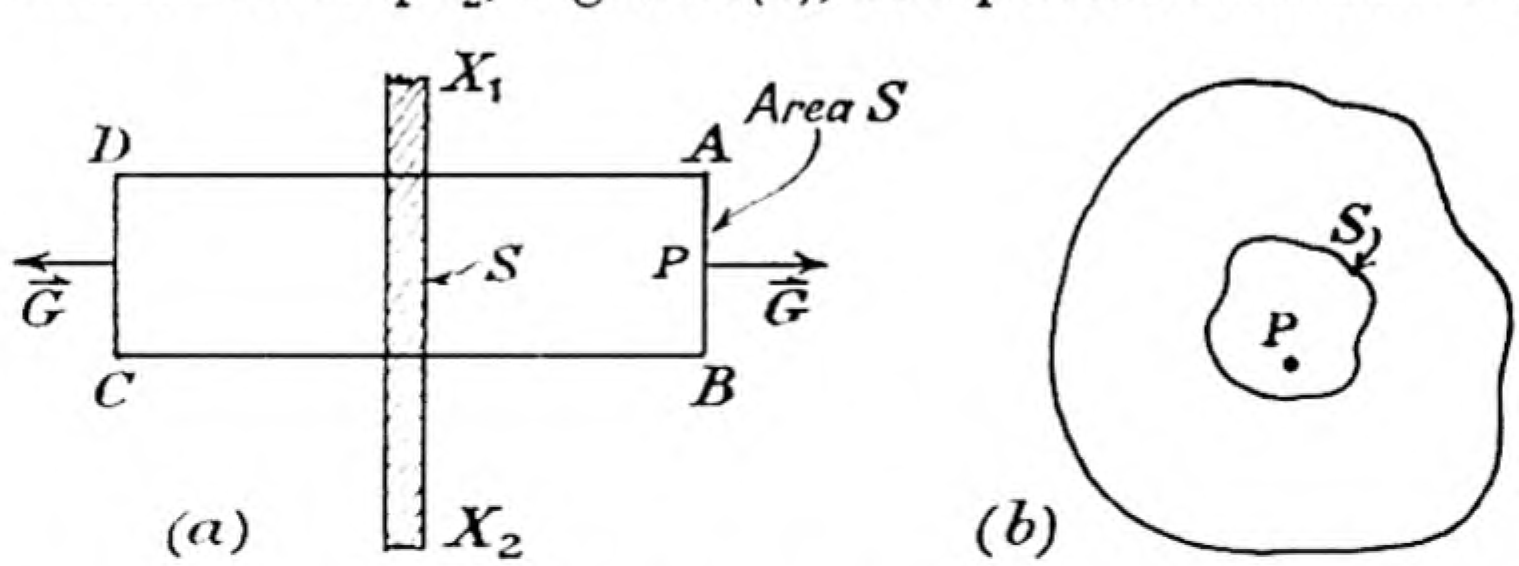


FIG. 6·10.—Gravitational field due to an infinite and uniform plate of matter.

of matter of constant density σ per unit area. Let P be the point at which the field is required. To obtain a suitable Gaussian surface, consider an area S of the plate, cf. Fig. 6·10(b); the projection of P on the plate must lie within the boundary of S. At each

point on the boundary of S draw normals to the plate and let these normals be truncated by planes AB and CD at equal finite distances from the mid-plane of the plate; AB passes through P. Then the flux will be zero across all portions of the Gaussian surface except AB and CD. Let $\overrightarrow{G}$ be the field intensity in the outward direction at any point on AB or CD. Then the flux of gravitational intensity across the surface considered is

$$(\overrightarrow{G}\cdot\hat{n})S + (\overrightarrow{G}\cdot\hat{n})S = 2(\overrightarrow{G}\cdot\hat{n})S.$$

By Gauss' theorem this is $-4\pi\gamma(\sigma S)$.

$$\therefore\ \overrightarrow{G}\cdot\hat{n} = -2\pi\gamma\sigma.$$

Thus $\overrightarrow{G}$ and $\hat{n}$ are antiparallel and the magnitude of the field is $2\pi\gamma\sigma$. It is independent of the distance of P from the sheet.

[N.B. In the corresponding electrical problem the Gaussian surface is made to terminate within the charged metal plate, i.e. where the field is zero; in the present problem this cannot usefully be done.]

Gravitational potential.—***The gravitational potential at a point in a gravitational field is defined as the work per unit mass done against the field in bringing up a small mass from infinity to the point considered.***

Thus if δW is the work done against the field in bringing up from infinity a small mass δm to the given point, the gravitational potential, U, at that point is given by

$$U = \lim_{\delta m \to 0} \frac{\delta W}{\delta m} = \frac{dW}{dm}.$$

Alternatively, we may define a difference in gravitational potential in the following way. If P and Q are two points in a gravitational field and δW is the work done against the field in taking a small particle of mass δm from P to Q, then

$$\lim \frac{\delta W}{\delta m} = \frac{dW}{dm} = dU,$$

and $\int_P^Q dU$ is the increase in potential in passing from P to Q.

Definition.—***A surface in a gravitational field such that at every point on it the gravitational potential has the same value, is known as an equipotential surface.***

Theorem.—*To show that the field strength at a point is normal to the equipotential which passes through that point.*

Let A and B, Fig. 6·11(a), be two points at a distance δs apart on an equipotential surface, U, and suppose that at A the field strength is $\vec{G}$ and that this vector makes an angle θ with $\vec{\delta s}$. The component of $\vec{G}$ along $\vec{\delta s}$ has a value G cos θ, so that the work done per unit mass against the field in passing from A to B is

$$-G \cos \theta \, \delta s,$$

the negative sign being necessary since the work is done by the

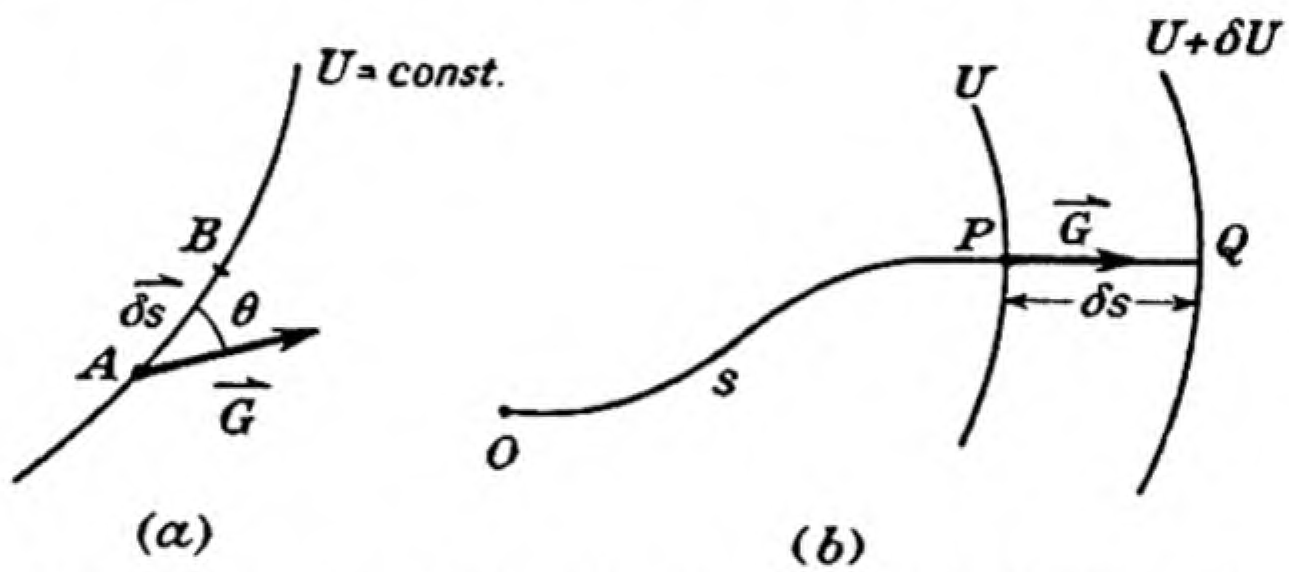


FIG. 6·11.—Gravitational potential and field strength.

field. This work is necessarily zero since A and B are points on an equipotential surface. Hence if $\vec{G}$ is not zero, cos θ must be zero, i.e. $\theta = \frac{1}{2}\pi$, or the field is normal to the equipotential at the point considered.

On the relation between gravitational potential and the field strength.—Let U and U + δU, Fig. 6·11(b), be two neighbouring equipotentials. Then if P is a point on the surface U, the field strength, $\vec{G}$, at P will be directed along the normal at P to the surface. Let this normal intersect the surface U + δU in Q. Then in taking a small particle of mass δm from P to Q, the work done per unit mass against the field is $-G(PQ) = -G\,\delta s$, where $PQ = \delta s$, s and $s + \delta s$ being the distances of P and Q from some fixed origin O.

Thus $$\delta U = -G\,\delta s,$$

or $$G = -\lim \frac{\delta U}{\delta s} = -\frac{dU}{ds}.$$

In vector notation this may be written

$$\vec{G} = -\hat{s}\frac{dU}{ds},$$

where $\hat{s}$ is the appropriate unit vector.

The gravitational potential due to a particle of matter.—Let m be the mass of a particle situated at O, Fig. 6·12(a). Let A be the point at which the gravitational potential due to the above particle is required. Call $\text{OA} = a$. Now the field strength at a

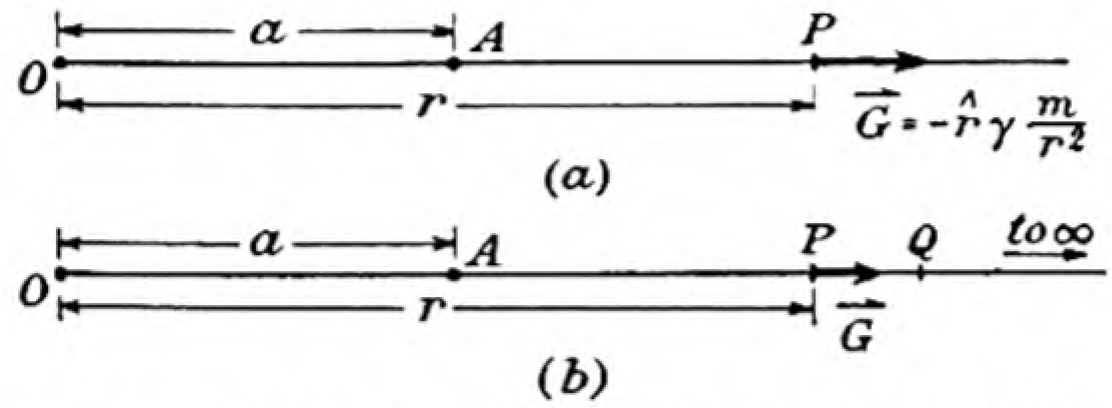


FIG. 6·12.—Gravitational potential due to a particle of matter.

point P at distance r from O, and measured in the direction of r increasing, is given by

$$\text{G} = -\gamma\frac{m}{r^2} = -\frac{d\text{U}}{dr},$$

where U is the gravitational potential at the point considered. Integrating this equation, we obtain

$$\text{U} = -\gamma\frac{m}{r} + \text{K},$$

where K is an arbitrary constant. It is customary to take the potential as zero at infinity so that

$$\text{U}_{r\to\infty} = -\gamma\frac{m}{\infty} + \text{K} = 0,$$

i.e.

$$\text{K} = 0,$$

$$\therefore\ \text{U} = -\gamma\frac{m}{r}.$$

If the potential is required at a point A, distance a from O, i.e. $\text{OA} = a$, we have

$$[\text{U}]_{r=a} = -\gamma\frac{m}{a}.$$

An alternative method of obtaining this equation is as follows. Let Q, Fig. 6·12(b), be a point at distance $r + \delta r$ from O and suppose δm is brought from infinity to the point, its path passing from Q to P and being along the straight line of which OP is part. Let $\vec{\text{G}}$ be the field strength at P; it is considered positive when it acts in the

direction of r increasing. If $PQ = \delta r$, the work done per unit mass against the field in passing from Q to P is

$$G \times (QP) = -G\,\delta r,$$

since $QP = -\delta r$. This is δU.

$$\therefore\ \delta U = -G\,\delta r.$$

But
$$G = -\frac{\gamma m}{r^2},$$

so that
$$\delta U = \frac{\gamma m}{r^2}\,\delta r.$$

$$\therefore\ U = \int_{\infty}^{a} \frac{\gamma m}{r^2}\,dr = -\frac{\gamma m}{a}.$$

Now if we return from P to infinity along some other path an equal amount of work must be done by the field, for otherwise energy could be gained or lost merely by taking a small particle along a closed path part of which is at infinity. Hence the potential at P must be independent of the path whereby the small element of matter is brought to P although, to evaluate this work, a straight line path through O and P is selected.

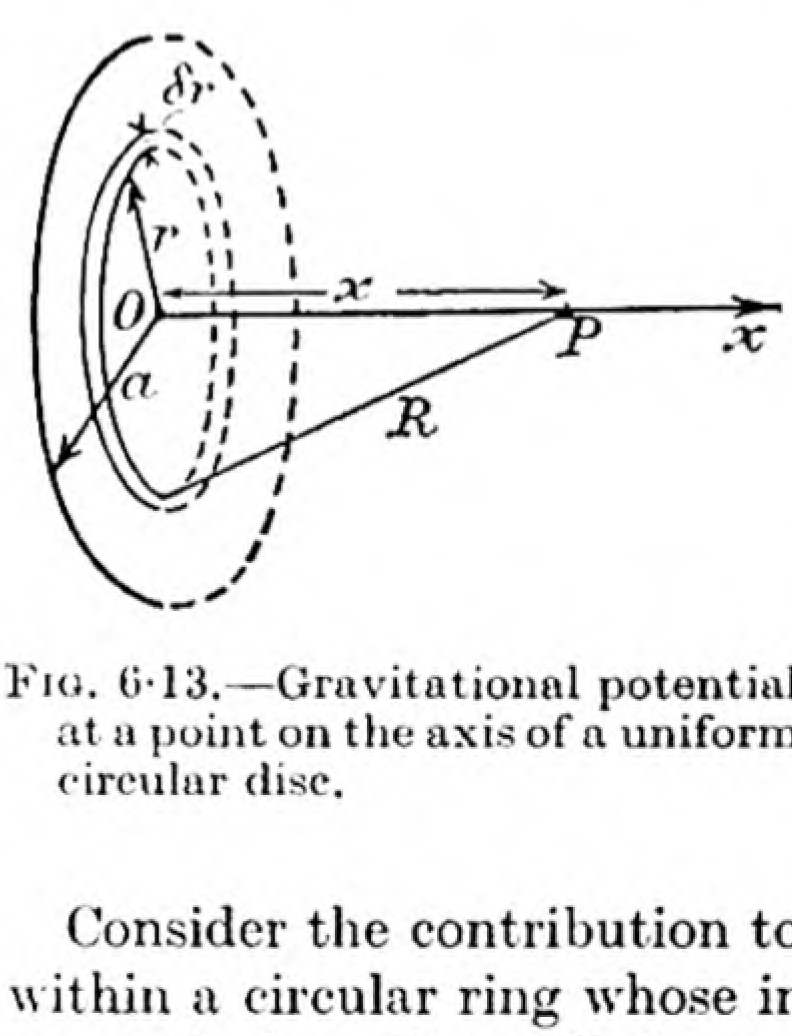


FIG. 6·13.—Gravitational potential at a point on the axis of a uniform circular disc.

The gravitational potential at a point on the axis of a uniform circular disc.—Let O, Fig. 6·13, be the centre of a uniform circular disc, radius a and mass per unit area σ. Let P be a point on the axis Ox, through O and normal to the plane of the disc.

Consider the contribution to the potential at P due to the matter within a circular ring whose inner and outer radii are r and $r + \delta r$, respectively. Since all elements of this ring are at the same distance from P, say R, we have

$$\delta U = -\gamma . \frac{2\pi r \sigma\,\delta r}{R}.$$

$$\therefore\ U = -2\pi\gamma\sigma \int_0^a \frac{r\,dr}{(r^2 + x^2)^{\frac{1}{2}}}$$

$$= -2\pi\gamma\sigma[(a^2 + x^2)^{\frac{1}{2}} - x].$$

$$\therefore\ G = -\frac{\partial U}{\partial x} = -2\pi\gamma\sigma\left[1 - \frac{x}{(a^2 + x^2)^{\frac{1}{2}}}\right].$$

When $a \to \infty$,

$$G_{a\to\infty} = -2\pi\gamma\sigma,$$

i.e. the field is constant at all points at a finite distance from the disc; cf. also p. 216.

On the gravitational potential due to a spherical shell.—Since at external points a spherical shell of matter produces the same gravitational field as if the whole mass, m of the shell were concentrated at its centre, the potential at external points, due to such a shell, is given by $-\gamma\frac{m}{r}$, where r is the distance of the point from the centre of the shell.

Inside the shell the field is zero, i.e. the potential is constant and, since potential is necessarily a continuous function, the potential inside the shell must be equal to that at its surface, viz.

$$U_{\text{in}} = -\gamma.\frac{m}{a},$$

where a is the radius of the shell.

On the gravitational potential due to an isolated uniform sphere.—Let O, Fig. 6·14(a), be the centre of a uniform sphere of

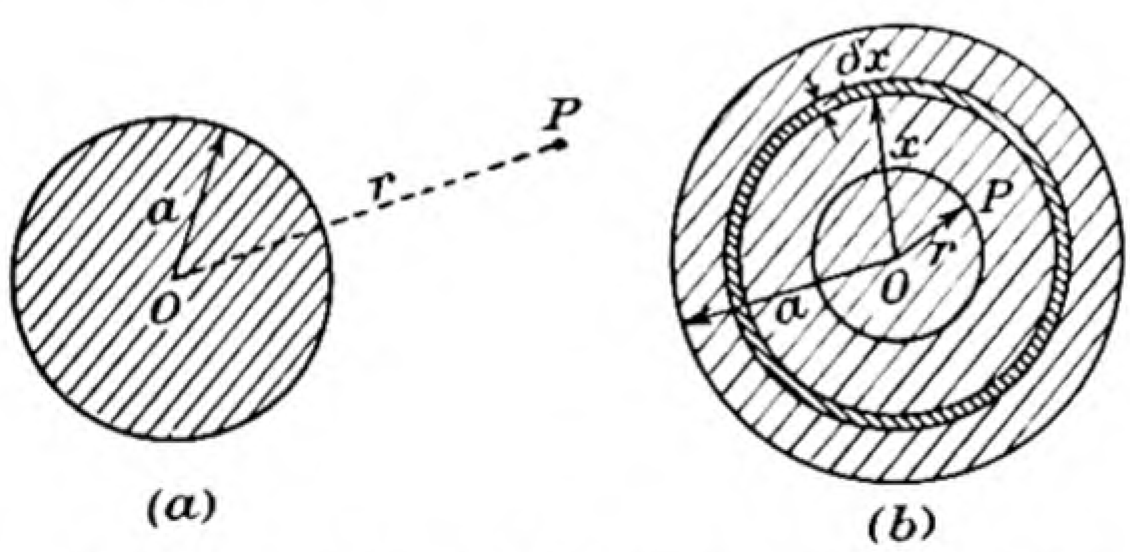


FIG. 6·14.—Gravitational potential due to an isolated uniform sphere.

radius a and mass m, and let P be the point at which the potential is required. Two cases arise.

(a) Let P be outside the sphere, i.e. $r > a$, where $r =$ OP. Then

$$U_{r\geqslant a} = -\gamma.\frac{m}{r},$$

since the mass of the sphere may be supposed concentrated at O.

In passing we note that

$$U_{r=a} = -\gamma.\frac{m}{a},$$

and

$$G_{r=a} = \left[-\frac{dU}{dr}\right]_{r=a} = \left[-\frac{d}{dr}\left(-\frac{\gamma m}{r}\right)\right]_{r=a} = \left[-\frac{\gamma m}{r^2}\right]_{r=a} = -\frac{\gamma m}{a^2}.$$

(*b*) If P is inside the sphere, i.e. $r < a$, as in Fig. 6·14(*b*), then the potential at P arises from the matter inside the sphere of radius r and from that outside it. Let these portions of the total potential be denoted by U_1 and U_2 respectively. Now the mass of the sphere of radius r is $\frac{4}{3}\pi\rho r^3$, so that the potential at P, a point on its surface, is given by

$$U_1 = -\frac{4}{3}.\frac{\pi\gamma\rho r^3}{r} = -\frac{4}{3}.\pi\gamma\rho r^2.$$

To calculate U_2, let us divide the portion of the whole sphere outside the sphere on which P lies into concentric shells, one of these being shown in Fig. 6·14(*b*). Let its radius be x and its thickness δx. Then the potential at P due to this shell is the same as that of a point just outside the shell, for the potential inside a shell is constant and equal to that at the surface. Thus

$$\delta U_2 = -\frac{4\pi\gamma x^2\rho\,\delta x}{x} = -4\pi\gamma\rho x\,\delta x.$$

Hence

$$U_2 = -4\pi\gamma\rho\int_r^a x\,dx = -2\pi\gamma\rho(a^2 - r^2).$$

Thus, by addition, potential being a scalar quantity,

$$\begin{aligned} U_{r<a} &= U_1 + U_2 = -\tfrac{4}{3}\pi\gamma\rho[r^2 + \tfrac{3}{2}(a^2 - r^2)] \\ &= -\tfrac{4}{3}\pi\gamma\rho a^3\left[\frac{3}{2a} - \frac{r^2}{2a^3}\right] = -\gamma m\left[\frac{3}{2a} - \frac{r^2}{2a^3}\right]. \end{aligned}$$

On differentiating, we find, for the field inside the solid sphere,

$$G_r = -\frac{\partial U}{\partial r} = -\frac{4}{3}\pi\gamma\rho r = -\frac{\gamma m}{a^3}r,$$

and

$$G_\theta = -\frac{1}{r}\frac{\partial U}{\partial\theta} = 0,$$

so that the field is entirely radial.

In addition, it may be noted that the potential at the centre of the sphere is to that at a point on its surface as 3 : 2, the values being $-\dfrac{3\gamma m}{2a}$ and $-\dfrac{\gamma m}{a}$, respectively.

On the gravitational potential due to a uniform thick spherical shell.—Let O, Fig. 6·15(*a*), be the centre of a uniform thick shell, the material of which has a density ρ, while a and b $(a < b)$ are the radii of the surfaces of the shell, as indicated. Then m, the mass of the shell, is given by

$$m = \tfrac{4}{3}\pi\rho(b^3 - a^3).$$

The potential has to be calculated at points lying in each of the three regions indicated. Let r be the distance of the point considered from O.

(*a*) $b \leqslant r$. The gravitational potential in this region is given by

$$U_{b\leqslant r} = -\tfrac{4}{3}\pi\gamma\rho(b^3 - a^3)\frac{1}{r} = -\gamma\frac{m}{r},$$

since as far as this region is considered, every elementary concentric

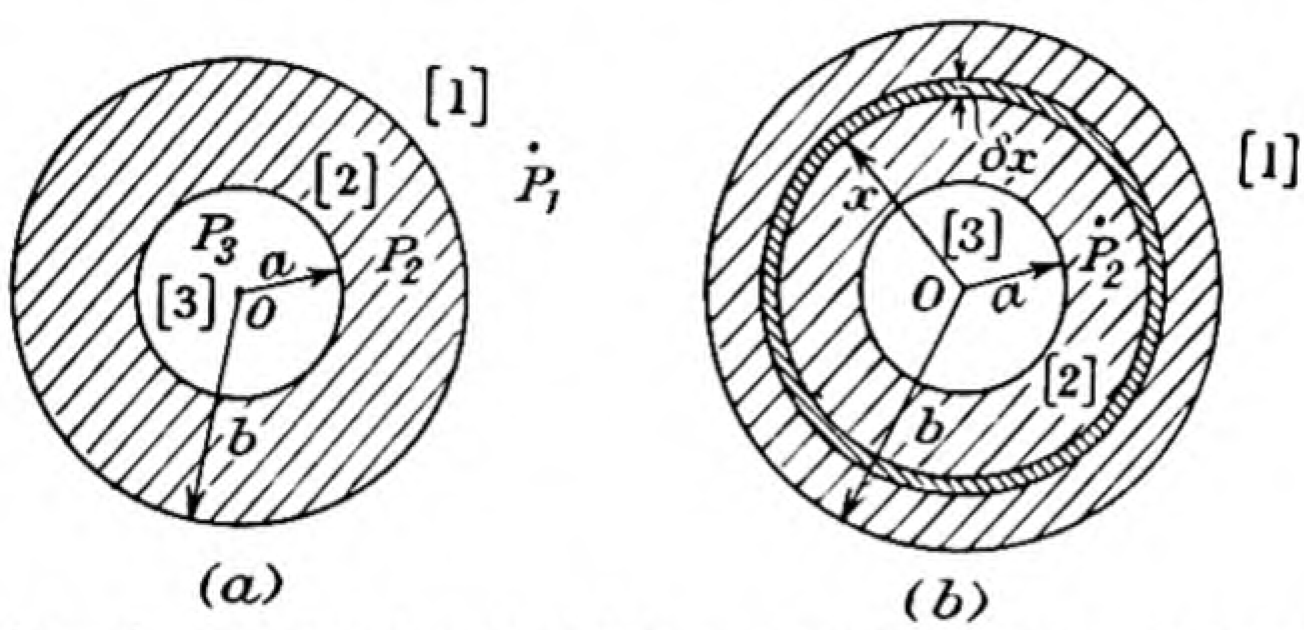


FIG. 6·15.—Gravitational potential due to a uniform thick spherical shell.

shell into which the thick shell may be assumed to be divided, and hence the whole shell itself, behaves as if its mass were concentrated at O.

(*b*) $a \leqslant r \leqslant b$. The gravitational potential at P_2, a point in this region and such that $OP_2 = r$, is due (*a*) to the matter lying within the shell of radii a and r and (*b*) to the matter lying in a shell of radii r and b. If we denote these potentials by U_1 and U_2 respectively, we have

$$U_1 = -\tfrac{4}{3}\pi\gamma\rho.\frac{r^3 - a^3}{r},$$

since P_2 is a point external to this part of the thick shell.

To calculate U_2, i.e. the contribution to the potential at P_2 of the matter within the shell defined by $r = r$ and $r = b$, let us consider the contribution from the shell defined by radii x and $x + \delta x$, where $r \leqslant x$, cf. Fig. 6·15(*b*). Then the potential of this shell due to itself is the contribution it makes to the potential at P_2, viz.

$$-\frac{4\pi\gamma\rho x^2\,\delta x}{x} = -4\pi\gamma\rho x\,\delta x.$$

Hence

$$U_2 = -\int_r^b 4\pi\gamma\rho x\,dx = -2\pi\gamma\rho(b^2 - r^2).$$

$$\therefore\ U_{a\leqslant r\leqslant b} = U_1 + U_2 = -\gamma m\frac{1}{b^3 - a^3}\left[\frac{3rb^2 - r^3 - 2a^3}{2r}\right].$$

(*c*) $0 \leqslant r \leqslant a$. In this region, free from matter, the potential is constant and equal to that at a point on the surface $r = a$; this is calculated by using the expression for the potential at a point in the region [2]. Thus

$$U_{0 \leqslant r \leqslant a} = -\frac{3}{2}\gamma m \frac{b+a}{b^2+ab+a^2}.$$

On the potential energy of a body at a height *h* above the earth's surface.—Let A be the point on the earth's surface and B a point at a height h above A. If r is the radius of the earth (assumed spherical), h is assumed small compared with r, i.e. $\frac{h}{r} \to 0$. If M is the mass of the earth and γ the Newtonian constant of gravitation, the potential energy of a particle of mass m at A is given by

$$W_A = -\gamma\left(\frac{M}{r}\right)m,$$

since the gravitational potential is $-\gamma\frac{M}{r}$.

Similarly,

$$W_B = -\gamma\left(\frac{M}{r+h}\right)m.$$

$\therefore\ W_B - W_A =$ increase in potential energy of the particle in passing from A to B

$$= -\gamma Mm\left[\frac{1}{r+h} - \frac{1}{r}\right] = -\gamma Mm\left[\frac{1}{r}\left(1 - \frac{h}{r}\right) - \frac{1}{r}\right]$$

$$= \gamma\frac{Mmh}{r^2} = mgh,$$

where g is the intensity of gravity, viz. $\left[-\overrightarrow{G}\right]_{r=a}$, or $\gamma\frac{M}{r^2}$.

On the attraction between two parts of the same body.—If the two parts of the body be denoted by A and B, cf. Fig. 6·16, the attraction of the whole on A is the attraction of B on A, plus the attraction of A on itself. But this latter attraction is zero. Hence the attraction between two parts of the same body is the same as the attraction of the whole on one part.

The gravitational field strength within an eccentric spherical cavity.—Let C, Fig. 6·17, be the centre of a spherical cavity, radius a, within a sphere whose centre is O and whose radius is b. Let ρ be the uniform density of the material filling the space between the two spherical surfaces. The eccentric shell thus formed may be considered as built up of a complete sphere of radius b and

density ρ together with a sphere of radius a and density $-\rho$. If P is a point inside the cavity, the field strength at that point will have components

$$-\tfrac{4}{3}\pi\gamma\rho\mathrm{OP} \text{ along OP},$$

and

$$-\tfrac{4}{3}\pi\gamma(-\rho)\mathrm{CP} \text{ along CP}.$$

These components are represented in magnitude and direction

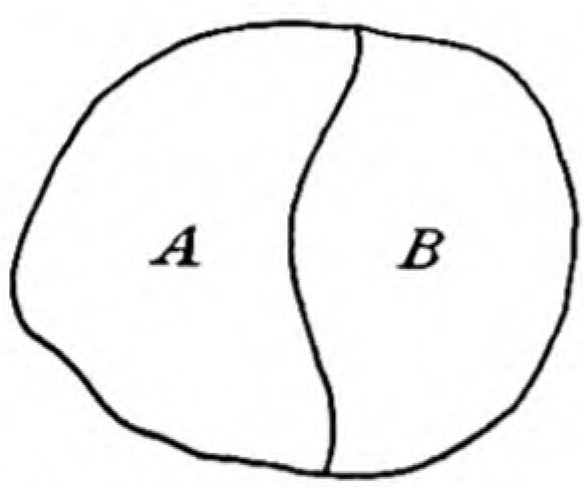


FIG. 6·16.—The attraction between two parts of the same body.

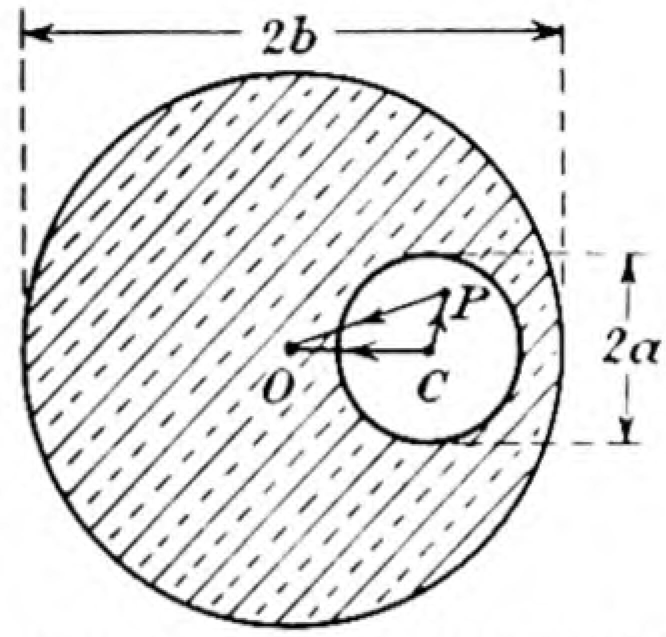


FIG. 6·17.—The gravitational field strength within an eccentric spherical cavity.

by the sides PO and CP of the Δ OCP. The resultant field is therefore represented in magnitude and direction by CO. Thus

$$\overrightarrow{\mathrm{G}}_{\mathrm{P}} = -\tfrac{4}{3}\pi\gamma\rho\,\overrightarrow{\mathrm{OC}} = \tfrac{4}{3}\pi\gamma\rho\,\overrightarrow{\mathrm{CO}},$$

and since this is independent of the position of P, the field throughout the cavity is uniform.

The attraction between two halves of a uniform isolated sphere.—If we consider an element of volume $r^2 \sin\theta\,\delta r\,\delta\theta\,\delta\phi$, cf. Fig. 1·08($b$), the force on it due to the rest of the sphere is along PO and has magnitude

$$\tfrac{4}{3}\pi\gamma\rho r\,.\,\rho\,.\,r^2 \sin\theta\,\delta r\,\delta\theta\,\delta\phi,$$

where ρ is the density of the material of the sphere; this is because the element is at a point on the surface of a sphere of radius r. The component of this force normal to the plane AOC, which is taken as the dividing plane of the sphere, is

$$\tfrac{4}{3}\pi\gamma\rho^2 r^3 \sin^2\theta \sin\phi\,\delta r\,\delta\theta\,\delta\phi.$$

This is obtained by considering the component along PN and the component of this normal to the plane AOC. For the hemisphere the force of attraction is obtained by integrating in turn between the limits $\phi = 0$, $\phi = \pi$, $\theta = 0$, $\theta = \pi$ and $r = 0$ to $r = a$, where

a is the radius of the complete sphere (not indicated). The result is easily shown to be

$$\frac{3}{16}\frac{\gamma m^2}{a^2},$$

where m is the mass of the sphere.

The above result may also be obtained as follows. If P is the point at distance y from the plane AOC, the attraction per unit mass at P is directed along PO and has magnitude $\frac{4}{3}\pi\gamma\rho r$, where $\mathrm{OP} = r$; its component normal to the plane AOC is $\frac{4}{3}\pi\gamma\rho y$ and this is the same for all points in a plane through P normal to Oy. The attraction on the circular disc between the planes $y =$ constant and $y + \delta y =$ constant is

$$\tfrac{4}{3}\pi\gamma\rho y \,.\, \pi\rho(a^2 - y^2)\,\delta y,$$

since $\sqrt{a^2 - y^2}$ is the radius of this disc. Hence the mutual attraction is

$$\tfrac{4}{3}\pi^2\gamma\rho^2\int_0^a y(a^2 - y^2)\,dy,$$

which gives the result already obtained.

The constant of gravitation; the mass and mean density of the earth.—The force exerted by the earth on a body just outside its surface is the special instance of a force which is accounted for by Newton's law of gravitation. Thus, if m is the mass of the body, M that of the earth and R its radius, then F, the force along the outward drawn normal at a point on the earth's surface, is given by

$$\mathrm{F} = -\gamma\,.\,\frac{\mathrm{M}m}{\mathrm{R}^2}\,.$$

The force along the inward drawn normal is $-\mathrm{F}$, i.e. $\gamma\,\frac{\mathrm{M}m}{\mathrm{R}^2}$, and this is equal to the weight of the body, viz. mg, where g is the intensity of gravity. Hence

$$g = \gamma\,.\,\frac{\mathrm{M}}{\mathrm{R}^2} = \tfrac{4}{3}\pi\gamma\mathrm{DR},$$

where D is the mean density of the earth.

Since g and R are known quantities, it follows that if either γ or M can be determined, the other may be calculated at once.

The rest of this chapter will describe experiments made to determine γ, the constant of gravitation; they fall naturally into two classes. In the first class some large terrestrial mass, such as a mountain of convenient size, shape and position, or the outer shell of the earth's crust is selected, and its mass determined from an ordnance survey and an examination of the distribution and density of the materials in it. Its attraction on a plumb-line on one side

of it or on a pendulum above or below it, is then compared with that of the whole earth on the same body. The mass of the earth is thus determined, whence γ may be deduced from the equation given above. In the second class, the experiment is conducted in a suitable laboratory, the attraction between smaller masses being measured. Such an experiment gives γ directly.

Intimately associated with a determination of the constant of gravitation is one in which the mean density of the earth is found. This is defined as the density of a homogeneous isotropic body which would have the same mass and the same volume as the earth itself.

As regards experiments of the first type, referred to above, MASKELYNE, who used this method, says 'Sir Isaac Newton gave us the first hint of such an attempt' when he wrote ' "That a mountain of a hemispherical figure, three miles high and six broad, will not, by its attraction, draw the plumb-line two minutes out of the perpendicular" '. But the honour of making the first attempt to solve this problem must be given to BOUGUER.

The experiments of Bouguer in Peru.—The first experiments on the attraction between terrestrial masses were made by BOUGUER who published an account of them in 1749. By means of experiments with a pendulum and with a plumb-line he endeavoured to determine the mean density of the earth. No quantitative significance can be attached now to the results he obtained, but the honour of showing that such an attraction did exist must be credited to this investigator. Only his plumb-line experiments will be described.

The mountain selected for this purpose was Chimboraço, which is four miles high though only two miles above the general level of the province of Quito in Peru. Two methods, amongst others, mentioned by Bouguer as a means of solving this problem are as follows. In the first the meridian altitude of a star is observed by a quadrant, whose zenith reading is fixed by a plumb-line, at a point on the north or south side of the mountain and then at a point east or west of the first station, but so far removed from the mountain that the effect of its attraction on the plumb-line would be negligible. This is the method he adopted, the quadrant having a radius of 2·5 feet. A second mode of procedure is to make observations on the north and south sides of the mountain; the difference between the astronomical and the geodetic or geographical latitude is double the deflexion due to the mountain.

The work of Bouguer was carried out under the most strenuous conditions; it was affected especially by the intense cold, for the observing stations were above the snow-line, and by the presence of a high wind, the full force of which was experienced at the southern station. However, Bouguer succeeded in making several

observations and, after making several small corrections, concluded that the angular deflexion of the plumb-line due to the mountain was 8″.

Bouguer concludes his account of these experiments by expressing the hope that a suitable hill might be found in France, or in England, and the experiment repeated. This desire was not fulfilled at once.

The work of Maskelyne.—In 1772, MASKELYNE,† the Astronomer Royal at that time, proposed to carry out an experiment on the lines of the above and a committee was appointed by the Royal Society 'to consider a proper hill whereon to try the experiment, and to prepare everything necessary for carrying the design into execution'. After rejecting several hills he says 'Fortunately, however, Perthshire affords us a remarkable hill, nearly in the centre of Scotland, of sufficient height, tolerably detached from other hills and considerably longer from east to west than from north to south'. This hill is known as 'Schiehallion', a word meaning 'Constant storm'.

Stations in the meridian and on the steeply sloping north and south sides of this hill were chosen; each station was about one-quarter of the way up the mountain in order to obtain maximum deflexions of the plumb-line. Maskelyne used a ten-foot zenith sector whose telescope turned on a horizontal axis at the objective end, and at the eye-piece end was furnished with a graduated arc several degrees in extent. The telescope had an achromatic objective. A plumb-line hung from the horizontal axis of the instrument and just came into contact with the graduated arc, thereby fixing the positions of the zenith. The instrument was placed with its plane in the geographical meridian, and the apparent zeniths of several stars were observed at the two stations, which were approximately one quarter of the way up the mountain. Corrections were made for precession, aberration and other small factors affecting the observations. Finally he showed that the apparent or astronomical difference of latitude between the two stations was 54·6″. The difference in geographical latitude was 42·94″, so that the effect of the mountain was to deflect the plumb-line at either station by $\frac{1}{2}(54{\cdot}6 - 42{\cdot}94'') = 5{\cdot}8''$.

To explain this, let A and B, Fig. 6·18(a), be the two stations, first supposing the mountain to be absent. Then the directions of the zeniths at these stations will be given by AZ_{A_1} and BZ_{B_1}, respectively. When these lines are produced backwards they will intersect at C_1, the centre of the earth. Let SA and SB be two parallel rays from a fixed star, S. Then the zenith distances of the above star are α_1 and β_1, these being the angles indicated. Actually these were small angles for the observations were confined to stars

† *Phil. Trans.*, 1775 and 1778.

near to the zenith, but in the diagram they must necessarily be enlarged. If λ_1 is the difference in geographical latitude between A and B, then it follows at once that

$$\alpha_1 - \beta_1 = \lambda_1.$$

Suppose now that the mountain is present; at the two stations

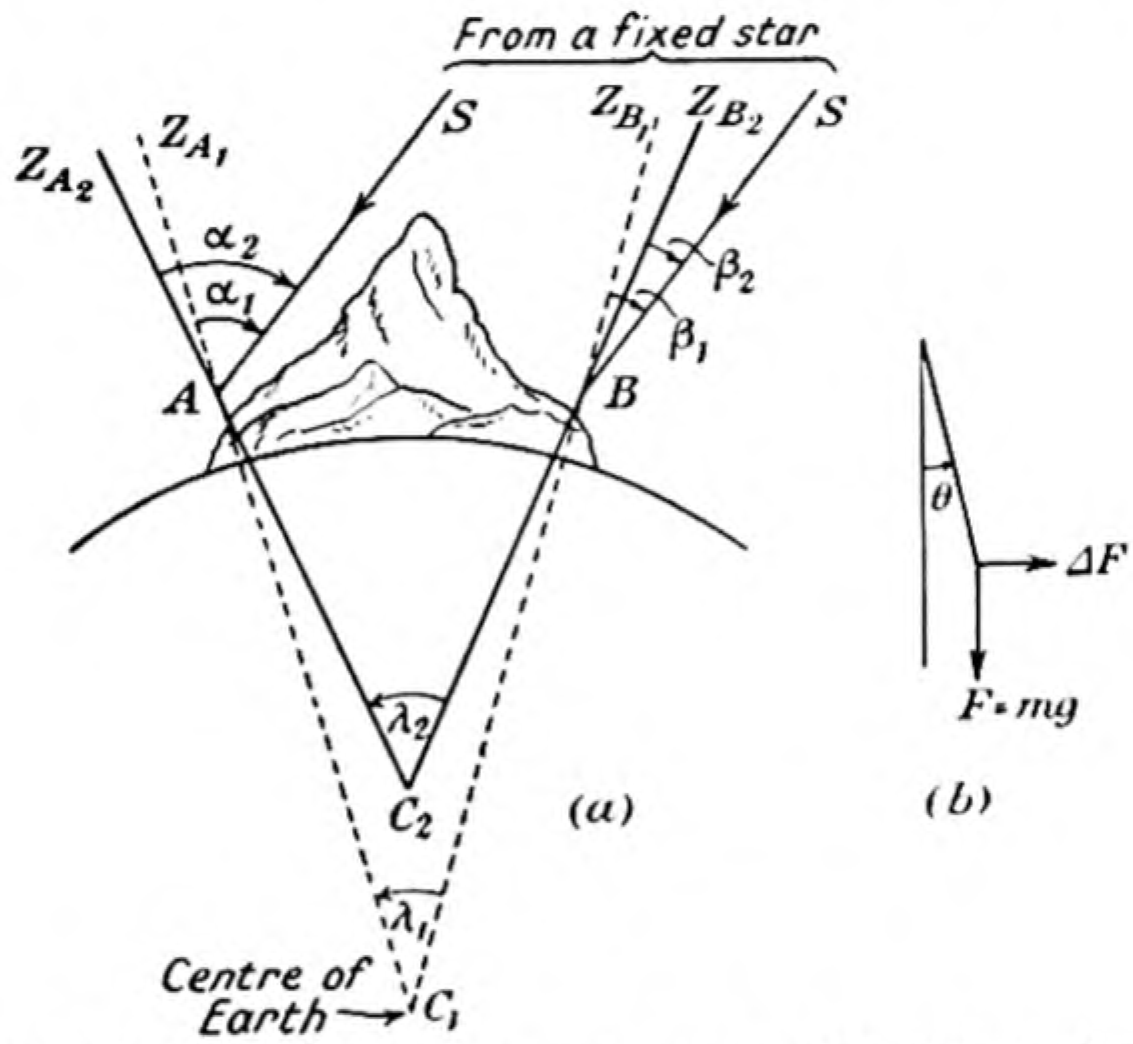


FIG. 6·18.—Plumb-line experiments on Mt. Schiehallion.

the plumb-line will set along the directions AZ_{A_2} and BZ_{B_2} respectively, and these lines when produced will intersect in C_2. The $A\widehat{C_2}B$ is λ_2, the difference in astronomical latitude between A and B, and is given by

$$\alpha_2 - \beta_2 = \lambda_2,$$

where α_2 and β_2 are the zenith distances of the star as observed from A and B. Maskelyne found, as already stated, that

$$\lambda_2 - \lambda_1 = 11{\cdot}6''.$$

Now let ΔF be the attraction of the mountain on the bob of the plumb-line along a horizontal direction in the meridinal plane, while F is the attraction of the earth on the bob, so that $F = mg$, where m is the mass of the bob. Then θ, the deflexion of the plumb-line from the true vertical, is given by

$$\frac{\Delta F}{F} = \tan\theta = \tan\tfrac{1}{2}(\lambda_2 - \lambda_1),$$

as is easily seen from Fig. 6·18(b). But $F = \gamma\frac{mM}{R^2}$, where M is the mass and R the radius of the earth. [In discussing experimental determinations of γ it is sometimes convenient to consider the

attraction between the bodies concerned and so avoid the minus sign occurring in the mathematical expression of Newton's law of gravitation.] Now ΔF can be calculated in terms of the mass of the mountain, the mean density of the materials in it, and of γ. Thus M can be calculated. These calculations were made by HUTTON, who gave as D, the mean density of the earth, a value $D = 4{\cdot}5$ gm.cm.$^{-3}$. Later on, after a careful survey of the mountain and district by PLAYFAIR, Hutton recalculated the mean density and found it to be $4{\cdot}9$ gm.cm.$^{-3}$.

In 1855, a similar experiment was made by JAMES and CLARKE at Arthur's Seat, Edinburgh. They found $D = 5{\cdot}42$ gm.cm.$^{-3}$, and showed that Hutton did not survey the land sufficiently far from Schiehallion.

These experiments are, historically, very important, and although the technique could probably be greatly improved at the present time, the inherent difficulty in the method lies in making an accurate estimate of the mass of the hill, and it does not seem likely that this will be overcome.

FIG. 6·19. Airy's method for finding γ.

Airy's experiments.—In 1826, Airy pointed out that the mean density of the earth might be determined by observing the difference in the rates of a pendulum at the top and bottom of a mine—the points A and B in Fig. 6·19. Let D be the mean density of the matter inside a sphere whose centre is that of the earth and whose surface passes through the bottom of the mine. Let h be the depth of the mine, and d the mean density of the matter in the shell of earth of thickness h. If G_A and G_B be the strengths of the gravitational field, and g_A and g_B the intensities of gravity at A and B respectively, then

$$G_B = -\gamma.\frac{\frac{4}{3}\pi r^3 D}{r^2} = -\tfrac{4}{3}\gamma\pi r D,$$

$$\text{and } G_A = -\gamma\left[\frac{\frac{4}{3}\pi r^3 D}{(r+h)^2} + \frac{\frac{4}{3}\pi\{(r+h)^3 - r^3\}d}{(r+h)^2}\right]$$

$$= -\tfrac{4}{3}\pi\gamma\left[\frac{r^3 D + (r^3 + 3r^2 h - r^3)d}{r^2\left(1 + \frac{2h}{r}\right)}\right] \qquad \text{[neglecting terms in } h^2 \text{ and } h^3]$$

$$= -\tfrac{4}{3}\pi\gamma\left[r\left(1 - \frac{2h}{r}\right)D + 3h\left(1 - \frac{2h}{r}\right)d\right] \qquad \left[\because \frac{h}{r} \to 0\right]$$

$$= -\tfrac{4}{3}\pi\gamma[(r - 2h)D + 3hd] \quad \text{[neglecting the terms in } h^2]$$

$$\therefore \frac{g_A}{g_B} = \frac{G_A}{G_B} = \left(1 - \frac{2h}{r}\right) + 3\frac{h}{r}.\frac{d}{D},$$

i.e. $$D = \frac{d}{\frac{2}{3} - \left[1 - \frac{g_A}{g_B}\right]\frac{r}{3h}} = \frac{d}{\frac{2}{3} - \left[1 - \left(\frac{T_B}{T_A}\right)^2\right]\frac{r}{3h}},$$

where T_A and T_B are the periods of the pendulum when at A and at B, respectively.

Airy made attempts to compare indirectly the values of the acceleration of a freely falling body at the top and bottom of a copper mine in Cornwall, a pendulum being used for this purpose. Two attempts failed owing to accidents to the apparatus. In 1854,† Airy, the Astronomer Royal, again attacked the subject and selected for the experiment Harton Pit, 1320 feet deep, in the county of Durham. Now his work was much facilitated by the use of electrical signalling in comparing the rates of the two clocks which were used, one at the top and the other at the bottom of the mine. The period of an invariable pendulum at each station was determined by the method of coincidences.

Airy found that '***Gravity below was greater than gravity above by $\frac{1}{19286}$th part, with an uncertainty of $\frac{1}{270}$th part of the excess.***' This gave for D a value $(6{\cdot}57 \pm 0{\cdot}04)$ gm.cm.$^{-3}$. Later on it was recalculated from Airy's observations by HAUGHTON and he obtained a value $D = (5{\cdot}48 \pm 0{\cdot}02)$ gm.cm.$^{-3}$. STERNECK, in Bohemia, using a similar method in 1883, found D to be about 5·7 gm.cm.$^{-3}$. The experiments were conducted in a mine 1000 metres deep.

In the above experiments the effects due to the ellipticity of the earth and its rotation are neglected. STOKES first showed that the correction on these accounts was very small.

Cavendish's determination of the constant of gravitation and of the mean density of the earth.—The mutual attraction between two lead spheres was measured by CAVENDISH‡ who used a method devised by MICHELL. This last named investigator constructed an apparatus for this purpose, but died before the experiment could be made. Cavendish modified, and in a great measure reconstructed, the apparatus but the principle remained unchanged.

Two equal balls of lead were suspended from the extremities of a light, horizontal rod or beam, this being hung from a torsion wire W, as in Fig. 6·19. The balls were subject to the force of gravity and practically to no other, so that in the absence of air currents the

† Airy, *Phil. Trans.*, **146**, 297, 1856.
‡ *Phil. Trans.*, **88**, 398, 1798.

suspended system would take up a position of rest for which there was no torsion in the wire. Two large equal balls of lead, M_1 and M_2, were then placed so that the centres of the four balls lay in a horizontal plane and each of the larger spheres was near to one of the smaller spheres but on opposite sides of the beam. Horizontal forces, due to attraction, were thereby exerted on the smaller spheres so that a couple was applied to the suspended system, which moved to a new position of equilibrium in which the above couple was balanced by a couple due to torsion in the wire. The deflexion of the suspended system was measured, and, when the torsional constant of the wire had been determined, the force of attraction between a large and a small sphere could be calculated in absolute units and thus γ, and hence D, determined.

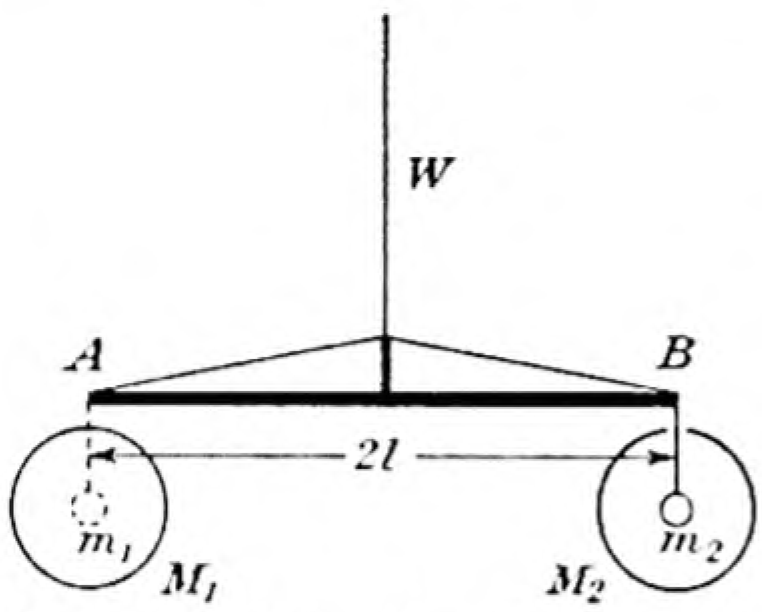


FIG. 6·20.—Principle of Cavendish's method for finding γ.

The mathematical analysis of the method, neglecting all corrections, is as follows: Let the mass of each suspended ball be m, while M is the mass of each of the larger spheres. Let $2l$ be the length of the beam. If ϕ is the steady angular deflexion of the rod from its zero position, the couple due to torsion in the wire is $b\phi$, where b is the couple for unit twist. The value of b may be determined from observations on T, the period of small oscillations of the beam and the masses suspended from it, for [cf. p. 292] it is known that

$$T = 2\pi\sqrt{\frac{I}{b}},$$

where I is the moment of inertia of the suspended system about its axis of rotation, i.e.

$$b = \frac{4\pi^2 I}{T^2}.$$

But since the force of attraction between each pair of spheres is $\gamma \cdot \frac{mM}{r^2}$, where γ is the gravitational constant and r the distance between the centres of attracting spheres, the couple on the beam is equal to

$$\gamma \cdot \frac{mM}{r^2}(2l).$$

$$\therefore\ 2\gamma l.\frac{mM}{r^2} = b\phi = \frac{4\pi^2 I}{T^2}.\phi,$$

i.e.
$$\gamma = \frac{2\pi^2 I r^2}{T^2 l m M}.\phi.$$

Now
$$g = \tfrac{4}{3}\pi\gamma RD, \qquad \text{[cf. p. 226]},$$

where R is the radius of the earth, and D its mean density. Hence

$$D = \frac{3}{4}.\frac{L}{\pi}\left(\frac{1}{2R}\right)\frac{lmM}{r^2}.\frac{T^2}{I\phi},$$

where L is the length of a pendulum beating seconds—this is introduced merely for convenience.

Cavendish's apparatus, according to Baily, a later worker on this subject, was erected in an outhouse in his garden on Clapham Common, in which he built an inner chamber to contain it. The motion of the beam was observed through telescopes fitted into the walls of the chamber—cf. Fig. 6·21(*a*). In this way disturbances due to air currents were much reduced. The torsion wire W was made of silvered copper, about a metre long, its diameter being such that the suspended system had a period of about 840 seconds; Cavendish soon changed this for one giving a period of about 420 seconds. The rod was made of deal, being about six feet long and braced by two wires *w*, *w*, stretched from its ends over a rigid vertical strut fixed to the centre of the rod. The silver wire was attached to a torsion head H, operated by a lever from outside the chamber containing the apparatus. The position of the beam was determined with the aid of two fixed ivory scales, the beam itself carrying a vernier at either end—cf. Fig. 6·21(*b*). The principal scale was divided into 20ths of an inch and the vernier enabled the position of the arm to be observed with an error less than 0·01 inch. The scales were illuminated by lamps, outside the room, and viewed through suitably mounted telescopes T_1 and T_2. The apparatus was levelled by means of screws.

The small balls were each about 2 inches in diameter; the larger ones 8 inches. These were mounted on a frame turning round a pivot in the roof of the chamber, so that they could be brought from one position shown by the full lines in Fig. 6·21(*c*), to a symmetrical position shown by the dotted lines. In this way the deflexion of the beam was doubled.

It was not found possible to observe the positions of static equilibrium of the beam, and in order to overcome this difficulty the following procedure was adopted. The two large balls were first arranged so that the line joining their centres was normal to the rod,

all twist being removed as far as possible from the wire. The large balls were then placed in their first position, the rod vibrating about its new position of equilibrium. Three consecutive turning positions of the beam were noted and from them the position of equilibrium

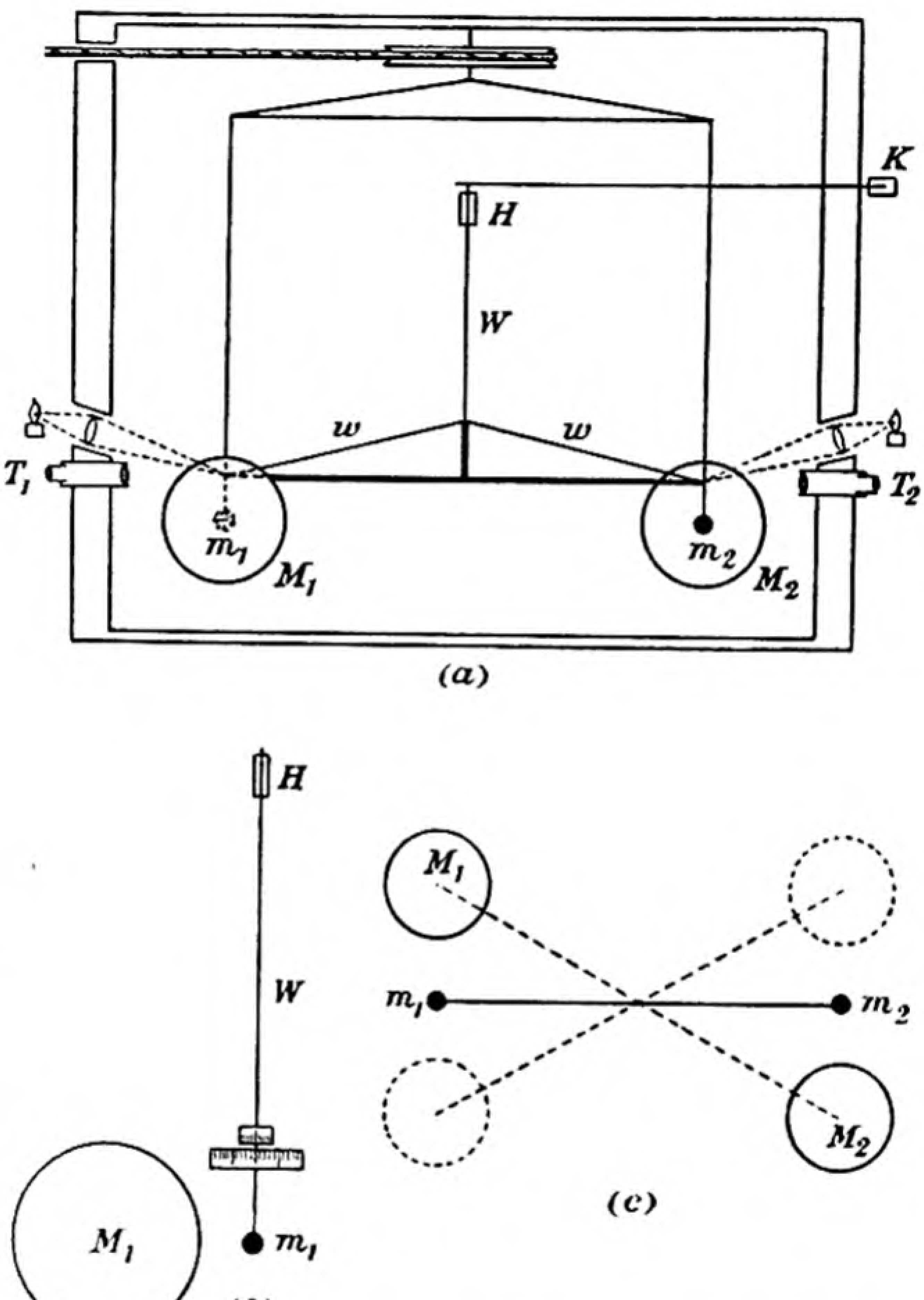


FIG. 6·21.—Cavendish's apparatus for determining the constant of gravitation.

was deduced. A similar set of observations was then made with the balls in their second position. The angular deflexion of the beam due to the attraction between the balls was then known, after a correction for the attraction between the larger spheres and the beam of the balances had been made.

It should be noted that theoretically it is more correct to use for m and M, the values determined by weighing in air, and not the

values reduced to weighing in a vacuum. For consider one small ball and its neighbouring large one. If these were removed and the space filled with air, then there would be a force of attraction between these masses of air which would be balanced by the attraction of the surrounding air if all the air is in equilibrium. When the balls are placed in position the surrounding air will exert this same action on them so that the resultant attraction will be due to the additional masses now introduced, i.e. to the mass of each as determined by weighing in air.

In all, twenty-nine results were obtained by Cavendish, the mean value for the density of the earth deduced from them being given by

$$D = 5{\cdot}45 \text{ gm.cm.}^{-3}.$$

[Owing to an arithmetical error, noted by Baily, the result was first given as $5{\cdot}48$ gm.cm.$^{-3}$.]

In the account of this experiment given by Poynting, in his renowned essay on the mean density of the earth, he says, 'An examination of Cavendish's work in this experiment, fully bears out the general opinion that he was a magnificent experimenter. It is true that details are not given of some of the measurements, an omission adversely criticized by Baily, but I think that we may trust to Cavendish's instinct for sound work, and take it for granted he would not go to so much trouble in the calculation of small corrections if his constants were not known with corresponding accuracy. Of course we can see now that the method might be improved in some ways, . . . , but considering that it was the first attempt to measure exactly forces of such an order the success obtained was most remarkable.'

Other determinations of γ with a torsion balance.—In 1841–2, Baily, at the recommendation of the Royal Society, repeated Cavendish's experiment. He used a mirror and telescope to determine the angular deflexion of the beam. The balance case was lined inside with tinfoil and wrapped outside with flannel to reduce the exchange of heat between the apparatus and its surroundings. He found $D = 5{\cdot}67$ gm.cm.$^{-3}$. For a long time this value was accepted, but Cornu and Baille pointed out several anomalies in his results. Moreover Baily corrected the masses to vacuum, a procedure wrong in principle, cf. above, but of negligible importance in practice. Baily, too, had noted that D decreased apparently as the mass of the large balls increased, a fact which he could not explain.

In 1870 Cornu and Baille introduced several improvements into the technique of this type of experiment. They reduced the dimensions of the apparatus to one-quarter that used by Cavendish, thereby diminishing the effect of temperature gradients within the

case. Amongst other improvements were (i) mercury was used for the attracting masses and this could be drawn from one vessel into another without the experimentalist coming near to the apparatus, (ii) the metal case containing the instrument was earthed to prevent electrical disturbances, (iii) the deflexions of the beam were recorded electrically. The final result was $D = 5{\cdot}56$ gm.cm.$^{-3}$.

Boys' determination of the constant of gravitation.—A description of the use of a torsion balance by several investigators to determine the constant of gravitation has already been given. The above experimentalists, on account of the small value of γ, aimed at increasing the sensitivity of the apparatus as much as possible. With this object in view, they used a long beam carrying masses at its ends, the whole being suspended by a fine wire. The attracting masses also were made as large as possible.

The great difficulty met in such work is the perpetual shifting of the position of static equilibrium of the suspended system, due in part to elastic fatigue in the wire, and partly, as Cavendish proved experimentally, to the effects of air currents set up by temperature gradients within the box housing the apparatus.

In 1889, C. V. Boys proposed to use a quartz fibre for the suspension, and showed that the linear dimensions of the apparatus could be much reduced without affecting its sensitivity, provided that a certain condition was fulfilled. This condition was that the period of the suspended system should not be changed. Thus, consider a piece of apparatus of this nature, having the same material as another, but whose linear dimensions have been reduced to one nth the corresponding dimensions of the other. This will be referred to as the small apparatus. Then the moment of inertia of the smaller suspended system will be n^{-5} times that of the other, since the mass is reduced n^3 times, and the square of the radius of gyration n^2 times. If the periods are to be identical for the two systems, the couple due to unit twist in the finer suspension must be reduced in the ratio $1 : n^5$. Now the masses of the balls in the small apparatus will each be reduced in the ratio $1 : n^3$, and since their distance apart is reduced n times, the forces of attraction in the small apparatus will be reduced in the ratio $1 : (n^6 \div n^2)$, or $1 : n^4$. The couple due to the forces of attraction will therefore be reduced in the ratio $1 : n^5$. Since the above moment is reduced in the same proportion as the torsion constant of the suspension, the angle of deflexion will be unchanged.

If, however, the length of the beam only is changed, and the attracting masses are moved until they are opposite and a fixed distance from the ends of the beam, then the moments of inertia will be altered in the ratio $n^2 : 1$, while the corresponding moments will only change in the ratio $n : 1$, and thus there is an advantage

in reducing the length of the beam until one of two things happens, either it is difficult to find a sufficiently fine torsion thread that will

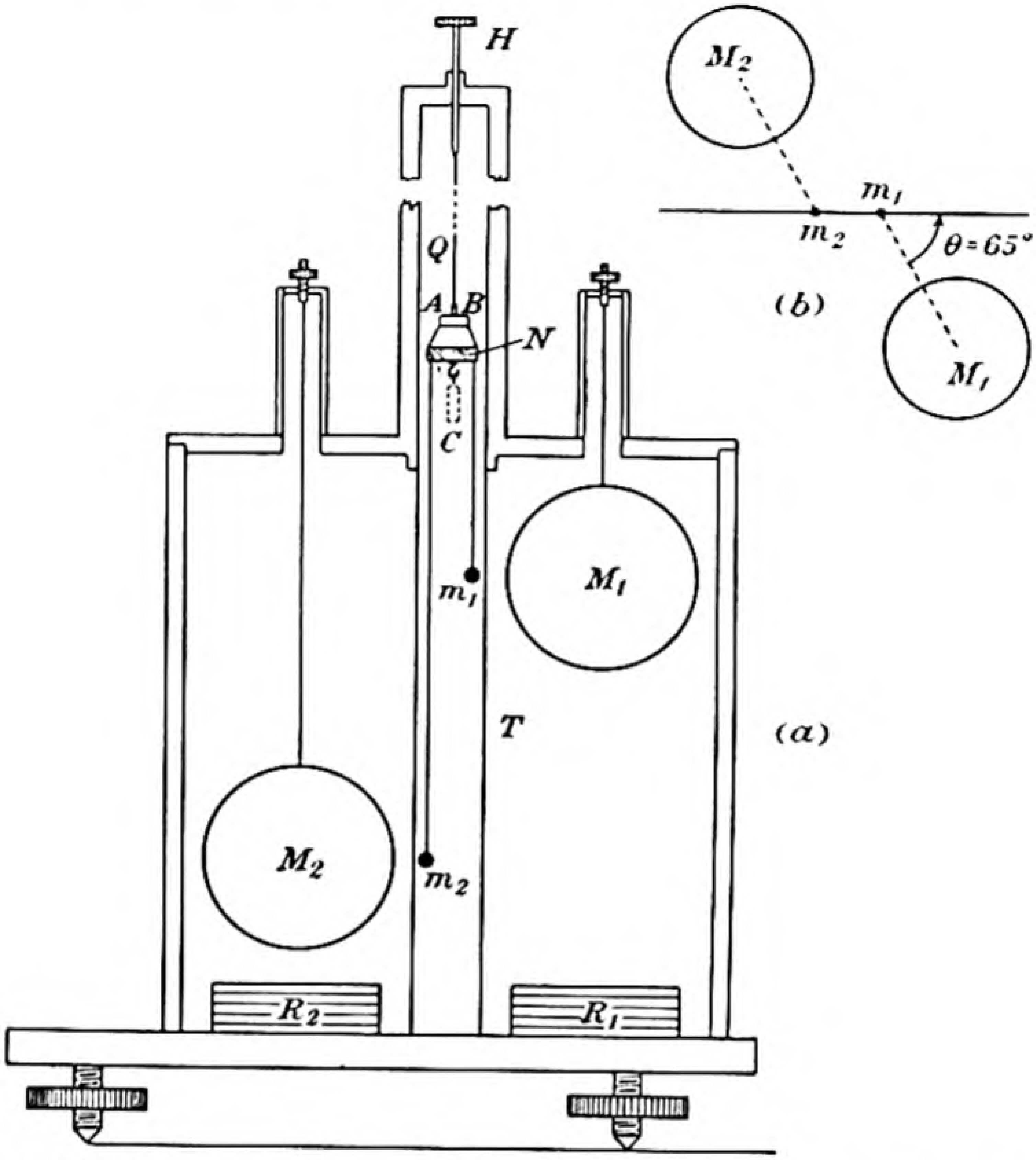


FIG. 6·22.—Boys' apparatus for determining the gravitational constant.

safely carry the beam and produce the required period . . . . or else, when the length of the beam is nearly equal to the diameter of the attracting balls, they then act with such an increasing effect on the suspended balls at the other end of the beam, that the balance of effect begins to fall short of that which would be due to the reduced dimension if the opposite ball did not interfere.'

The apparatus finally used by Boys is shown digrammatically in Fig. 6·22(a). The two small spheres, m_1 and m_2, were suspended by fine gold wires from a small beam AB carried by a quartz fibre Q, attached to a torsion head H. These small spheres and the suspension were contained in a vertical brass tube T, of circular section. This protected the moving system from extraneous disturbances and, moreover, if the suspended system is hung centrally, the

attraction of the tube produces no effect. The large attracting masses, M_1 and M_2, which must be outside the tube, must be capable of taking alternate positions on the two sides of the beam, so that it is deflected first in one direction and then in the other. For this purpose they were suspended from the lid of an outer brass case in the manner shown, the lid being capable of revolving about a vertical axis coincident with that of the central tube. To reduce the effect of M_1 on m_2, and that of M_2 on m_1, the spheres were arranged as in the diagram, care being taken to see that the centres of neighbouring spheres were in a common horizontal plane.

The plane mirror N, rigidly attached to the moving system, enabled the angular deflexion of the beam to be measured. The vertical edges of this mirror had grooves cut in them; the wires suspending m_1 and m_2 rested in these grooves, the horizontal distance between the wires thereby being maintained constant. R_1 and R_2 were pads of india-rubber to prevent damage to the apparatus should the larger masses fall.

Now there are two vertical planes, one containing the beam and the other normal to it, in which the centres of the balls will lie when they produce no deflexion of the beam. At some intermediate position the moment of the forces of attraction will be a maximum. If Fig. 6·22(b) is a plan of the centres of the balls, Boys proved that the above moment was a maximum when θ, the angle indicated, was 65°. The balls were therefore arranged so that this condition was fulfilled.

The principal dimensions of Boys' apparatus were as follows:

Distance from centre to centre of the large balls in plan, 6 in.

Distance from centre to centre of the small balls in plan, 1 in. (about).

Diameter of large balls, 4·25 (or 2·25) in.

Diameter of small balls, 0·2 (or 0·25) in.

Difference in level between centres of upper and lower pairs, 6 in.

Length of quartz fibre, 17 in.

Other important data are:—

The large spheres were made by pouring molten lead into an accurately turned spherical cavity; when the metal in the neck of the container was just solid, they were subjected to a large hydraulic pressure. Cavities were therefore impossible. Each had a mass of about 7408 gm.

The small spheres were made of pure gold. Those of 0·25 in. in diameter had a mass of 2·65 gm.

The period was about 165 sec.

The mathematical theory of Boys' experiment.—Fig. 6·23 shows the positions of the centres of the attracting balls with respect

to those hanging from the beam and the axis of rotation. First consider the moment due to the force of attraction between m_1 and M_1. The actual force is numerically equal to $\gamma.\frac{m_1M_1}{D_1^2}$, and is directed along m_1M_1, i.e. its line of action makes an angle α_1 with

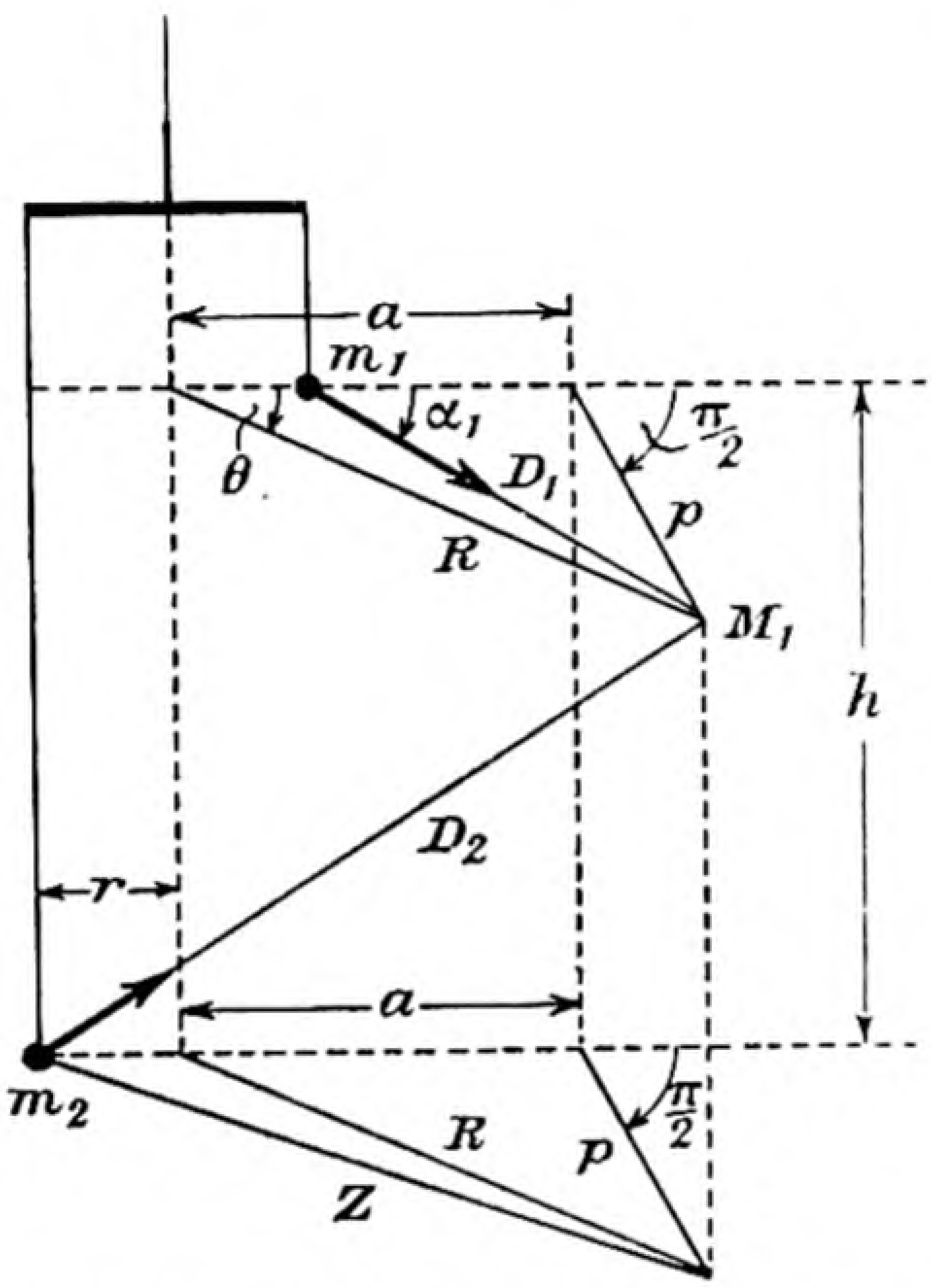


FIG. 6·23.—Theory of Boys' method for finding γ. [θ is the angle through which the lead balls are rotated from the vertical position in the plane of the beam, minus the angle through which the gold balls are deflected.]

the vertical plane containing the beam. The moment of this force about the axis of rotation is therefore

$$\left[\gamma.\frac{m_1M_1}{D_1^2}.\sin\alpha_1\right]r = \gamma.\frac{m_1M_1}{D_1^3}.pr,$$

where p and r are the distances indicated.

Now consider the effect due to M_1 on m_2. The force of attraction is $\gamma.\frac{m_2M_1}{D_2^2}$, the horizontal component of this force in the vertical plane containing m_2 and M_1 being $\gamma.\frac{m_2M_1}{D_2^2}.\frac{Z}{D_2}$, where Z and D_2 are the distances indicated. The component of this horizontal

component of the attraction normal to the plane of the diagram is

$$\gamma.\frac{m_2 M_1}{D_2^{\,2}}.\frac{Z}{D_2}.\frac{p}{Z} = \gamma.\frac{m_2 M_1}{D_2^{\,3}}.p.$$

The moment of this force about the axis of suspension is $-\gamma.\frac{m_2 M_1}{D_2^{\,3}}.pr$, the negative sign being inserted, since the direction is opposite to that of the moment due to the attraction between m_1 and M_1.

Hence for all the balls, the total moment about the axis of suspension is

$$\gamma\left[\frac{m_1 M_1}{D_1^{\,3}}.pr + \frac{m_2 M_2}{D_1^{\,3}}.pr - \frac{m_2 M_1}{D_2^{\,3}}.pa - \frac{m_1 M_2}{D_2^{\,3}}.pr\right] = \gamma Q \text{ (say)}.$$

This is equal to $b\phi$, where ϕ is the angular deflexion of the beam, and b is the couple in the quartz fibre due to unit twist in it. In order to find b it was necessary to know the moment of inertia of the suspended system about its axis of rotation. Apparently this could be obtained in the usual manner [cf. p. 392], by using a bar of known moment of inertia about the above axis. In the present instance, however, care must be taken to see that in the two timing experiments the suspended mass is the same so that the quartz fibre shall be stretched by the same amount, for the torsional constant depends upon the amount of stretch given to the fibre [cf. p. 296].

Boys therefore determined the period of the suspended system—call it T_1. The small gold spheres and the wires attached to them were then removed. A cylindrical counterpoise of known and very small moment of inertia about its axis, and having the same mass as the gold spheres and their attachments, was then suspended from a hook attached to the beam, cf. C, Fig. 6·22(a), where this cylinder is indicated by its dotted outline. Under these circumstances the fibre was stretched by the same amount. Let T_2 be the observed period.

Let I_0 be the moment of inertia of the mirror and beam about a vertical axis, I_1 that of the gold spheres and the wires fixed to them about the axis of rotation of the suspended system—this could be calculated. Let I_2 be the moment of inertia of the cylinder about a vertical axis. Then

$$T_1 = 2\pi\sqrt{\frac{I_0 + I_1}{b}}, \qquad \text{and} \qquad T_2 = 2\pi\sqrt{\frac{I_0 + I_2}{b}}.$$

Hence
$$b = \frac{4\pi^2(I_1 - I_2)}{T_1^{\,2} - T_2^{\,2}}.$$

The calculation of γ in Boys' experiment.—The equation expressing the condition for the equilibrium of the beam is

$$\gamma Q = \text{moment of forces, about a vertical axis, acting on the beam,}$$
$$= b\phi = b.\frac{\sigma}{4\rho},$$

where σ is the deflexion in scale divisions caused by changing the positions of the large spheres from one side of the beam to the other, and ρ is the distance of the mirror from the scale used to determine the deflexion. The factor 4 is introduced owing to the fact that the angle is doubled on account of its deflexion being observed with the aid of a beam of light reflected from it, and again doubled by moving the large balls in the manner just stated.

The results obtained were published in 1895; Boys found

$$\gamma = 6{\cdot}6576 \times 10^{-8}\ \text{gm.}^{-1}\text{cm.}^{3}\text{sec.}^{-2},$$

and $$D = 5{\cdot}5270\ \text{gm.cm.}^{-3}.$$

Braun's experiment on γ.—In 1896. BRAUN, at Mariaschein, Bohemia, devised a torsion balance which was enclosed in a fairly good vacuum, the pressure being about 0·005 atmosphere, so that air currents were much reduced. His apparatus was much larger than that of Boys and the suspension wire was made of metal. Later on, a quartz fibre was used. He obtained a value for D identical with that of Boys.

Heyl's work on the gravitational constant.—The torsion balance, since its inception by Cavendish for the determination of the gravitational constant, has been used by BOYS and BRAUN for the same purpose although each employed a different arrangement. There are two different methods of using a torsion balance; the 'direct deflexion' method and the 'time of swing' method. Boys used only the former method whereas Braun used both methods and mounted his apparatus in a partial vacuum in order to diminish the effects of cross currents. In the time of swing method, the time of swing of a suspended system is measured in two positions which may be called 'near' and 'far'. Plans of these arrangements for the second method are shown in Fig. 6·24(a) and (b). In the 'near' position the attraction of the larger masses upon the suspended system is such that the time of swing is diminished, while in the far position it is increased; in Braun's work this difference was about 46 seconds.

A study of the work of Boys and Braun led HEYL to the conclusion that only the second method could lead to a result of greater accuracy.

Three series of measurements were made, using small masses of gold, platinum and optical glass. The timing observations were

made visually and recorded on a chronograph, standardized against signals from an observatory. The observation room was a constant temperature vault, at the Bureau of Standards, divided into two rooms and situated 12 metres below ground. The torsion balance was set up in the small room; the observing apparatus was in the large room.

Steel, containing 0·90 per cent. carbon, was employed as the material for the large masses, this being selected as it had a most

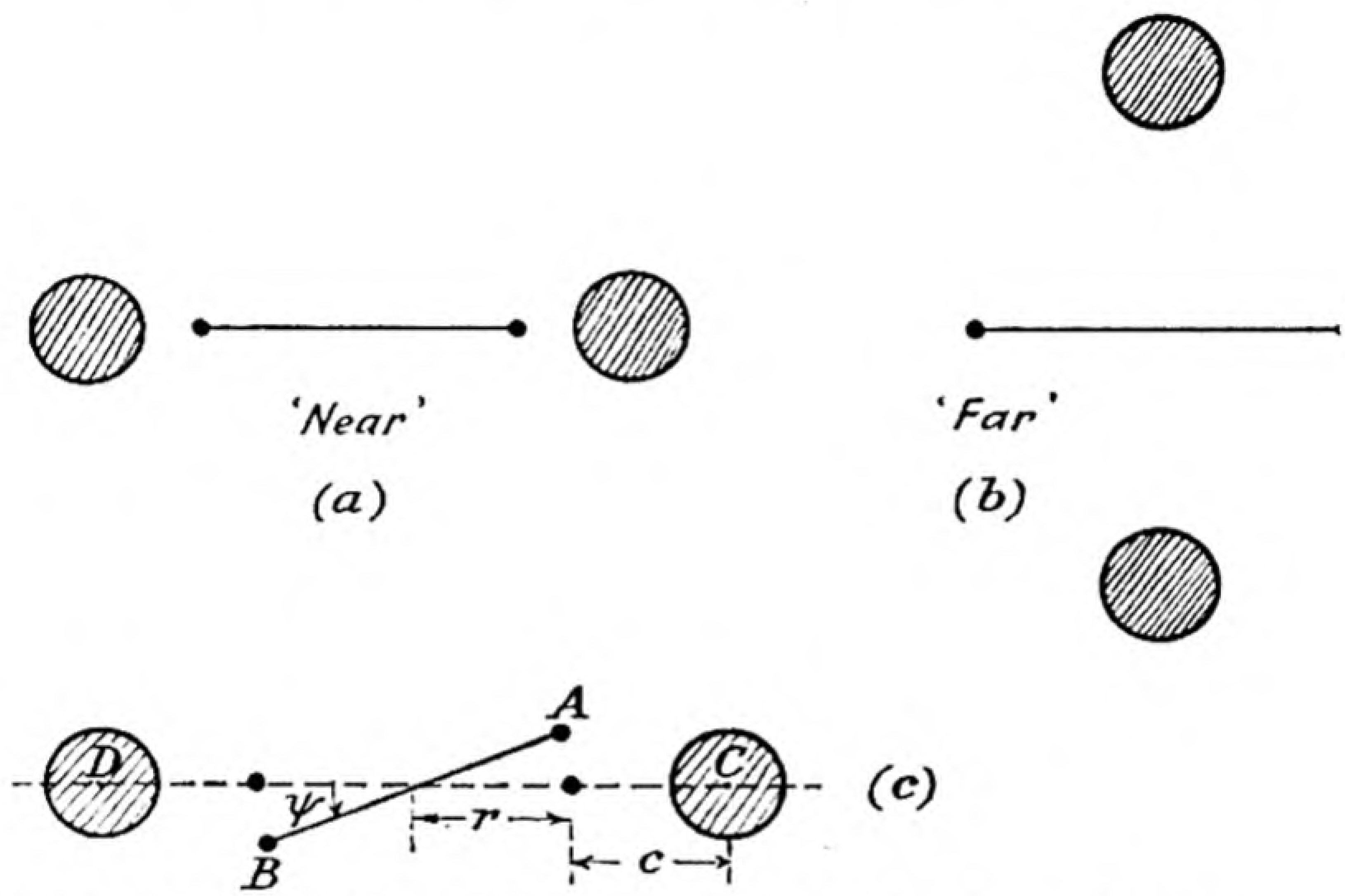


FIG. 6·24.—Principle of Heyl's method for the determination of the gravitational constant.

uniform structure and was least liable to segregation. To avoid blow-holes and pipes, a large ingot 30 cm. in diameter was forged down to 20 cm. and annealed before machining. Nearly all previous experimenters had used masses of spherical form because of the simplicity of the calculations involved. The practical difficulties of machining a sphere of mass 60 kgm. appeared to be so great that it was decided to adopt a cylindrical form, 'thus shifting the burden to the broad shoulders of the mathematician'.

The small masses were spherical in shape. The use of gold, platinum and glass was not prompted by any suspicion that a specific difference in attraction might be found; it was the result of circumstances. Gold was used by Boys and there can be no objection to his work on these grounds since he did not employ a vacuum but in Heyl's work it was found that the gold balls gradually absorbed mercury, probably from the mercury gauges used. The gold spheres were replaced by platinum ones, thinly coated with lacquer. The third set was made of glass so that internal holes could be detected visually.

A tungsten wire was used to carry a light aluminium rod from the ends of which hung the small spheres. To cause the system to be deflected from its position of static equilibrium two bottles, each holding about 2 kgm. of mercury, were placed in positions of maximum attraction for the pendulum. After 15 minutes the system had been twisted to its maximum extent; the bottles were then placed to reverse the attraction. The process was repeated until the suspended system had the required amplitude.

The simplified theory is as follows. Consider the total energy of the suspended system when the large masses are in the 'near' position. At an instant when the suspended beam makes an angle ψ with its position of static equilibrium, cf. Fig. 6·24(c), the kinetic energy of the system is $\frac{1}{2}I\dot{\psi}^2$ and the potential energy stored in the suspension $\frac{1}{2}b\psi^2$, where I is the moment of inertia of the system about its axis of rotation and b the torsional constant for the tungsten wire. In addition it is necessary to consider the potential energy of the small masses for A is near to C and D, and B is near to D and C. We may write this as $\frac{1}{2}\gamma A_1\psi^2$, where A_1 is a constant involving the product of one large mass and a purely geometrical function of the distances r and c. Hence

$$\tfrac{1}{2}[I\dot{\psi}^2 + b\psi^2 + \gamma A_1\dot{\psi}^2] = \text{constant}.$$

Differentiating with respect to time, we find

$$I\ddot{\psi} + (b + \gamma A_1)\psi = 0,$$

so that the period, T_1, of small oscillations is given by

$$T_1 = 2\pi\sqrt{\frac{I}{b + \gamma A_1}}\,.$$

In a similar way, when each large mass occupies a 'far' position, we find

$$I\ddot{\psi} + (b + \gamma A_2)\psi = 0,$$

so that

$$T_2 = 2\pi\sqrt{\frac{I}{b + \gamma A_2}}\,.$$

These equations give

$$\gamma = \frac{4\pi^2 I(T_2^2 - T_1^2)}{(A_1 - A_2)T_1^2T_2^2}\,.$$

The following values for γ were obtained:—

Varnished platinum spheres	$6{\cdot}664 \times 10^{-8}$ gm.$^{-1}$cm.^{3}sec.$^{-2}$
Glass spheres	$6{\cdot}674 \times 10^{-8}$,, ,, ,,
Mean	$6{\cdot}669 \times 10^{-8}$,, ,, ,, .

This value is a little higher than the values obtained by Boys and

Braun and the cause of the discrepancy in Heyl's results has not been explained.

Zahradníček's resonance method for determining γ.—The apparatus used by Zahradníček in 1933 to determine a value for the

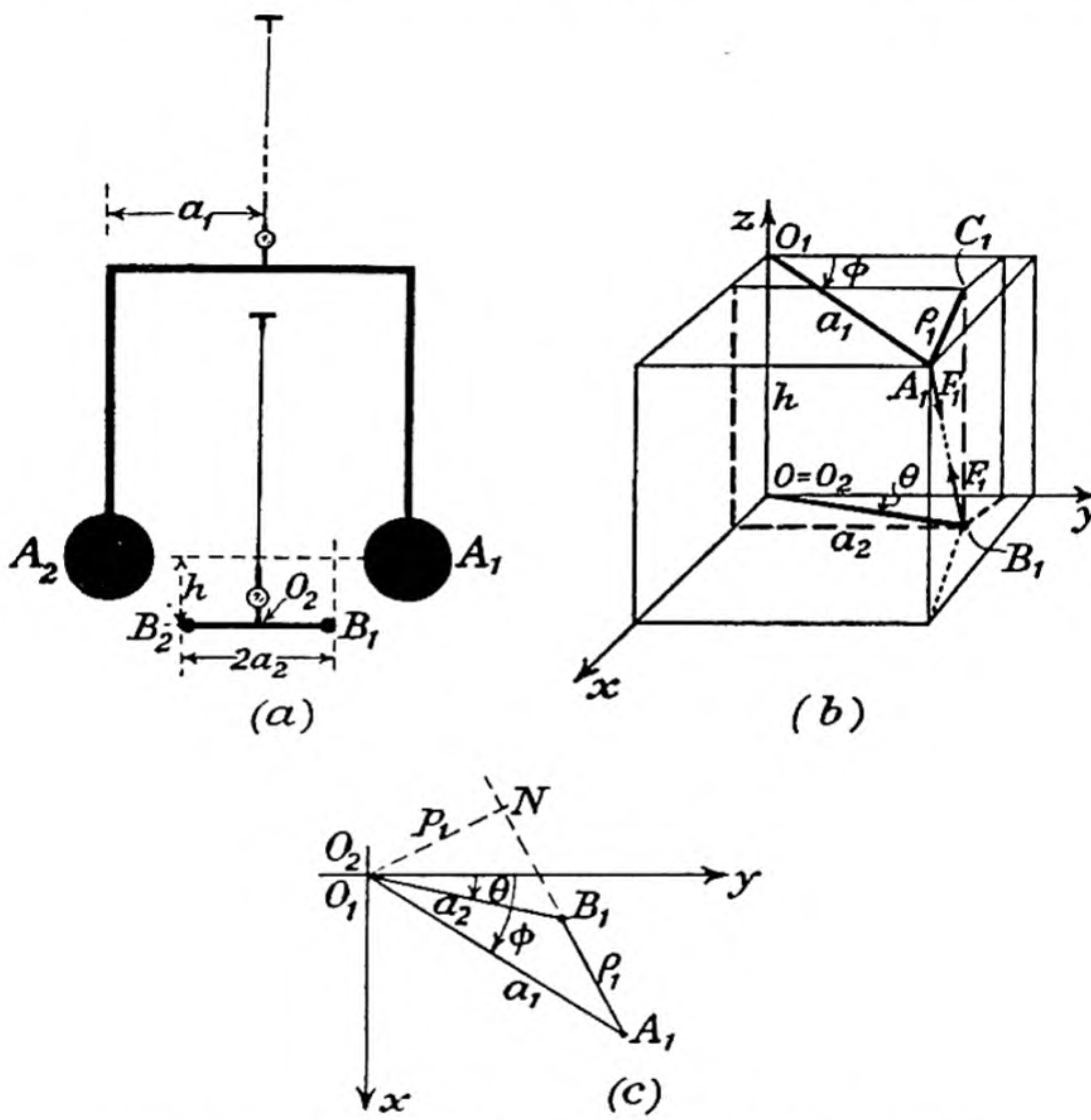


FIG. 6·25.—Zahradníček's resonance method for finding γ.

gravitational constant, γ, consists of two coaxial torsion balances, which will be termed 'primary' and 'secondary' respectively. The former is relatively robust and consists of a steel wire, diameter 0·04 cm., carrying a short rod at its lower end; a concave mirror with a large radius of curvature is fixed to this rod which supports a horizontal beam. This consists of a horizontal portion about 60 cm. long and from each end of it there hangs vertically downwards a slightly narrower arm. The entire beam is thus in the form of ⊓ and is made from brass tubing. Two equal lead spheres, mass 5–10 kgm., are mounted on the same level at the lower ends of the vertical arms as shown in Fig. 6·25(a). The secondary balance is much smaller; its axis is vertically below that of the

primary balance and when each is in its position of static equilibrium all supporting wires and tubes are coplanar. The beam of the secondary balance is simply a stout piece of aluminium wire, about 30 cm. long, with small equal lead spheres, mass 10 gm., at its ends; this balance also carries a concave mirror and the oscillations of each balance are registered photographically on a rotating drum covered with silver bromide paper.

Each balance, when displaced and then released, executes damped oscillations about its position of static equilibrium, since forces and couples are exerted on each system due to mutual attractions between the various lead spheres. The damping of the primary system is very small and in the discussion which follows will be neglected. To exclude draughts the secondary balance is enclosed in a wooden case fitted with a glass window to permit the oscillations of the system to be recorded. The main experiment consists in the adjustment of the two systems until the condition of resonance between them is established, i.e. the two periods of oscillation are equal. The theory which follows shows that a value for the gravitational constant may be calculated when the masses of the spheres and the linear dimensions of each balance are known.

Let O_2, the centre of the beam of the secondary balance, be taken as O, the origin of a system of rectangular coordinates, as shown in Fig. 6·25(b). Then Oy is the axis of the secondary beam when it is not executing oscillations about the vertical axis Oz. When the primary beam is displaced through an angle ϕ the coordinates of the centres of the large spheres will be

$$x_1 = a_1 \sin\phi, \quad y_1 = a_1 \cos\phi, \quad z_1 = h,$$

and $(-x_1, -y_1, h)$, where $2a_1$ is the distance apart of the centres of these spheres and h is the vertical distance between the planes in which the centres of the spheres lie.

Similarly, when the secondary beam of length $2a_2$ is displaced through an angle θ, the centres of the spheres it carries will have coordinates

$$x_2 = a_2 \sin\theta, \quad y_2 = a_2 \cos\theta, \quad z = 0,$$

and $(-x_2, -y_2, 0)$.

The attractions between unlike spheres are given by

$$F_1 = \gamma\frac{m_1 m_2}{r_1^2} \quad \text{and} \quad F_2 = \gamma\frac{m_1 m_2}{r_2^2},$$

where $A_1B_1 = r_1$ and $A_1B_2 = r_2$. Their values are given by

$$r_1^2 = (a_1\sin\phi - a_2\sin\theta)^2 + (a_1\cos\phi - a_2\cos\theta)^2 + h^2$$
$$= a_1^2 + a_2^2 - 2a_1a_2\cos(\phi - \theta) + h^2,$$

and $$r_2^2 = a_1^2 + a_2^2 + 2a_1a_2\cos(\phi - \theta) + h^2.$$

Now suppose that the vertical plane through A_1 and B_1 intersects the horizontal plane through A_1 in the straight line A_1C_1. Then $B_1C_1 = h$ and if we write $A_1C_1 = \rho_1$, we have

$$\rho_1^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$$
$$= (a_1 \sin \phi - a_2 \sin \theta)^2 + (a_1 \cos \phi - a_2 \cos \theta)^2.$$

Similarly,

$$\rho_2^2 = (a_1 \sin \phi + a_2 \sin \theta)^2 + (a_1 \cos \phi + a_2 \cos \theta)^2.$$

The horizontal component of F_1 in a direction parallel to C_1A_1 is $F_1\left(\frac{\rho_1}{r_1}\right)$; for F_2 it is $F_2\left(\frac{\rho_2}{r_2}\right)$. The former component gives rise to a torque $F_1\left(\frac{p_1}{r_1}\right)p_1$, where, cf. Fig. 6·25($c$), p_1 is the perpendicular distance of O_1 from the vertical plane through A_1 and B_1. Since, however,

$$p_1\rho_1 = 2\,\Delta A_1O_1B_1$$
$$= a_1a_2 \sin(\phi - \theta),$$

the torque on the secondary beam due to both horizontal components is

$$2F_1 . \frac{\rho_1 p_1}{r_1} = \frac{2F_1a_1a_2 \sin(\phi - \theta)}{r_1}.$$

Similarly, the two forces F_2 give rise to a torque

$$\frac{2F_2a_1a_2 \sin(\phi - \theta)}{r_2},$$

which opposes that due to F_1. Thus the resultant torque, in the sense of that due to F_1, is

$$2a_1a_2 \sin(\phi - \theta)\left[\frac{F_1}{r_1} - \frac{F_2}{r_2}\right] = 2\gamma m_1m_2a_1a_2 \sin(\phi - \theta)\left[\frac{1}{r_1^3} - \frac{1}{r_2^3}\right],$$

and if ϕ and θ are sufficiently small, this expression becomes

$$2\gamma m_1m_2a_1a_2(\phi - \theta)\,\Delta,$$

where

$$\Delta = \frac{1}{[(a_1 - a_2)^2 + h^2]^{1\cdot5}} - \frac{1}{[(a_1 + a_2)^2 + h^2]^{1\cdot5}},$$

since $\cos(\phi - \theta) \to 1$, when both ϕ and θ are small.

If I is the moment of inertia of the secondary beam and its attachments about the axis of rotation, the motion of the secondary system, in the absence of the primary, may be represented by

$$I\ddot{\theta} + P\dot{\theta} + Q\theta = 0,$$

where P and Q are constants. In the presence of the primary system, the equation expressing the motion of the secondary system is

$$I\ddot{\theta} + P\dot{\theta} + Q\theta = H(\phi - \theta),$$

where $H = 2\gamma m_1 m_2 a_1 a_2 \Delta$. Then

$$\ddot{\theta} + \alpha\dot{\theta} + \beta\theta = \mu\phi,$$

where $\beta = \dfrac{Q + H}{I}$, $\mu = \dfrac{H}{I}$, etc. If we assume $\phi = \hat{\phi}\cos\omega_1 t$, i.e. the motion of the primary is undamped, so that $\mu\phi = \hat{\chi}\cos\omega_1 t$, then

$$\ddot{\theta} + \alpha\dot{\theta} + \beta\theta = \hat{\chi}\cos\omega_1 t,$$

and the particular integral, cf. p. 37, which is all we have to consider, is

$$\theta = \frac{\hat{\chi}}{\sqrt{(\beta - \omega_1^2)^2 + \alpha^2\omega_1^2}}\cos(\omega_1 t - \psi),$$

where $\tan\psi = \dfrac{\alpha\omega_1}{\beta - \omega_1^2}$. If we write $\omega_0^2 = \beta$, so that $\omega_0 = 2\pi f_0$, where f_0 is the free period of the secondary system, its amplitude will be a maximum when

$$\omega_0^2 - \omega_1^2 = \tfrac{1}{2}\alpha^2,$$

a result obtained by differentiating $(\omega_0^2 - \omega_1^2)^2 + \alpha^2\omega_1^2$ with respect to ω_1 and equating to zero, etc. If θ_0 is this maximum amplitude, we have

$$\theta_0 = \frac{\hat{\chi}}{\sqrt{(\omega_0^2 - \frac{1}{2}\alpha^2 - \omega_0^2) + \alpha^2\omega_1^2}} = \frac{\hat{\chi}}{\alpha\sqrt{\frac{1}{4}\alpha^2 + \omega_1^2}}$$

$$= \frac{\mu\hat{\phi}}{\dfrac{2\delta}{T_2}\sqrt{\dfrac{1}{4}\cdot\dfrac{4\delta^2}{T_2^2} + \omega_1^2}},$$

where T_2 is the period of the secondary system and $\delta = \frac{1}{2}\alpha T_2$, the decrement per cycle of the oscillations of the secondary system, cf. Vol. V, p. 347.

Hence
$$\theta_0 = \frac{\dfrac{H}{I}\phi_0}{\dfrac{2\delta}{T_2}\sqrt{\dfrac{\delta^2}{T_2^2} + \omega_1^2}} = \frac{HT_2}{2I\delta}\cdot\frac{\phi_0}{\sqrt{\dfrac{\delta^2}{T_2^2} + \omega_1^2}}$$

$$= \frac{2\gamma m_1 m_2 a_1 a_2 \Delta . T_2}{2I\delta . \sqrt{\dfrac{\delta^2}{T_2^2} + \omega_1^2}}\phi_0$$

$$\therefore \gamma = \frac{\theta_0}{\phi_0} \cdot \frac{I\delta\sqrt{\dfrac{\delta^2}{T_2^{\,2}} + \omega_1^{\,2}}}{m_1 m_2 a_1 a_2 \,\Delta \,. T_2}.$$

From observations on the turning points in the motions of the two systems a value for $\dfrac{\theta_0}{\phi_0}$ is obtained; then $\omega_1 = \dfrac{2\pi}{T_1}$, where T_1 is the period of oscillation of the primary balance. An important correction is applied for the attraction on the beam and other parts of the secondary balance, while a small correction is also made for the almost negligible damping of the primary. The method is an accurate one and enables a large number of values for γ to be obtained in a relatively short time. Zahradníček gives

$$\gamma = (6{\cdot}65_9 \pm 0{\cdot}02)\ \text{gm.}^{-1}\text{cm.}^3\text{sec.}^{-2}.$$

EXPERIMENTS WITH A COMMON BALANCE

The work of von Jolly.—In 1881, von Jolly† described some work in which he had determined the gravitational constant with the aid of a common balance. This was erected at the top of a tower at the University of Munich. The principle of the experiment was to counterpoise in a pan of the balance a large mass hung several metres below the other pan. A large mass of lead was then placed below the lower pan and the balance again equilibrated. From the increase in the mass in the upper pan necessary to restore equilibrium the force of attraction between the lower masses was deduced, and γ then calculated. Fig. 6·26 is a diagrammatic sketch of von Jolly's apparatus. From the pans of a common balance other pans were carried by wires about 21 metres long. Two practically equal globes each containing 5×10^3 gm. of mercury were then placed, one on an upper pan and the other on a lower pan, as indicated, and the balance equilibrated. A large sphere M, built up from thick sheets of lead, and about 1 metre in diameter, its mass being $5{\cdot}775 \times 10^6$ gm., was then placed below the lower pan containing the globe

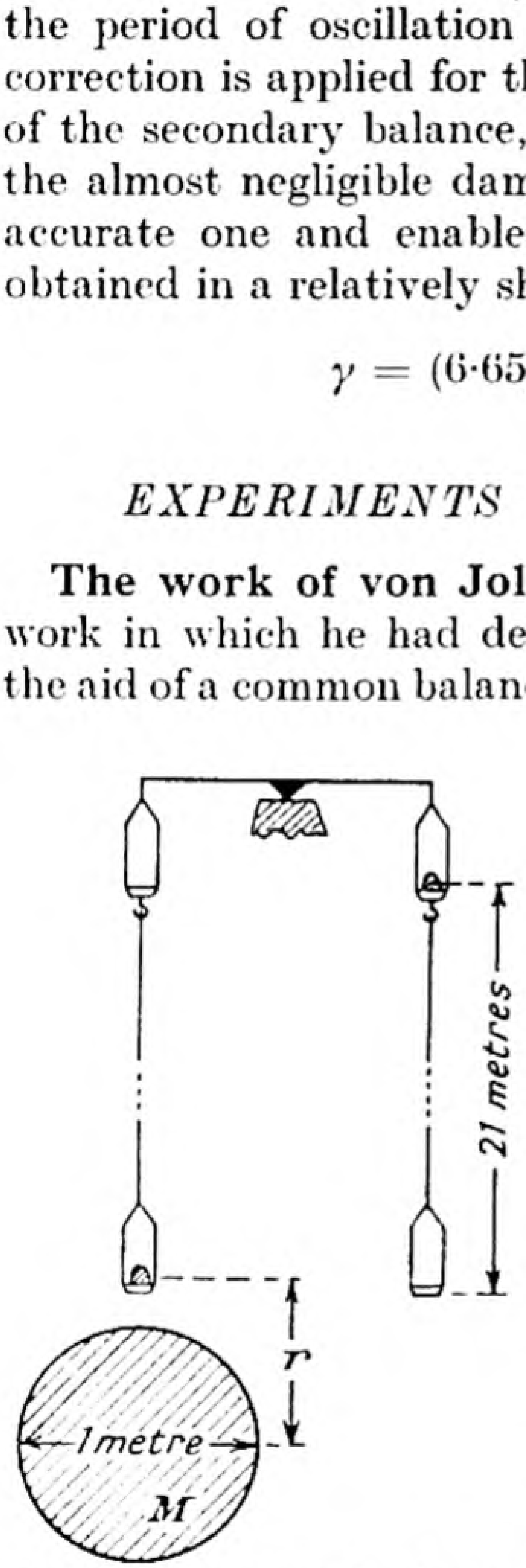


FIG. 6·26.—Principle of von Jolly's apparatus for finding γ.

† *Wied. Ann.*, **14**, 331, 1881.

filled with mercury. Let μ be the additional mass required in the upper pan to restore equilibrium. Then

$$\mu g = \gamma . \frac{m\mathrm{M}}{r^2} ,$$

where m is the mass of the globe containing the mercury, M that of the lead sphere, and the other symbols have their usual meanings. Thus γ could be determined.

In the actual experiment the procedure was not quite so simple. Four equal globes were obtained, two only being filled with mercury, but all were hermetically sealed. To begin the experiment the two heavy globes were placed in the upper pans, the light ones in the lower pans. The balance was equilibrated. In this way variations in the density of the air between the higher and lower levels were of no consequence. The two globes on one side of the balance were then interchanged. The variation with height of the intensity of gravity caused the equilibrium of the balance to be upset. From the change in the mass of the upper pan necessary to restore equilibrium the formula for the variation of g with height was verified except in so far as it was disturbed by local irregularities. The large lead sphere was then placed in position and the experiment repeated. From the difference in gain in weight with and without the sphere, viz. $0{\cdot}589 \times 10^{-3}$ gm.-wt., γ was calculated.

The final result was $\mathrm{D} = (5{\cdot}69 \pm 0{\cdot}07)$ gm.cm.$^{-3}$, but the method, as used by von Jolly, cannot be susceptible of great accuracy on account of convection currents in the tower.

Poynting's balance experiment.—The balance used in this experiment to determine the constant of gravitation was a bullion balance with a four foot beam. It is shown diagrammatically in Fig. 6·27. Two equal balls, of masses m_1 and m_2, were suspended from the beam of the balance. Below the balance was a turntable T, its axis of rotation passing through the fulcrum of the balance. Upon it rested two large masses M_1 and M_2—spheres made from a lead–antimony alloy for the sake of hardness.† M_1 was twice the mass of M_2. It is necessary to use the smaller mass M_2 to prevent the table from tilting and so causing a variation of its attraction on the beam. At first Poynting did not use M_2, and his results were anomalous. The beam of the balance was deflected through a small angle after it had been equilibrated, on account of the couple due to the attraction between the various spheres. The table was then rotated through two right angles so that M_1 came below m_2, and the change in the angular position of the beam was noted.

† C. V. Boys criticizes the use of an alloy in this experiment since one cannot be sure that its density is uniform—if it is not, the centre of gravity of a sphere does not necessarily coincide with its geometric centre.

With the results thus obtained it was not possible to calculate γ, owing to the unknown value of the pull of the table and the loads it carried on the beam. To eliminate this the masses m_1 and m_2 were each raised one foot and the experiment was repeated. The following approximate analysis—the 'cross-pulls' between the

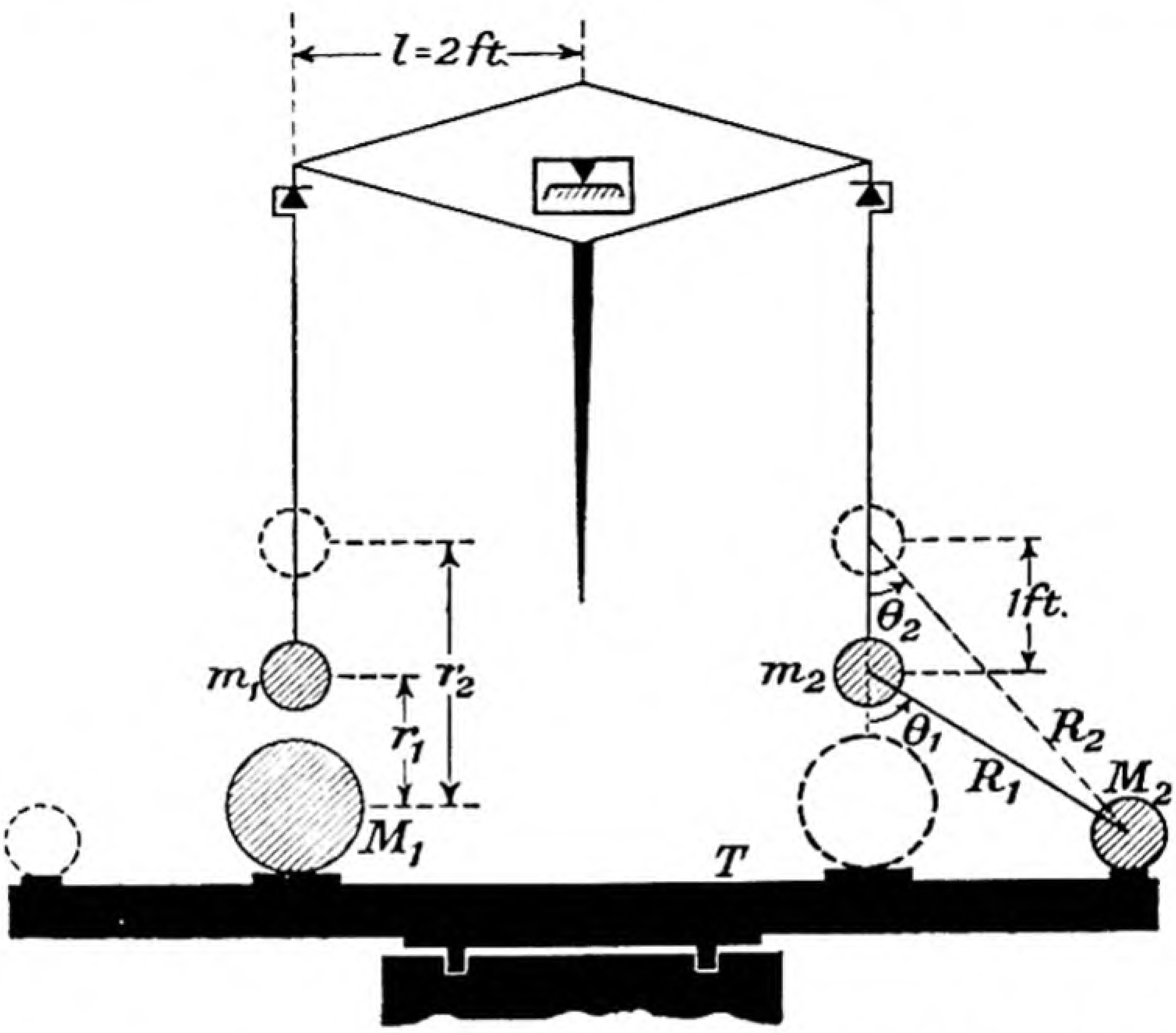


FIG. 6·27.—Poynting's balance experiment for finding the gravitational constant and the mean density of the earth.

different spheres being neglected, for example—shows how γ was obtained. An additional approximation arises from the fact that perfect symmetry in the geometry of the system is assumed.

When the masses are arranged as shown by the full lines in Fig. 6·27, the equilibrium of the balance is expressed by the equation

$$\left[m_1 g + \gamma \cdot \frac{m_1 M_1}{r_1^2}\right] l = \left[m_2 g + \gamma \cdot \frac{m_2 M_2}{R_1^2} \cdot \cos\theta_1\right] l + \kappa_1 + a\psi_1,$$

where κ_1 is a constant representing the turning effect of the attraction on the beam and ψ_1 is the angular deflexion of the beam, a being a constant.

When the table is turned through an angle π, the corresponding equation is

$$\left[m_1 g + \gamma \cdot \frac{m_1 M_2}{R_1^2} \cdot \cos\theta_1\right] l = \left[m_2 g + \gamma \cdot \frac{m_2 M_1}{r_1^2}\right] l + \kappa_2 + a\psi_2.$$

Subtracting these two equations we have

$$\gamma\left[\frac{m_1 M_1}{r_1^2} - \frac{m_1 M_2}{R_1^2}.\cos\theta_1\right]l$$
$$= \gamma\left[\frac{m_2 M_2}{R_1^2}.\cos\theta_1 - \frac{m_2 M_1}{r_1^2}\right]l + (\kappa_1 - \kappa_2) + a\beta_1,$$

where β_1 is written for $(\psi_1 - \psi_2)$, a quantity actually observed.

When the smaller spheres are raised through a distance of one foot, the corresponding equation is

$$\gamma\left[\frac{m_1 M_1}{r_2^2} - \frac{m_1 M_2}{R_2^2}.\cos\theta_2\right]l$$
$$= \gamma\left[\frac{m_2 M_2}{R_2^2}.\cos\theta_2 + \frac{m_2 M_1}{r_2^2}\right]l + (\kappa_1 - \kappa_2) + a\beta_2.$$

Eliminating κ_1 and κ_2 from these equations we have

$$\gamma\left[m_1 M_1\left(\frac{1}{r_1^2} - \frac{1}{r_2^2}\right) - m_1 M_2\left(\frac{\cos\theta_1}{R_1^2} - \frac{\cos\theta_2}{R_2^2}\right)\right.$$
$$\left. - m_2 M_2\left(\frac{\cos\theta_1}{R_1^2} - \frac{\cos\theta_2}{R_2^2}\right) + m_2 M_1\left(\frac{1}{r_1^2} - \frac{1}{r_2^2}\right)\right]l$$
$$= a(\beta_1 - \beta_2).$$

In order to find the value of the constant a, the attracting spheres were removed and the balance was equilibrated with the small masses hanging from the appropriate pans, and the deflexion ϕ of the beam caused when a rider of mass μ was shifted through a distance s on the beam observed. Then

$$a\phi = \mu g s.$$

Thus γ, and hence D, could be calculated. Poynting took great precautions not to introduce temperature gradients in the experimental chamber which was housed in a cellar at the Mason College, Birmingham, by making all final adjustments from the outside—the deflexions were observed with the aid of a telescope. All distances were measured very accurately. The final result was

$$\gamma = 6{\cdot}6984 \times 10^{-8}\ \text{gm.}^{-1}\text{cm.}^3\text{sec.}^{-2},$$
$$D = 5{\cdot}4934\ \text{gm.cm.}^{-3}.$$

A method for observing small angular deflexions.—To measure the small angular deflexions of the beam in the experiment described above, Poynting used the double-suspension mirror

method proposed by KELVIN. M, Fig. 6·28, is the mirror supported by two strings [not necessarily parallel to each other, although Poynting used parallel strings 3 or 4 mm. apart] from the end of a horizontal screw S and from the end of an arm Q in line with the screw. The arm Q is attached at right angles to the pointer of the balance, which swings in a plane perpendicular to that of the diagram. Suppose that the pointer swings so that Q moves forward a distance a. Then the mirror will rotate through an angle $\frac{a}{d}$, where d is the distance between the ends of S and Q, i.e. the sensitivity of the arrangement is inversely proportional to the distance d. In the actual apparatus used by Poynting the deflexion of the mirror was 150 times that of the beam of the balance. It is always necessary to damp the motion of the mirror by attaching to it four vanes moving in a dash-pot containing a light lubricating oil.

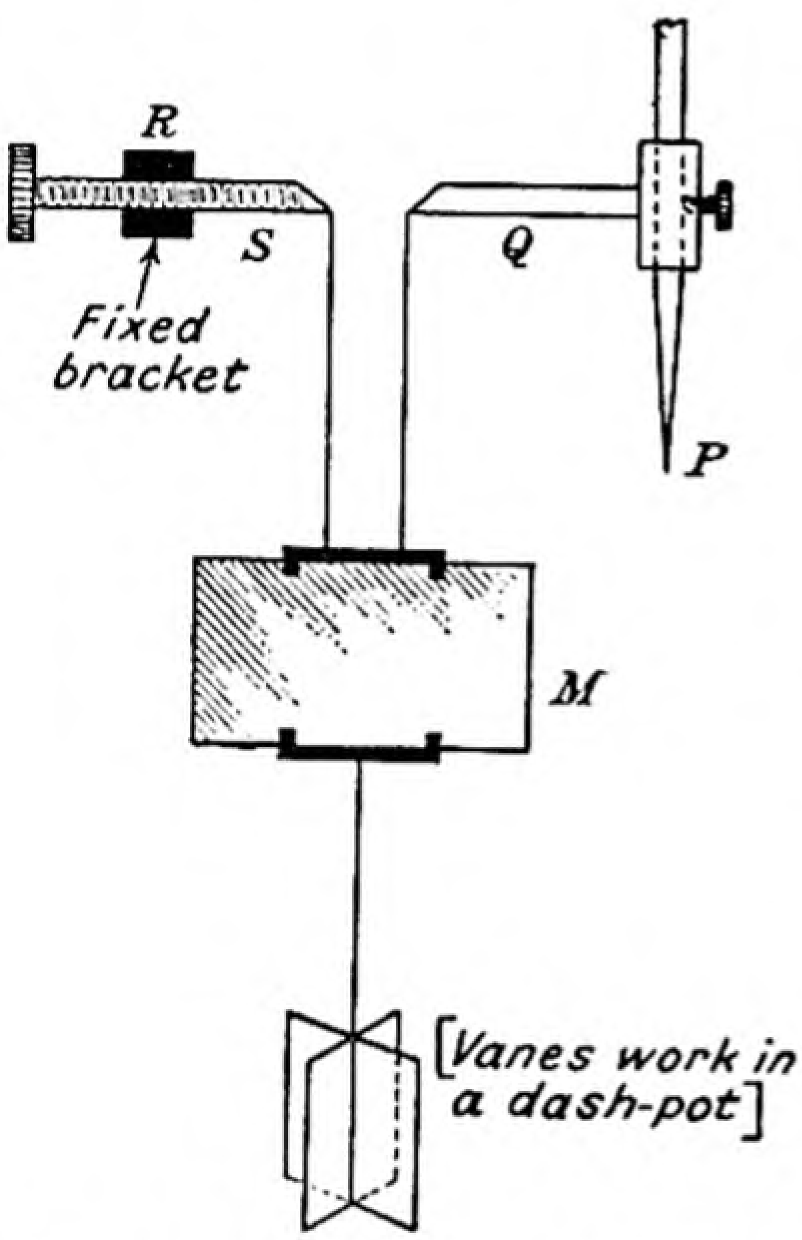


FIG. 6·28.—Principle of Kelvin's method for determining a small angular deflexion.

An approximate method for comparing the masses of two planets.—This method is only applicable when one of the planets has a satellite and the orbits concerned may be taken as circles. For example, we may compare the mass of the earth with that of the sun.

For this purpose let S, Fig. 6·29, be the sun, mass M, and E the earth, mass m. Let μ be the mass of the moon, L. Then if R and r are respectively the mean radii of the orbits of the earth and moon, we have, if ω_1 is the angular velocity of the earth and γ the gravitational constant,

$$\gamma \frac{Mm}{R^2} = m\omega_1{}^2 R,$$

$$\therefore \gamma M = \omega_1{}^2 R^3.$$

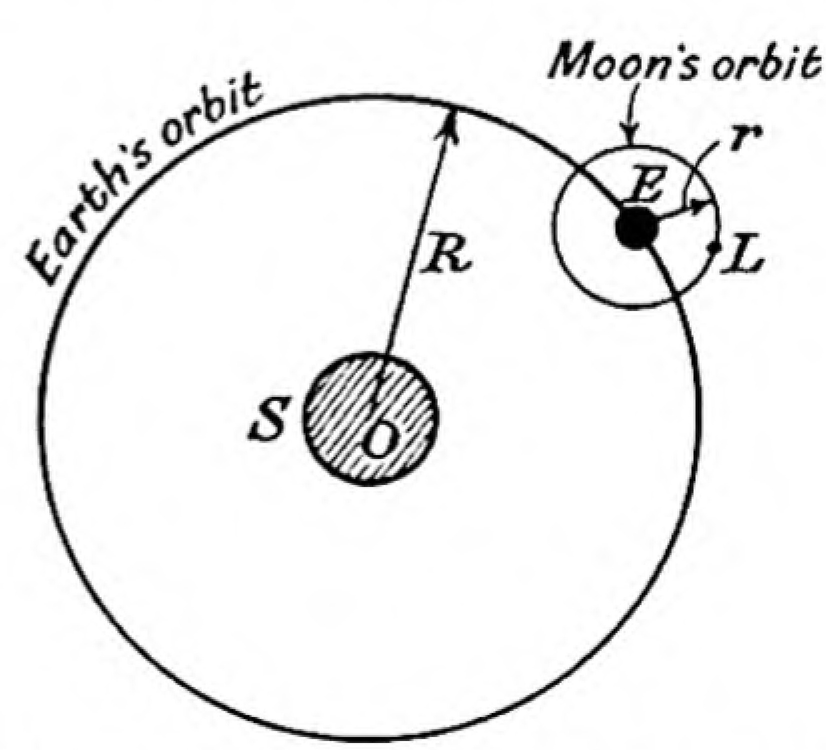


FIG. 6·29.—The orbits of the earth and moon. [Assumed circular and not to scale.]

Similarly,

$$\gamma m = \omega_2{}^2 r^3.$$

$$\therefore\ \frac{m}{M} = \left(\frac{\omega_2}{\omega_1}\right)^2 \left(\frac{r}{R}\right)^3 = \left(\frac{T_1}{T_2}\right)^2 \left(\frac{r}{R}\right)^3,$$

where, in general, $\omega = 2\pi T^{-1}$, T being the period. Thus from the observed periods of the earth and moon, and their mean distances from the sun and earth respectively, the masses of the earth and sun may be compared.

The range of gravitation.—All experimental evidence points to the fact that the law of gravitation is valid when the distance between the attracting bodies is as large as that between the planets, or as small as that separating the balls in the experiments by Boys and by others. When the distance becomes microscopic so that the bodies are in 'contact', it is possible that the law is modified, if it is maintained that adhesion is due to gravitational forces, for adhesion between surfaces is a variable factor.

Gravitational permeability and the attraction of crystals.—It is well known that the behaviour of a crystalline substance, such as quartz, in an electric field depends upon the orientation of the axes of the crystal with respect to the field; but the weight of such a crystal as determined by weighing does not appear to depend upon the orientation of the crystallographic axes with respect to the field, for the weight is constant. Also, the force of attraction between electrically charged bodies, or between magnetic poles, depends upon the nature of the medium in which they are embedded or which occupies the space between them. Nothing corresponding to this has ever been detected in connexion with gravitational fields of force. AUSTIN and THWING† tested this question experimentally, using a modified form of Boys' torsion balance. Screens of lead, mercury, glycerol, etc. were interposed between the balls but changes in the deflexion of the beam carrying the suspended balls were not detected with any degree of certainty.

Theories of gravitational attraction.—No satisfactory explanation, other than that in terms of the general theory of relativity, of the force of attraction between material bodies has been given, but a review of other theories which have been made might not be without profit.

(i) ***Le Sage's theory:*** LE SAGE, of Geneva, in 1782, published a theory to account for the force of gravitation. According to it innumerable small particles were imagined to be flying about through space in all random directions. Ordinary matter is very highly, but not completely, permeable to these particles. A single

† *Physical Review*, **5**, 294, 1897.

material body, in free space, and therefore freely exposed to a bombardment by these particles, would, on the whole, acquire no velocity in virtue of the bombardment, since it would receive as many blows on one side as on the other. When two bodies are present, however, the bombardment on one side of each will be less than that on another and the bodies will move towards each other. In order to explain the law of gravitation quantitatively by this theory it is necessary to assume, what we now know to be valid, that the inter-atomic distances in all matter are very great compared with the sizes of the atoms themselves, so that only a very small proportion of the bombarding particles are stopped even by the densest of materials.

MAXWELL objected to this theory on the following grounds:

(*a*) The velocities of the corpuscles must be enormously greater than that of the heavenly bodies, for otherwise they would constitute a medium offering resistance to the motion of the planets.

(*b*) The transmission of gravitational forces would not be instantaneous, e.g. the earth's acceleration towards the sun at a given instant would be in the direction which the sun had a little time previously, i.e. at a time when some of the corpuscles now reaching the earth were passing near to the sun.

(*c*) The rate at which energy would have to be dissipated by the corpuscles to account for the attractive forces on one pound of matter would be about 10^{12} ft.lb.-wt.sec.$^{-1}$; such a rate of supply of energy is sufficiently great to raise very quickly the whole of the material universe to a white heat.

(ii) ***Challis' theory:*** This theory was based on the fact that waves in an elastic fluid, impinging upon a body immersed in that fluid, tend to cause the body to move towards (or away from) the centre of the disturbance according as the wavelength is very large (or very small) compared with the linear dimensions of the body.

(iii) ***Kelvin's theory:*** According to this theory, all space was filled with an incompressible fluid and all material bodies were continuously generating and emitting it at a constant rate, the fluid flowing off to infinity; or else always absorbing it, in which case it flowed in from infinity and was annihilated in the body. In either instance Kelvin proved that there was a force of attraction between bodies varying inversely as the square of the distance between them.

Maxwell remarks, 'All these theories require the expenditure of work. According to them we must regard the processes of Nature, not as illustrations of the great principle of the Conservation of Energy but as instances in which, by a nice adjustment of powerful agencies not subject to this principle, an apparent Conservation of Energy is maintained. Hence we are forced to conclude that the explanation of gravitation is not by these hypotheses.' Such was

Maxwell's opinion, but nowadays we have come to regard the idea of a perfectly elastic collision in connexion with the kinetic theory of gases as commonplace; hence this objection cannot be raised against Le Sage's theory.

(iv) ***The stress theory:*** Maxwell showed that there exists in an electrical field a tension along the tubes of force and a pressure at right angles to them. Such a set of stresses, with signs reversed, would explain the attraction between material bodies.

(v) ***The electrical theory:*** Every atom of matter consists of a positively charged nucleus surrounded in its normal state by electrons, whose number is such that the total negative charge on them just neutralizes the positive charge on the nucleus. If the force of attraction between unit positive and unit negative charges be very slightly in excess of the force of repulsion between unit charges of the same sign, then the difference between these two forces would be sufficient to account for gravitational attraction.

(vi) ***Einstein's relativity theory and gravitation:*** The accuracy with which the motions of celestial objects can be accounted for, and even predicted, by means of analysis based on Newton's laws of motion and his theory of attraction, is so amazing that for many years any question throwing doubt upon them would have been regarded as 'sacrilege'. Towards the end of the last century, however, difficulties began to arise and as time went on new ones appeared. In the first place, mass is found to be a variable depending upon the velocity of the body; secondly, the conception of length is not so simple as it first appears since the distance between two points is found to depend upon the motion of the observer.

Moreover, classical mechanics makes no attempt to discover how the force of attraction between two bodies arises and yet it is by means of this force that a body is able to produce effects on another body even when the distance apart is considerable. All these difficulties, and others, have been explained by EINSTEIN (1915) in his theory of relativity. According to Einstein, a planet, while describing its orbit about the sun, is in the same condition as a body moving in a straight line with uniform velocity. The orbit is the result of something analogous to curvature impressed by the sun upon the surrounding space-time continuum. In this 'relativistic space' the path of a body under the influence of no constraint is no longer linear. The whole idea of 'force' is dropped and the non-linear motion of the planet is attributed entirely to the modified space.

The theory shows that Newton's law of gravitation is, strictly speaking, only applicable to weak gravitational fields. Also, according to Einstein's theory, the major axis of the planetary orbit

should rotate slowly. It is only in the case of the planet Mercury that this rotation can be measured. This apparent anomaly among the motions of the planets was known before the advent of relativity and was a real stumbling block for classical mechanics. The rotation is found to be 43″ in a century, a value which agrees with that predicted by the theory of relativity whereas Newton's theory cannot account for it.

Again, according to Einstein, a ray of light has a mass which, on account of the large velocity involved, is by no means negligible. In consequence rays of light should be deflected as they pass through a strong gravitational field. If light were corpuscular in its nature, as Newton suggested, then a ray of light should suffer a deflexion θ, as it passed near to the sun; according to Einstein's theory the deflexion should be 2θ. The total eclipse of May, 1919, provided an occasion for testing Einstein's theory. The result of an expedition sent out for this purpose was published in November, 1919, and was at once a sensational triumph for the new theory.

Finally, Einstein showed that the frequency of the light emitted by an excited atom situated on the surface of a celestial body should be somewhat less than the frequency of the light emitted from a similar excited atom situated in free space or on the surface of a smaller body. Thus a displacement towards the red end of the spectrum ought to take place for spectral lines associated with stellar atoms as compared with the spectral lines of the same element when these lines are produced in the laboratory. The amount of this displacement is given by

$$1 - \frac{f}{f_0} = \frac{\gamma}{c^2}\frac{M}{r},$$

where f_0 is the frequency (terrestrial), f the frequency of the light from the 'stellar' atom, γ the Newtonian constant, c the velocity of light, M the mass of the heavenly body and r its radius. This displacement, termed the 'red shift', predicted for the sun is about 0·008 Å for light of wavelength 4000 Å. In 1925 ADAMS, from observations on the companion of Sirius, found a shift of 0·32 Å. Using the above formula and this observed shift, the radius of the star turns out to be 18,000 km., a value which agrees with that found by an independent method.

This effect has also been detected in the spectra of the remarkable type of stars known to astronomers as **white dwarfs**; they appear to consist of matter with a density 10,000 times that of ordinary matter and accordingly possess abnormally large values of $\frac{M}{r}$.

In conclusion, it must be noticed, however, that Newton's law of gravitation is not removed from the scene by Einstein's relativity

theory. For all engineering purposes, and indeed most physical ones too, Newton's theory is sufficiently accurate and, probably for ever, Newton's laws of motion will provide the basis on which to build an elementary study of mechanics and physics.

EXAMPLES VI

6·01. A small particle of mass m is placed on the axis of a thin circular metal disc at a distance z from its centre. If the disc is of radius a, uniform thickness t and the density of its material is ρ, find the gravitational attraction between the disc and the mass m.

$$\left[2\pi\gamma\rho t m\left\{1 - \frac{z}{(a^2 + z^2)^{0\cdot5}}\right\}\right]$$

6·02. A triangle is formed of three uniform rods whose masses per unit length are λ_1, λ_2 and λ_3 respectively. If O is a point inside the triangle and at distances p_1, p_2 and p_3 from the sides of the triangle, show that if

$$\frac{\lambda_1}{p_1} = \frac{\lambda_2}{p_2} = \frac{\lambda_3}{p_3},$$

a particle at O will be in equilibrium.

6·03. Assuming the orbits of the earth and moon to be circular, calculate the relative masses of the sun and the earth given that the period of the moon is $\frac{1}{13}$ year and that the distance from the sun to the earth is 390 times as great as the distance from the earth to the moon. $[0{\cdot}352 \times 10^6]$

6·04. A straight narrow tunnel is bored from the surface to the centre of a sphere of radius a and uniform density ρ. Determine the work required to take a small mass m from the centre to the surface of the sphere (a) by considering changes in potential, (b) by some other method. $[\frac{2}{3}\pi m\gamma\rho a^2]$

6·05. Find the gravitational potential and the field strength at a point on the axis of a thin circular loop of mass m and radius a if z is the distance of the point from the centre of the loop. Hence show that the potential is numerically a maximum at $z = 0$, and that the field strength is numerically a maximum at $z = \dfrac{a}{\sqrt{2}}$.

6·06. A smooth straight narrow tunnel is bored through an isolated uniform sphere at rest. A small particle moves from rest in this tunnel. If ρ is the density of the material of the sphere and γ the gravitational constant, find the period of the motion and show that it is independent of the size of the sphere and the direction of the tunnel. $\left[\mathrm{T} = \sqrt{\dfrac{3\pi}{\gamma\rho}}.\right]$

6·07. The density of the material of a sphere varies inversely as the distance from the centre. Show that the field strength at a point within the sphere is independent of its position. [If $\rho = kr^{-1}$, $\mathrm{G} = -2\pi k\gamma$.]

6·08. Find the gravitational potential and field strength due to a thin circular disc of radius a and mass μ per unit area, at a point distant z from the centre of the disc and equidistant from all points on its rim.

$$\left[-2\pi\gamma\mu[\sqrt{a^2 + z^2} - z],\ -2\pi\gamma\mu\left\{1 - \frac{z}{(z^2 + a^2)^{0\cdot5}}\right\}.\right]$$

6·09. Define ***flux of gravitational intensity*** and find an expression for it across an egg-shaped surface surrounding a material body.

If the mean density of the earth may be taken as 5·5 gm.cm.$^{-3}$ and the gravitational constant as $6{\cdot}6 \times 10^{-8}$ gm.$^{-1}$cm.^{3}sec.$^{-2}$, find a value for the distance of the moon from the earth in terms of the earth's radius. The period of the moon may be taken as 28 days. [61]

6·10. State Gauss' theorem as applied to gravitation and apply it to show that when two uniform spheres attract each other, each acts as if its mass were concentrated at its centre.

Describe, briefly, some experiment in which use is made of this result.

6·11. State Newton's law of gravitation and define the Newtonian gravitational constant.

Show that the gravitational intensity due to a uniform spherical shell of matter, at a point inside ths shell, is zero. Write down a simple expression for the intensity at a point outside the shell.

6·12. Show that the gravitational field strength at the vertex of a right circular cone of the frustum of the cone intercepted between two planes normal to the axis is $-2\pi\gamma\rho l(1 - \cos\alpha)$, where l is the thickness of the frustum measured along its axis, α the semi-angle of the cone, ρ the density of its material and γ the constant of gravitation.

6·13. Show that the observed latitudes of stations due north and south and at the foot of a hemispherical hill of radius a and density ρ differ by $\frac{a}{R}\left(2 + \frac{\rho}{\rho_0}\right)$, where R is the radius of the earth and ρ_0 is the mean density of its material.

6·14. State (a) Kepler's laws of planetary motion, (b) the conclusions which can be drawn from them.

Estimate the distance of the moon from the earth from the following data:—intensity of gravity = 980 cm.sec.$^{-2}$; mean diameter of the earth = $1{\cdot}273 \times 10^4$ km., and lunar month = 27·32 days. [Assume that the moon moves in a circular orbit round the earth.]

[$3{\cdot}83 \times 10^5$ km.]

6·15. State Newton's fundamental law of gravitation.

Describe, and give the theory of, Boys' method of determining the Newtonian constant.

6·16. Two small spheres, each of mass 20 gm., are supported by light threads each 10 metres in length; the centres of the spheres are in a horizontal plane. If the threads are 2 cm. apart at the upper ends, by how much is the distance between the centres of the spheres less than 2 cm.? [Assume a value for γ.] [$6{\cdot}6 \times 10^{-7}$ cm.]

6·17. Assuming the value of the gravitational constant to be $6{\cdot}6 \times 10^{-8}$ gm.$^{-1}$cm.^{3}sec.$^{-2}$, obtain a value for the period of a small satellite revolving in a circular orbit close to a spherical planet of unit density. [$1{\cdot}20 \times 10^4$ sec.]

6·18. Compare the minimum speed with which a particle would move round the earth near the surface and in a plane passing through its centre with the minimum velocity with which the particle must be projected in order to escape from the earth's gravitational field. indicate why your solution is only approximate. [$1 : \sqrt{2}$.]

6·19. Define the ***Newtonian constant of gravitation***, γ, and describe a laboratory method by which the value of this constant has been accurately determined.

Given that in c.g.s. units $\gamma = 6{\cdot}7 \times 10^{-8}$, the radius of the earth = $6{\cdot}4 \times 10^8$, and its mean density = 5·5, calculate the intensity of gravity at the earth's surface. [988 cm.sec.$^{-2}$]

6·20. Define the ***constant of gravitation*** and explain how the mean

density of the earth may be calculated, using the value of this constant.

Outline the principle of the Cavendish experiment and explain the advantages of the modifications in the apparatus as used by Boys.

6·21. A uniform sphere has a radius of 1·0 cm. Find the percentage increase in its weight when a second sphere of radius 25 cm. and density 10·4 gm.cm.$^{-3}$ is placed immediately below and nearly touching it. [$6 \cdot 7 \times 10^{-6}$]

6·22. Two small spheres are fixed at the opposite ends of a light horizontal beam 160 cm. long which is supported by a wire, the period of torsional oscillations being 400 sec. Estimate the deflexion of the beam caused by placing one of two spheres each of mass 2×10^5 gm. opposite each small sphere so as to produce the maximum possible deflexion, the distance between the centres of the attracting spheres being 20 cm.

[Neglect any 'cross-effect' between the attracting masses and explain why in your calculation it is not necessary to know the mass of each suspended sphere and take $\pi^2 = 10$.] [$1 \cdot 7 \times 10^{-3}$ radian]

6·23. If $\gamma = (6 \cdot 670 \pm 0 \cdot 005)\ 10^{-8}$ dyne.cm.^{2}gm.$^{-2}$(gm.$^{-1}$cm.^{3}sec.$^{-2}$), what is the force between two spheres, each of mass 5 kg., placed 30 cm. apart?

Assuming the above value for γ, and the radius of the earth to be $6 \cdot 37 \times 10^8$ cm., obtain a value for the mean density of the earth.

[$(1 \cdot 853 \pm 0 \cdot 002) \times 10^{-3}$ dyne., 5·54 gm.cm.$^{-3}$]

6·24. Prove that if the earth were spherical and uniform in content, the decrease in gravitational attraction as one rose in a balloon to a height z would be nearly twice the decrease as one descended in a mine to an equal depth.

6·25. A sensitive bullion balance has a sensitivity such that when 1 mg. is moved 1 cm. along the balance arm the deflexion of the balance is 600 scale divisions. Two spheres each of mass 400 gm. and radius 2 cm. are counterpoised on the balance, their centres being 60 cm. apart and a lead sphere of radius 12 cm. being situated immediately below one of the small spheres and almost in contact with it. The sphere is then moved to a similar position below the other sphere, when the balance deflexion is 192 scale divisions. Obtain a value for the gravitational constant. [$6 \cdot 86 \times 10^{-8}$ gm.$^{-1}$cm.^{3}sec.$^{-2}$]

[Assume $g = 1000$ cm.sec.$^{-2}$, density of lead $= 11 \cdot 37$ gm.cm.$^{-3}$.]

6·26. Give an account of an accurate method for the determination of the Newtonian constant of gravitation.

Assuming that the planets move in approximately circular orbits round the sun as centre, prove Kepler's third law that the square of a planet's year is proportional to the cube of its distance from the sun.

If a planet has a satellite moving in a circular orbit about it with a known period and at a known distance from it, show how the ratio of the mass of the planet to that of the sun may be obtained, the year of the planet and its distance from the sun being known. (G)

6·27. Describe an accurate method for determining the constant of gravitation.

Assuming the value of this constant to be $6 \cdot 7 \times 10^{-8}$ dyne.cm.^{2}gm.$^{-2}$, calculate the mass of the sun on the assumption that the earth travels in a circular orbit of radius $1 \cdot 5 \times 10^{11}$ metres with a speed of 3×10^4 metre.sec.$^{-1}$. [$2 \cdot 01 \times 10^{30}$ kgm.]

6·28. Describe Airy's mine experiment. Find the acceleration of

gravity at the bottom of a shaft of 0·5 miles depth if the average density of the appropriate outer shell of the earth is 2·5 gm.cm.$^{-3}$. The mean density of the earth is given as 5·5 gm.cm.$^{-3}$, the radius of the earth as 4000 miles and the intensity of gravity as 981·2 cm.sec.$^{-2}$.

[981·28 cm.sec.$^{-2}$]

6·29. Prove that the value of the intensity of gravity at a point on an elevated table-land of height h is given by

$$g\left[1 - \left(2 - \frac{3\rho}{2\rho_0}\right)\frac{h}{a}\right],$$

where g is the intensity of gravity at sea-level, a the earth's radius, ρ_0 the mean density of the earth and ρ the density of the material in the table-land. [Hint: The table-land may be regarded as a thin circular plate of surface density ρ; also $g = \frac{4}{3}\pi\gamma\rho a$ and at the top of the table-land the intensity of gravity due to the 'sphere' is $g\,\dfrac{a^2}{(a+h)^2}$.]

6·30. Von Jolly found that when a mass of 5 kg. is raised 21 metres above the surface of the earth its weight diminishes by 31·8 milligram.-wt. Use this observation to calculate the diminution of gravitational attraction on 1 gm. of matter due to a displacement of 1 cm. above the surface of the earth. If $g = 980$ cm.sec.$^{-2}$ and $r = 6{\cdot}4 \times 10^8$ cm., compare the value obtained from Jolly's observation with that calculated theoretically.

$\left[\text{Hint, if } F = \dfrac{\gamma M}{r^2}, \dfrac{dF}{dr} = -\dfrac{2\gamma M}{r^3}, \text{etc.}\right]$ [2·97 × 10^{-6} dyne; 1: 1·02]

6·31. A sensitive balance carries four pans, one pair being 20 metres below the other pair. In each of the upper pans a sphere of mass 5 kg. was placed and the balance equilibrated. When one sphere was transferred to the pan below, its weight increased and the balance was equilibrated by adding a mass of 32 mg. to an upper pan.

When a lead sphere, 50 cm. radius, was placed directly below the sphere in the lower pan, an additional mass of 0·69 mg. had to be added to an upper pan to restore equilibrium. Assuming the density of lead to be 11·4 gm.cm.$^{-3}$ and the radius of the earth 6·4 × 10^8 cm., obtain a value for the mass of the earth. [6·0 × 10^{24} kgm.]

6·32. Calculate a value for the period of the earth if this were rotating so quickly that the weight of a body at the equator were zero. [Assume $r = 6{\cdot}4 \times 10^8$ cm.] [$\frac{1}{17}$ day.]

6·33. Describe a laboratory method for the determination of the Newtonian constant of gravitation.

The ratio of the times of revolution of Mars and the earth round the sun is 1·88. Find the ratio of the mean distances of these planets from the sun. Prove the formula used. [1·52]

6·34. Derive an expression for the gravitational potential at a point outside a uniform sphere. Calculate the energy that has to be supplied by a rocket in order to move a projectile of total mass 10,000 kg. up to a height of 1,000 km. above the earth's surface. Frictional forces in the atmosphere may be neglected. Assume that the intensity of gravity at the earth's surface $g = 9{\cdot}80$ m.sec.$^{-2}$ and the earth-radius $R = 6{\cdot}37 \times 10^6$ m. [6·2 × 10^5 joule.]

6·35. If a uniform sphere has a mass m and radius a, show that the attraction which one hemisphere exerts upon the other is $\dfrac{3}{16}\dfrac{\gamma m^2}{a^2}$.

6·36. A uniform sphere of radius a, and density ρ, is divided into two

parts by a plane at distance b from its centre. Prove that the mutual attraction between the two parts is

$$\tfrac{1}{3}\pi^2\gamma\rho^2(a^2 - b^2)^2.$$

6·37. Show that two equal spheres, radius a, and density equal to the mean density of the earth, starting from rest at a great distance apart will collide with a common speed equal to $a\sqrt{\dfrac{g}{2\mathrm{R}}}$, where g is the intensity of gravity and R is the radius of the earth. [Assume that the spheres are subject only to their mutual attraction.] [Hint: The decrease in potential energy of one sphere in the field of the other when both are in contact is equal to the kinetic energy of both spheres.]

6·38. Assuming that the mean angular diameter of the sun, as viewed from the earth, is $9{\cdot}4 \times 10^{-3}$ radian and that the gravitational constant is $6{\cdot}7 \times 10^{-8}$ cm.^{3}gm.$^{-1}$sec.$^{-2}$, estimate the mean density of solar matter. [Take 1 year $= 3{\cdot}1 \times 10^7$ sec.] [1·41 gm.cm.$^{-3}$.]

6·39. The density ρ at a point distance r from the centre of a sphere of radius a is given by

$$\rho = \frac{\mathrm{M}}{\pi a^3}\left(\frac{3}{2} - \frac{r}{a}\right).$$

Show that M is the mass of the sphere. Also if g is the intensity of gravity at the surface of the sphere and powers of $\dfrac{h}{a}$ above the first are neglected, show that at a height h from the surface the intensity is

$$g\left(1 - \frac{2h}{a}\right),$$

but that its value at a depth h below the surface is unaltered.

B.Sc., A.M., (L).

CHAPTER VII

ELASTICITY; STRESS, STRAIN AND THE STRENGTH OF MATERIALS

Introduction.—A study of the subject of mechanics begins with the conception of an infinitely small but massive particle and discusses its behaviour under the influence of impressed external forces; later the idea of the ***rigid body*** is mooted. Such a body possesses the property that the distance between any two points in it is invariable however the forces to which it is subjected may vary. Actually no body is perfectly rigid and the behaviour of ordinary materials under the action of forces constitutes the study of elasticity.

Strain.—When a body is in equilibrium solely under the action of internal molecular forces, that body is said to be in its ***natural state***. If external forces act on such a body so that the material is stressed, and the body undergoes change of shape or size, or changes in both, then the body is said to be in a ***state of strain*** or to be ***strained***. Now according to MAXWELL [*Theory of Heat*, p. 295], ***a perfectly rigid body***, if it existed, is one which would experience no strain when acted upon by external forces setting up a state of stress in the body. Actually no body fulfils such conditions, i.e. all known materials are ***elastic***, i.e. stresses within a body are always accompanied by strain. ***A perfectly elastic body***, according to Maxwell, ***is one which, when subjected to a given stress at a given temperature, experiences a strain of definite amount, which does not increase when the stress is prolonged, and which disappears completely and instantaneously*** † ***when the stress is removed***. It sometimes happens that when the external forces are removed, the new size and shape of the body is retained; the material is then said to be ***perfectly plastic***. It must be noted at once, however, that the substances termed perfectly elastic are only so provided that the deforming forces are not excessive, and that plastic substances may possess a small degree of elasticity if the forces applied are exceedingly small. The behaviour of substances stressed beyond the limit of perfect elasticity will be discussed more fully at a later stage.

Tensile strain and tensile stress.—Suppose that AB, Fig.

† This word is now added to Maxwell's original statement.

7·01(*a*), is a uniform rod of material under the action of two tractive forces F, F, acting along its longitudinal axis. Let CD be a plane, normal to the longitudinal axis of the rod and dividing this into two portions. If internal forces were not called into play as soon as the stretching forces became operative, the rod would increase in length and continue to do so until it ruptured. Now the portion AC will exert a force across CD on the portion DB; this force will be the resultant of the elementary forces, arising from the above cause, and acting on each element of CD—cf. Fig. 7·01(*b*). For CD to remain in equilibrium the above resultant must be equal and opposite to the force F acting at the end B. Similar remarks apply to the portion AC.

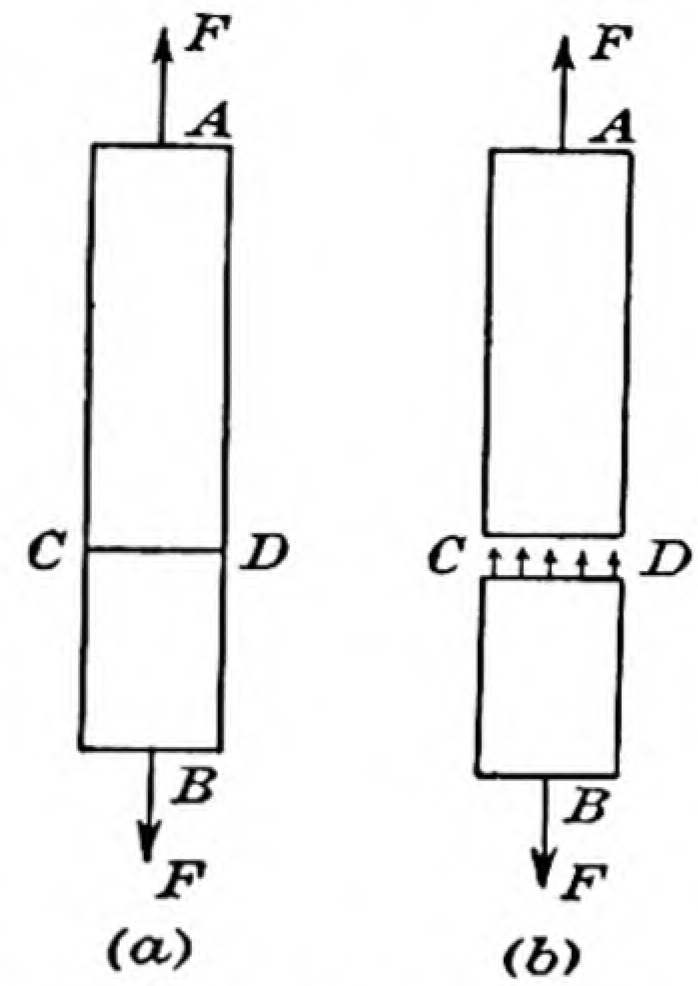


Fig. 7·01.—Internal stress caused by a longitudinal pull.

In virtue of the equal and opposite forces acting across any such plane dividing the body into two portions, the body is said to be ***stressed***, or in a ***state of stress***. The term stress is therefore used to denote the mutual action between two bodies in contact, or between two portions of the same body, whereby each one, or each portion, exerts a force upon the other.

If the force acting across any portion of the surface CD is directly proportional to the area of that portion, the stress is said to be uniform. Thus, if the area CD is S, and the stress is uniformly distributed across that area, the quantity $\frac{F}{S}$ is termed the ***extensional or tensile stress***, across the section of the body considered. If the forces are reversed the body is subject to a thrust and the corresponding stress is a ***compressive stress*** of amount $\frac{F}{S}$.

If the stress is not uniform, let δF be the force acting across and normally to an element of area δS. Then p, the stress at a point on the surface δS, is given by

$$p = \lim_{\delta S \to 0} \frac{\delta F}{\delta S} = \frac{dF}{dS}.$$

If l is the original length of the rod and Δl is the increase in length then $\frac{\Delta l}{l}$ is the ***tensile strain*** in the rod. Strain is therefore a dimensionless quantity.

Since a stress is a force per unit area, it follows that stresses are measured in the absolute systems of units either as dyne.cm.$^{-2}$, or as poundal.ft.$^{-2}$. Other common units for stress are ton.-wt.in.$^{-2}$, kgm.-wt.cm.$^{-2}$, and atmospheres; the first two are gravitational units.

Shear strain and shear stress.—A shear stress exists between two parts of a body in contact when each part exerts an equal and opposite force laterally on the other part and in a direction tangential

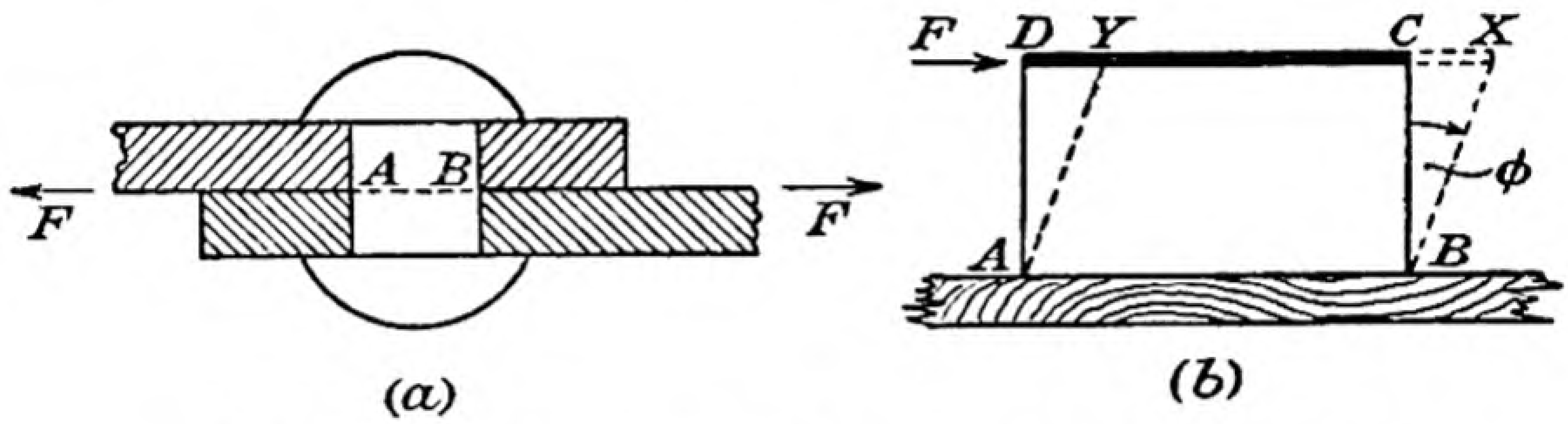


FIG. 7·02.—(*a*) Shear stress and strain. (*b*) Shearing strain.

to the surface of contact separating the two parts. Thus, suppose a rivet holds together two plates which sustain a pull F, F, across the section AB, Fig. 7·02(*a*). Under these conditions the lower portion of the rivet exerts a force parallel to AB on the upper portion, preventing it from moving to the left; similarly, the upper part exerts a force on the lower. The rivet is said to be in a ***state of shear*** across the plane AB, and if S is the area of the section AB, the ***shear stress*** is defined as $\frac{F}{S}$.

To discover how the strain is measured when a state of shear exists, let us consider ABCD, Fig. 7·02(*b*), the cross-section of a block of india-rubber glued to a table along that face of which AB is the trace. Imagine that a piece of sheet brass glued to the upper surface is urged forward by a force F parallel to AB. When equilibrium is reached let the plate be in a position XY, i.e. the plate will have suffered a displacement CX with respect to the lower face. The block is now said to be sheared, and the shearing strain is equal to the circular measure of the small angle $\widehat{CBX} = \phi$, say. Since $\phi = \frac{CX}{BC}$, it will be seen that the shearing strain is the ratio of the relative lateral displacement CX of two horizontal layers at distance BC apart to that distance, i.e. it is equal to the numerical value of the relative lateral displacement of the two horizontal layers at unit distance apart.

If S is the area of the upper face the shearing stress is $\frac{F}{S}$.

It is important to note the following distinction between strain due to stretching [or compressing] forces and that due to shearing forces, for in the first instance both the volume and shape of the body may alter, whereas in the second it is the shape alone which changes, the volume remaining constant. A particular instance in which a change in volume but no change in shape occurs is when a cube of material which is isotropic, i.e. has properties the same in all directions, is subjected to a uniform pressure.

Complementary stresses due to shear.—Theorem: ***A shear stress in a given direction cannot exist without an equal shear stress existing at right angles to it.*** To prove this, let us consider the rectangular body of sides, a, b and c, shown in Fig. 7·03. Let F_1, F_1, be the forces tending to displace the upper face with respect to the lower. The area of each of these faces is ab, so

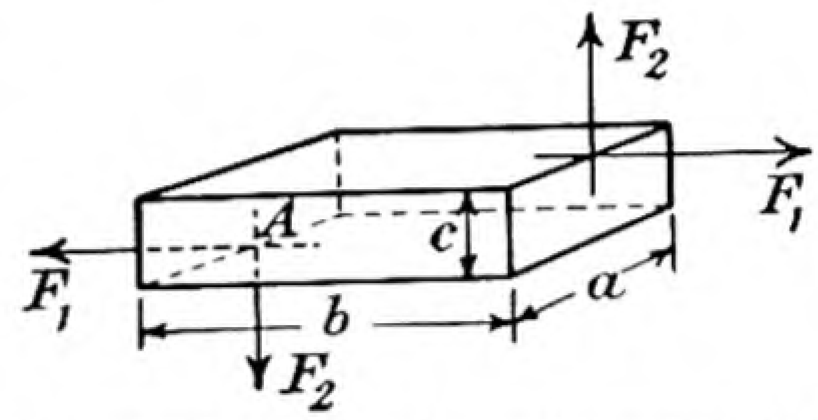


FIG. 7·03.—Complementary stresses due to shear.

that the shear stress is $\frac{F_1}{ab}$. Let F_2, F_2 be shearing forces at right-angles to the above. Then the corresponding stress is $\frac{F_2}{ac}$. For equilibrium, the moment of all the forces about any point in their plane—say A—must be zero, i.e. $F_1c = F_2b$. Dividing throughout by abc, we have

$$\frac{F_1}{ab} = \frac{F_2}{ac},$$

i.e. the stresses are equal.

Volume strain.—When either solid or fluid matter is subjected to a change in pressure the volume of the substance changes. The ratio of the change in volume to the original volume measures the strain in this instance.

Normal and tangential stresses.—Hitherto, it has been assumed that the direction of the tractive forces acting on a body is normal to the surface across which the stress is calculated; but, in general, there will be a stress across any plane drawn in the body. For let HK, Fig. 7·04(a), be a plane across a uniform rod, the normal to the plane making an angle θ with the axis of the rod. Let the stretching forces be F, F, as shown. If we consider the equilibrium of the lower portion of the rod, the forces acting on it are as shown in Fig. 7·04(b). Now the force F on HK may be resolved into two components F_n and F_t, cf. Fig. 7·04(c), which are respectively normal and tangential to the section HK; their values are F cos θ

and F sin θ. The normal stress across HK is therefore given by

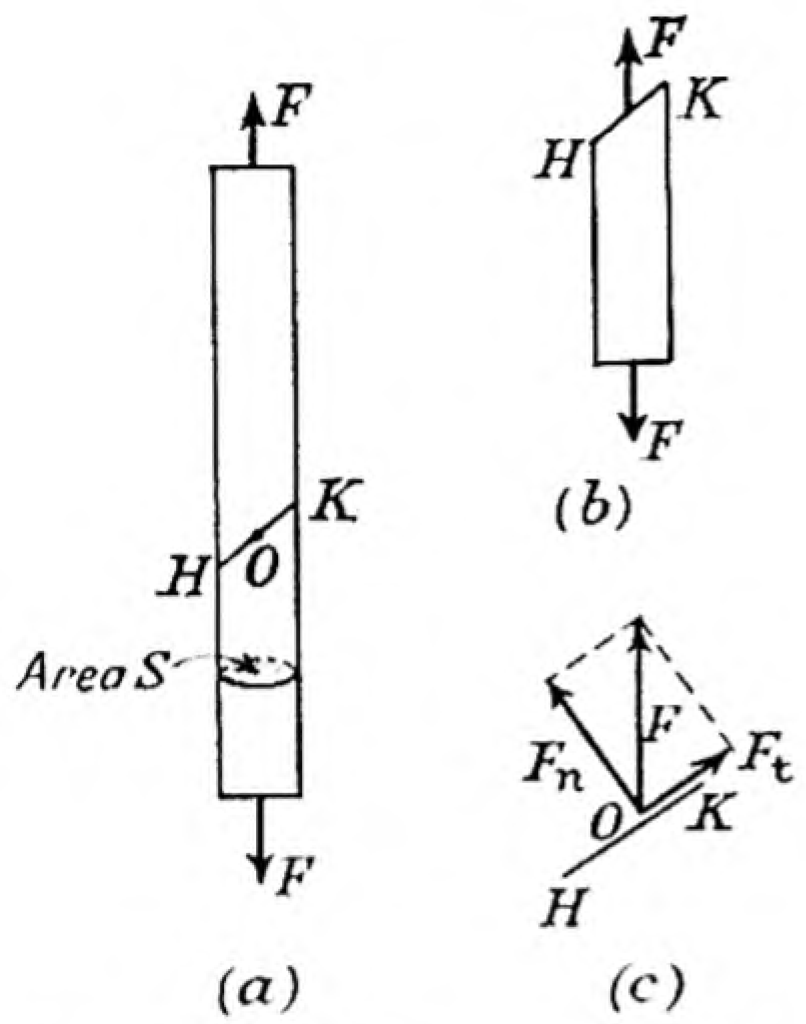


FIG. 7·04.—Stress on an oblique section of a rod.

$$p_n = \frac{F_n}{\text{area HK}} = \frac{F_n}{S \sec \theta},$$

where S is the cross-sectional area of the rod. Since $F_n = F \cos \theta$, we have

$$p_n = \frac{F \cos \theta}{S \sec \theta} = \frac{F}{S} \cos^2 \theta.$$

The tangential stress is likewise given by

$$p_t = \frac{F_t}{\text{area HK}} = \frac{F}{S} \sin \theta \cos \theta$$

$$= \frac{1}{2} \frac{F}{S} \sin 2\theta.$$

The tangential stress at a point in a plane within the material is therefore a maximum when sin 2θ is unity, i.e. when $\theta = \frac{1}{4}\pi$ or $\frac{3}{4}\pi$. In the actual testing of rods of material by stretching them, it is frequently found that yielding takes place, at least in part, by shearing across surfaces inclined at an angle $\frac{1}{4}\pi$ to the direction of the tractive forces.

Since p_n and p_t are vectors, the resultant stress across the section HK is parallel to F and given by

$$p = [p_n^{\,2} + p_t^{\,2}]^{0\cdot 5} = \left[\left(\frac{F \cos \theta}{S \sec \theta}\right)^2 + \left(\frac{F \sin \theta}{S \sec \theta}\right)^2\right]^{0\cdot 5}$$

$$= \frac{F}{S \sec \theta}$$

$$= \frac{F}{\text{area HK}}.$$

A similar argument applies to any small elementary area within a body. Thus if δF is the resultant of all the elementary forces acting on an elementary surface δS, Fig. 7·05(*a*), it may be resolved into two components $\delta F \cos \theta$ and $\delta F \sin \theta$, where θ is the angle a normal to δS makes with δF. Hence

$$p_n = \lim_{\delta S \to 0} \frac{\delta F \cos \theta}{\delta S} = p \cos \theta,$$

and

$$p_t = \lim_{\delta S \to 0} \frac{\delta F \sin \theta}{\delta S} = p \sin \theta.$$

These stresses are shown in Fig. 7·05(*b*).

Stresses normal to a surface may be due to forces tending to stretch or to compress a body; in the latter instance the length of the body must be small to avoid 'buckling'. The stress due to the

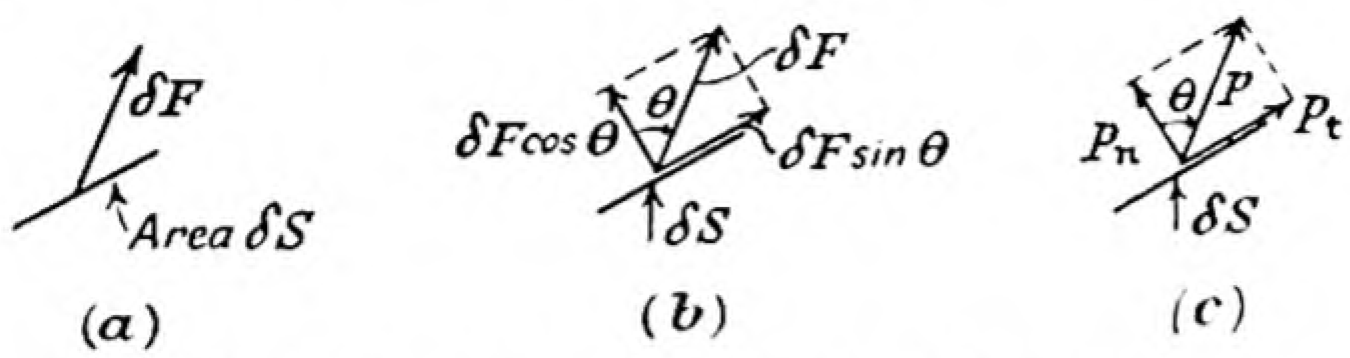


FIG. 7·05.—Normal and tangential stresses.

former set of forces is considered positive, i.e. the stress is positive when the two portions of the body, one on either side of an imaginary surface across which the stress is acting, tend to come apart.

Tangential stresses, i.e. those tending to make one part of a body slide across the other part, are termed ***shearing stresses***, cf. Fig. 7·06(a).

Shear; another point of view.—Let ABCD, Fig. 7·06(a), be a cross-section of a cube of isotropic material after shearing: A_0B_0CD is the original section. The shear strain is measured by the angle ϕ, and the stress is $\frac{F}{S}$, where S is the area of a face of the cube. An inspection of the diagram shows that the diagonal DB_0 has increased in length to DB, while the other diagonal A_0C has been reduced to AC. Consider the former diagonal. With centre D and radius DB_0 describe an arc to cut DB in X. Since B_0B is small it follows that $B_0\widehat{B}X = \frac{1}{4}\pi$ and that $B_0\widehat{X}B = \frac{1}{2}\pi$. Hence $BX = B_0X$.

The strain along DB is

$$\frac{DB - DB_0}{DB_0} = \frac{BX}{DB_0}$$

$$= \frac{B_0B}{\sqrt{2}} \cdot \frac{1}{a\sqrt{2}} = \tfrac{1}{2}\phi,$$

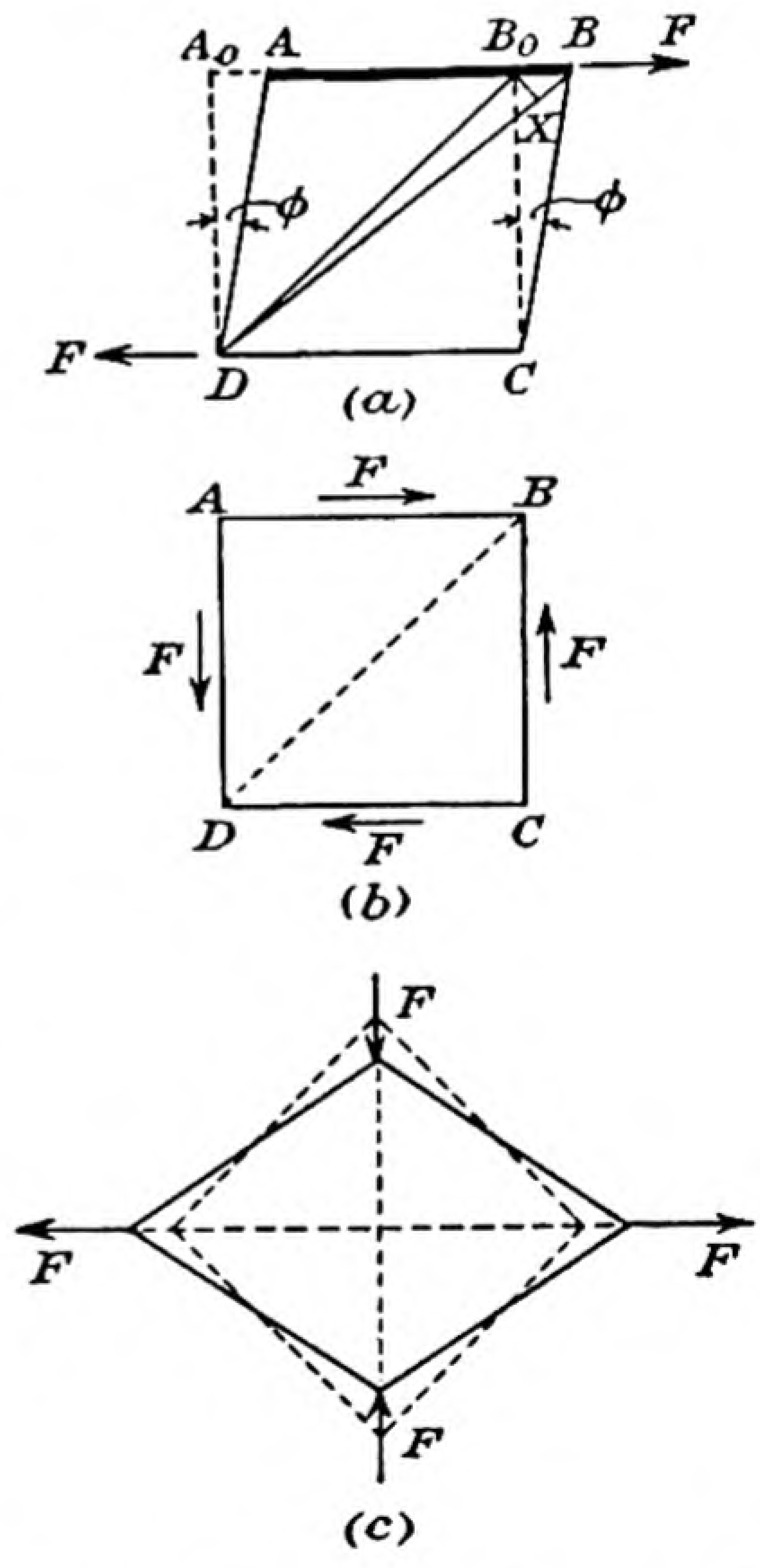


FIG. 7·06.—Shear; another point of view.

a being the length of the edge of the cube when it is not strained.

Similarly the strain along AC is a compressive one of amount $\frac{1}{2}\phi$.

The fact that it has been established that there are strains along DB and AC suggests that stresses will exist in these directions. From the proposition established on p. 265, it follows that for the cube under consideration to be in equilibrium the system of forces shown in Fig. 7·06(b) must be operative upon its faces. The stress across each face is $F \div S = p$. Now the portion BCD is in equilibrium under the action of the forces F, F, along CB and CD respectively, together with the force F_x which the portion ABD exerts upon BCD. Now the resultant of F, F is a force $\sqrt{2}.F$ along CA. Consequently F_x must be a force $\sqrt{2}.F$ along AC. This is distributed over an area $\sqrt{2}.S$ so that the corresponding stress is

$$p_x = \frac{\sqrt{2}.F}{\sqrt{2}.S} = p.$$

It is a compressive stress.

Similarly, by considering the equilibrium of the part ABC it may be shown that there is a tensile stress p_y along BD, also equal in magnitude to p. Hence a shearing stress p is equivalent to a tensile stress p in one direction combined with an equal compressive stress in a perpendicular direction, cf. Fig. 7·06(c).

On the combination of two simple stresses in directions at right angles to each other.—(a) ***Like stresses:*** Consider a block of material subjected to pairs of stretching forces F_x, F_x; F_y, F_y, as shown in Fig. 7·07(a). Let p_x and p_y be the stresses produced at an internal point O by the forces F_x and F_y on planes normal to their respective directions, which are taken as the axes of rectangular coordinates Ox, Oy. Consider the stresses normal and tangential to any plane HK passing through O, the normal to this plane making an angle θ with the x-axis. Since p_x and p_y may vary within the material let us consider the equilibrium of a small element of the material, HKL, of unit depth perpendicular to the plane of the diagram, HL and KL being parallel to the axes Ox, Oy, respectively. Consider the forces on the faces of this element due to the rest of the material. We may omit all consideration of the forces on the end faces parallel to the plane of the diagram, since these are equal and opposite. The forces on the other faces are given by

$$\delta F_x = p_x . \mathrm{KL},$$

$$\delta F_y = p_y . \mathrm{HL},$$

$$\delta F_n = p_n . \mathrm{HK},$$

and

$$\delta F_t = p_t . \mathrm{HK},$$

where p_n and p_t are the normal and tangential components of the stress on the face HK, and δF_n and δF_t the components of the resultant force on the face HK due to the action of the rest of the body on it.

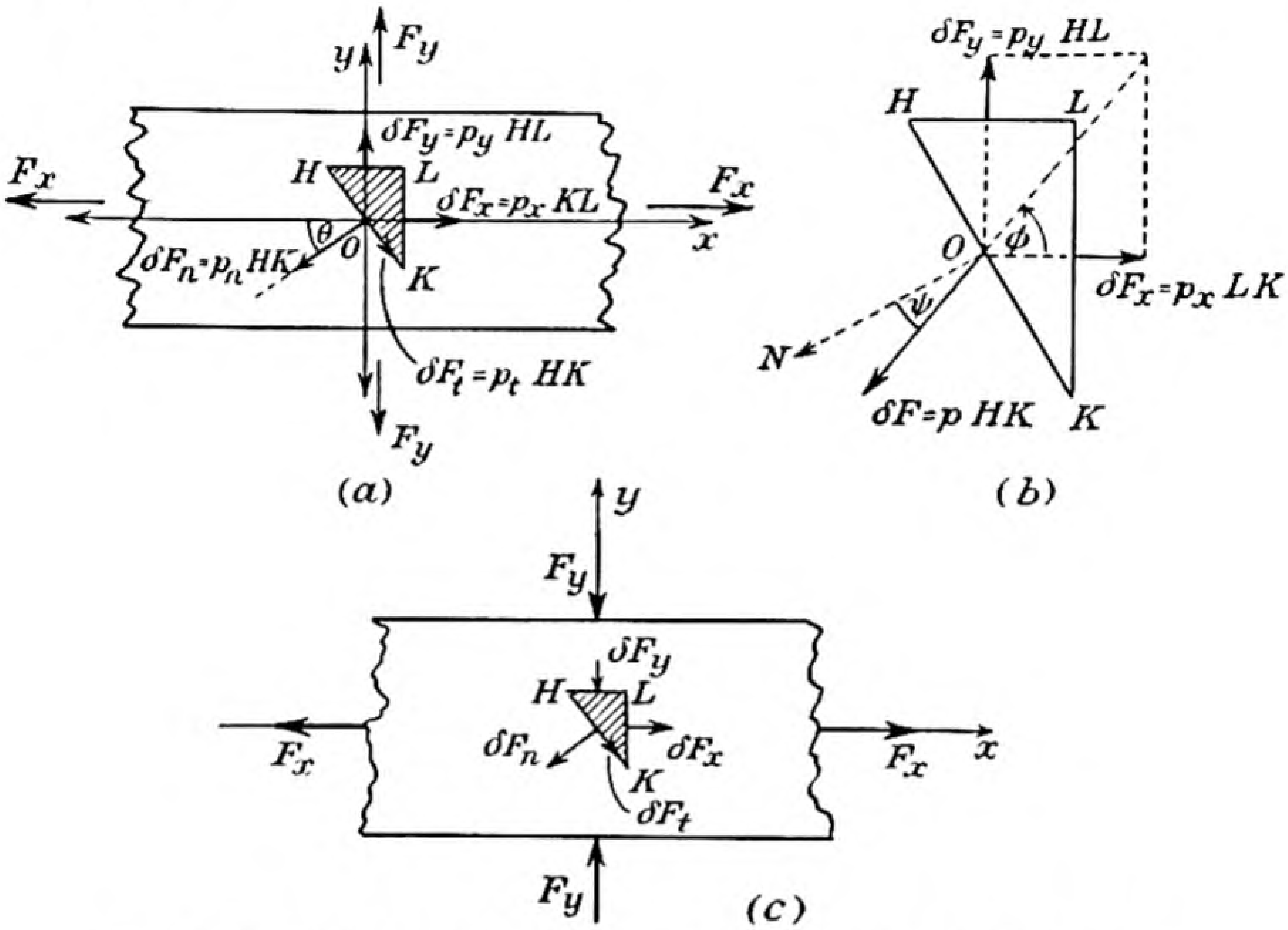


FIG. 7·07.—Combination of two stresses mutually at right angles.

Resolving forces perpendicular to and along HK, we have, for equilibrium,

$$p_n.\mathrm{HK} = p_x.\mathrm{KL}.\cos\theta + p_y.\mathrm{HL}.\sin\theta,$$

and
$$p_t.\mathrm{HK} = p_x.\mathrm{KL}.\sin\theta - p_y.\mathrm{HL}.\cos\theta.$$

$$\therefore\ p_n = p_x.\cos^2\theta + p_y.\sin^2\theta,$$

and
$$p_t = (p_x - p_y)\sin\theta.\cos\theta = \tfrac{1}{2}(p_x - p_y)\sin 2\theta.$$

The last equation gives the shear stress; it is a maximum when $\theta = \frac{1}{4}\pi$ or $\frac{3}{4}\pi$, the corresponding normal stress being

$$[p_n]_{\theta=\frac{1}{4}\pi} = p_x\cos^2\tfrac{1}{4}\pi + p_y\sin^2\tfrac{1}{4}\pi = \tfrac{1}{2}(p_x + p_y).$$

Let p be the resultant stress at any point on HK—cf. Fig. 7·07(b). Then

$$\begin{aligned} p.\mathrm{HK} &= \text{resultant force on HK} \\ &= \sqrt{(\delta F_x^{\,2} + \delta F_y^{\,2})} \\ &= \sqrt{(p_x.\mathrm{KL})^2 + (p_y.\mathrm{HL})^2} = \mathrm{HK}\sqrt{p_x^{\,2}\cos^2\theta + p_y^{\,2}\sin^2\theta}. \end{aligned}$$

$$\therefore\ p = \sqrt{(p_n^{\,2} + p_t^{\,2})}.$$

To specify the direction of the resultant stress on the face HK, either ϕ, the angle the resultant makes with OX, or ψ, the angle it makes with the normal to the face HK, must be calculated. Now

$$\tan\phi = \frac{\delta F_y}{\delta F_x} = \frac{p_y . HL}{p_x . KL}$$

$$= \frac{p_y}{p_x} . \tan\theta.$$

Also $p_n = p . \cos\psi$, and $p_t = p . \sin\psi$, so that

$$\tan\psi = \frac{p_t}{p_n} = \frac{(p_x - p_y)\sin\theta . \cos\theta}{p_x . \cos^2\theta + p_y . \sin^2\theta}.$$

(*b*) ***Unlike stresses:*** Suppose that the forces F_y are such that they tend to compress the material. Again consider an element HKL, the forces on the faces being HL and KL, directed as shown in Fig. 7·07(*c*), and given by

$$\delta F_x = p_x . KL,$$

$$\delta F_y = p_y . HL.$$

Let p_n and p_t be the normal and tangential components of the stress across HK, the force resultant on this face therefore having components $p_n . HK$ and $p_t . HK$ respectively. For the equilibrium of this element, we have, as before,

$$p_n . HK = \delta F_n = \delta F_x . \cos\theta - \delta F_y . \sin\theta,$$

and

$$p_t . HK = \delta F_t = \delta F_x . \sin\theta + \delta F_y . \cos\theta.$$

Hence $$p_n = \frac{\delta F_n}{HK} = p_x . \cos^2\theta - p_y . \sin^2\theta,$$

and $$p_t = \frac{\delta F_t}{HK} = (p_x + p_y)\sin\theta . \cos\theta = \tfrac{1}{2}(p_x + p_y)\sin 2\theta.$$

Again the tangential stress is a maximum when $\theta = \frac{1}{4}\pi$ or $\frac{3}{4}\pi$, the corresponding normal stress being $\frac{1}{2}(p_x - p_y)$.

When the unlike forces discussed above are equal in magnitude, so that $|p_x| = |p_y| = p$, say, we have a type of stress of great practical importance, for then the normal stress on any plane parallel to the planes defined by $\theta = \frac{1}{4}\pi$ and $\theta = \frac{3}{4}\pi$ is zero, the tangential stress being p. The body is then said to be in a ***state of simple shear.***

An ellipse of stress.—To construct this ellipse, with O as centre, Fig. 7·08, and radii $OA = p_x$ and $OB = p_y$, we draw two circles. Let CD be the trace of the plane across which the stress is to be determined; let ON, the normal to this plane at O, make an angle θ with

the x-axis. If ON cuts the two circles in M and N, then straight lines through M and N and parallel to Ox and Oy respectively will intersect at a point R on the ellipse whose major and minor semi-axes are OA and OB.

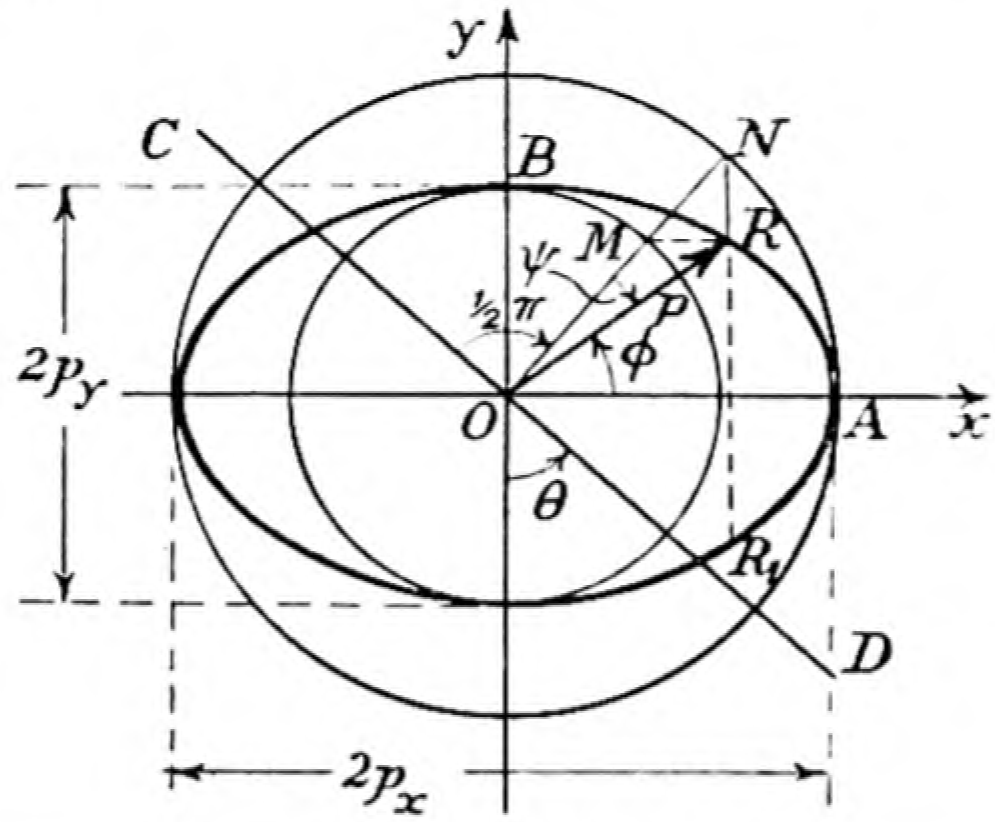


FIG. 7·08.—An ellipse of stress.

The coordinates of R are $(p_x \cos\theta, p_y \sin\theta)$; hence

$$OR = \sqrt{p_x^2 \cos^2\theta + p_y^2 \sin^2\theta} = p,$$

the resultant stress on CD. Moreover, if OR makes an angle ϕ with Ox,

$$\tan\phi = \frac{p_y}{p_x}.\tan\theta.$$

while if ψ is $\widehat{RON}$,

$$\tan\psi = \tan(\theta - \phi) = \frac{(p_x - p_y)\tan\theta}{p_x + p_y\tan^2\theta} = \frac{p_t}{p_n}.$$

Thus OR gives the magnitude and direction of the resultant stress on CD.

[N.B. If one of the stresses is reversed, the stress on CD will be given by OR_1 where R_1 is the 'image' of R.]

Principal planes and principal stresses.—When a body is under the action of a system of applied forces which causes wholly normal or wholly tangential stresses across different known planes, the state of stress across another known plane may be determined by adding vectorially the various tangential components to give the resultant tangential stress; similarly for the normal components.

The planes through a point within a material such that the resultant stress across each is wholly normal are called ***principal planes***; the normal stresses across them are the ***principal stresses***.

However complex the state of stress at a point within a material it may be shown that three mutually perpendicular principal planes

always exist; thus the stress at the point concerned may always be resolved into three principal stresses. Physical considerations, as the sequel will show, often enable us to locate the directions of the three principal stresses.

Hooke's law and the limit of perfect elasticity.—For many substances it is found experimentally that the deformation produced by a given stress is directly proportional to that stress; the stress at which this linear relationship ceases to be valid is termed the ***elastic limit*** of the material under examination for the particular type of stress applied. The existence of this linear relationship for stresses below the elastic limit was first discovered by HOOKE in 1679 and is known as Hooke's law. If a substance is subjected to a stress below the elastic limit it recovers completely when the stress is removed, but if the stress applied exceeds the elastic limit, then the substance does not return to its original state when the stress is removed—the substance is then said to have acquired a ***permanent set.*** [At the present time engineers often refer to the elastic limit as defined above as the stress at the limit of proportionality and define the elastic limit as the stress when the permanent set in the material amounts to a very small arbitrary amount; the difference is small and will be neglected in all subsequent discussion.]

Another definition of the elastic limit is as follows: it is that stress which produces the maximum amount of recoverable deformation. To determine its value experimentally the specimen is loaded and the distance between two fiducial marks measured after the stress has been removed. This process is repeated with increasing stresses until a permanent set occurs for each stress that is applied. A graph of the permanent set against the load is then plotted and the stress at which the permanent set begins is deduced. For most solids these two definitions are practically equivalent.

In discussing the behaviour of materials stressed beyond the elastic limit our attention will be confined to specimens which are subjected to gradually increasing stretching forces.

The behaviour of a solid when the applied stress exceeds the elastic limit.—(*a*) ***Brittle materials in tension:*** Cast iron, hardened iron, Portland cement, stone and brick are examples of brittle substances. Fig. 7·09(*a*) shows the relation between the strain and stress for such a substance. OA is linear so that the stress at A is the elastic limit, but beyond A the graph is curved. The point B represents the stage when the substance breaks.

(*b*) ***Ductile materials in tension:*** The stress-strain diagram for a 0·25 per cent carbon steel, obtained autographically, is shown in Fig. 7·09(*b*). A is the elastic limit and beyond this point the graph curves until the point Y is reached. Then comes the portion YZ of

the curve, representing the stage during which there is a large increase in the strain with practically no increase in stress; on a self-recording sensitive extensiometer the portion YZ appears as an irregular wavy line the stress corresponding to Z being less than that at Y. Y is called the ***yield-point***, the corresponding stress being the ***yield-stress***. In the stage AY the stretch is partly elastic and partly due to plastic flow in minute particles distributed throughout the material under test. Beyond Z the elongation becomes plastic;

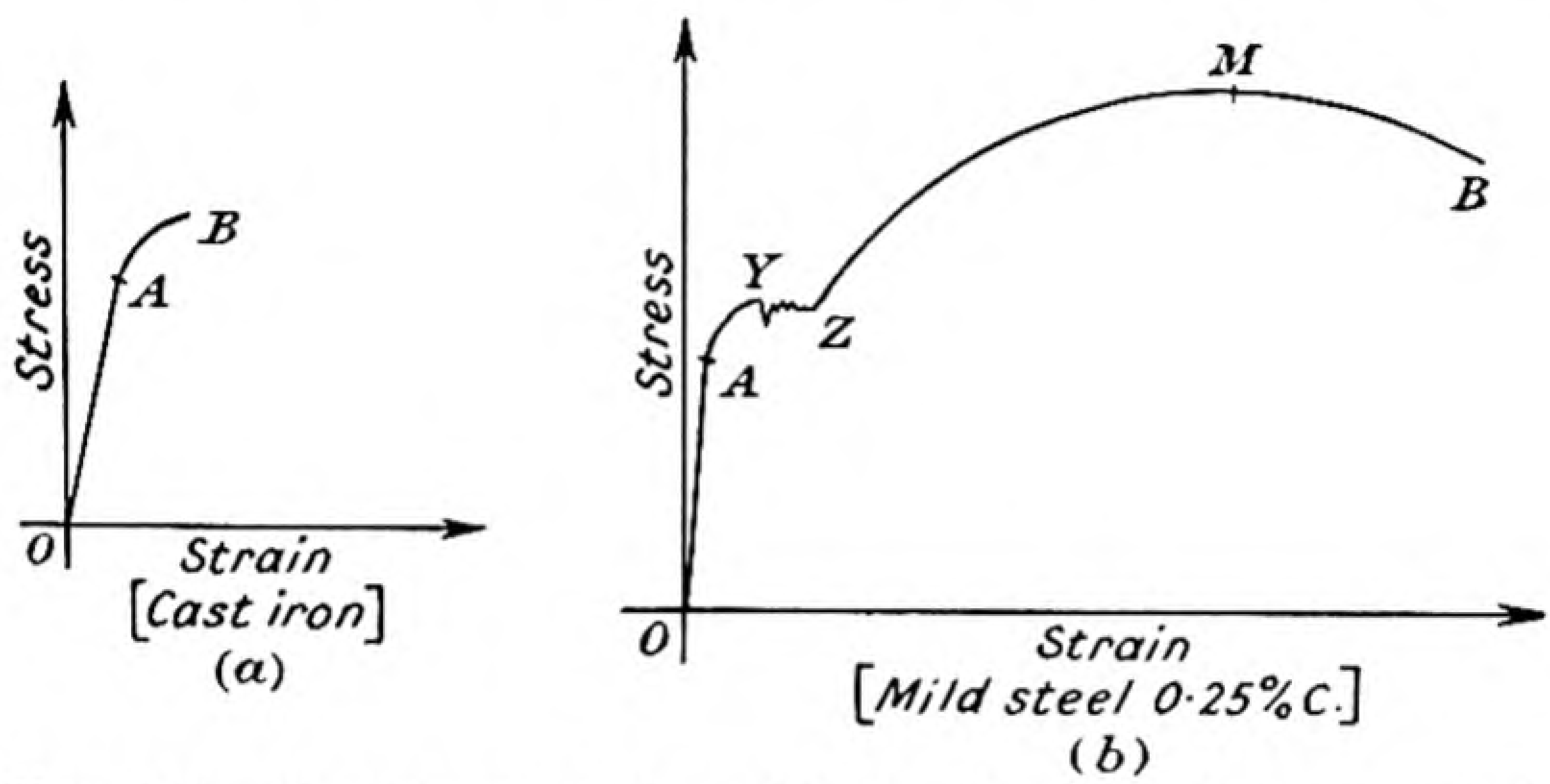


FIG. 7·09.—The behaviour of a solid when the applied stress exceeds the elastic limit.

during the elastic stage the stretch is caused by simple tension but in the plastic stage shear stress becomes predominant so that the stretch is mainly due to local shear taking place throughout all parts of the specimen. As the stress is increased the stretch proceeds steadily until the bar is about to fracture. Then a stage with marked instability sets in and the piece becomes considerably thinner at one point, i.e. the specimen exhibits a local constriction and a marked roughening of the hitherto smooth machined surface of the material appears. The specimen exhibits a phenomenon known as 'necking'. Immediately this occurs the stress decreases automatically and the portion MB of the curve is obtained; the break finally occurs at B. The stress corresponding to M is called the ***ultimate strength*** or ***tensile strength*** of the material under test. Thus the tensile strength of a material is the maximum force to which the test specimen may be subjected by slowly increasing the load, divided by the original cross-sectional area of the test specimen. Steel (0·18 per cent carbon) has a tensile strength of 20 ton.-wt.in.$^{-2}$, 0·32 per cent carbon steel has a tensile strength of 34 ton.-wt.in.$^{-2}$, while among timbers, British oak, 'that synonym for strength and durability', with a tensile strength of about 7 ton.-wt.in.$^{-2}$, stands supreme if certain foreign woods are excluded. Unfortunately oak

contains an acid which corrodes iron and steel. It is for this reason that copper rivets are used in the construction of wooden ships.

Substances such as certain manganese and nickel steels, cupro-alloys do not show a yield-point so that the transition from the elastic state to the plastic state is gradual.

In connexion with any given stress-strain diagram it has to be remembered that in the stage beyond the yield-point the greater part of the strain for a given stress increment occurs almost as soon as the stress is changed but there is a much smaller part of the strain which only appears with time, the stress being kept constant. EWING refers to this phenomenon as 'creeping'. It is therefore necessary to specify in a tensile test the rate at which the stress is applied for this influences the amount of non-elastic strain. It is found that beyond the yield-point (or the commercial elastic limit as it is sometimes called) the gradual reduction in the cross-section of the bar is directly proportional to the strain so that the volume of the specimen remains constant.

Actual stress-strain and nominal stress-strain diagrams.—Engineers find it a matter of convenience to specify the tensile strength of a material in terms of the original cross-section of the specimen. Moreover, self-recording machines always give load-strain diagrams and the mass of the load is only proportional to the stress if the cross-section of the specimen remains constant. If F is the stretching force, S the cross-sectional area when $F \to 0$, then the nominal stress $p = \frac{F}{S}$. If, however, S_0 is the actual cross-sectional area, $\frac{F}{S_0} = P$, the actual stress. For stresses below the elastic limit, $S - S_0 \to 0$, so that the difference between the nominal and the actual stresses is evanescent. In the case of ductile materials when stressed beyond the elastic limit, the difference between S and S_0 may be considerable. It is therefore necessary to distinguish between a nominal stress-strain diagram and an actual stress-strain diagram. Such diagrams for a mild steel are indicated in Fig. 7·10. The actual stress-strain diagram can only be constructed as far as M_1, the point corresponding to M; but the position of B_1, the last point on such a curve, may be calculated by dividing the breaking load by the section of the fracture at break. The actual stress at breaking is much greater than the nominal stress.

The cold-working of metals and alloys.—Suppose that a piece of ductile material has been strained so that its state is represented by a point W, Fig. 7·11, on a normal stress-strain diagram. If the stress is then reduced to zero and a tensile test carried out almost at once, it will be found that the range of proportionality between stress and strain has been reduced and that there is a new

yield-point at a stress somewhat higher than that previously applied to the specimen.

Again, when X is reached, let the stress be maintained constant for several hours; when the stress is increased a new yield point is found. The result of such mechanical treatment while the metal is cold is to give greater tensile strength to it and to reduce the ultimate

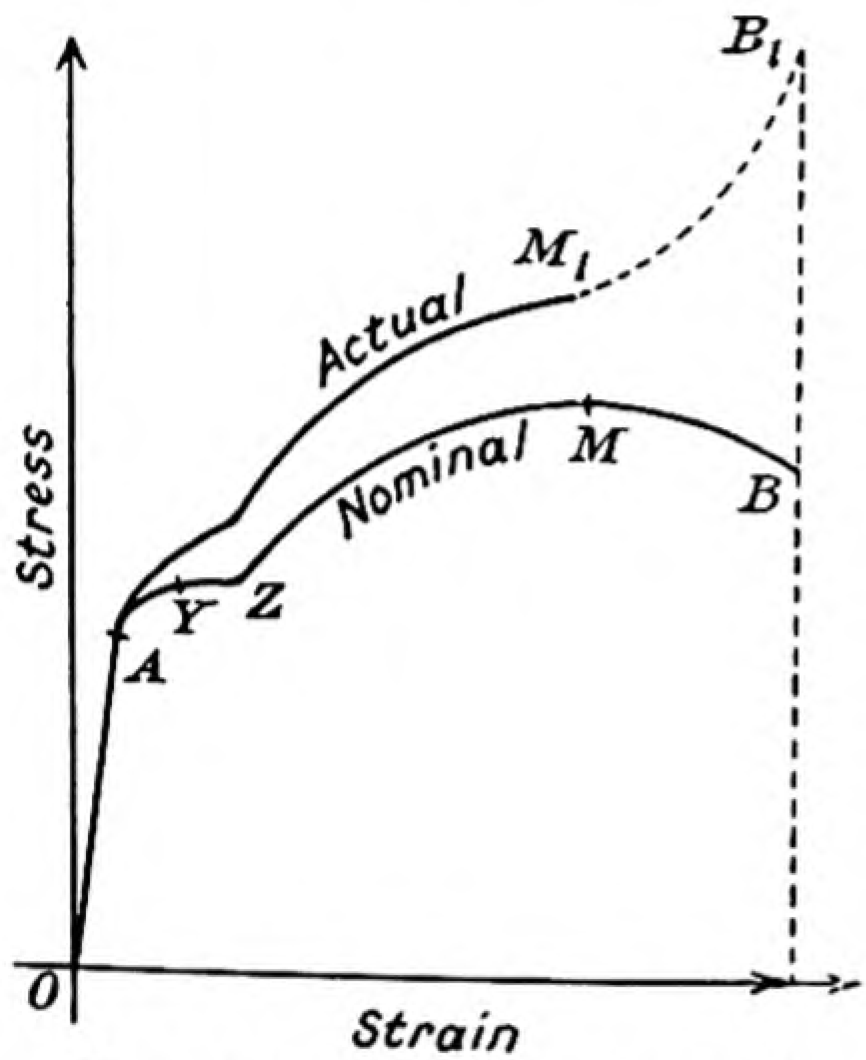


FIG. 7·10.—Nominal and actual stress-strain diagrams.

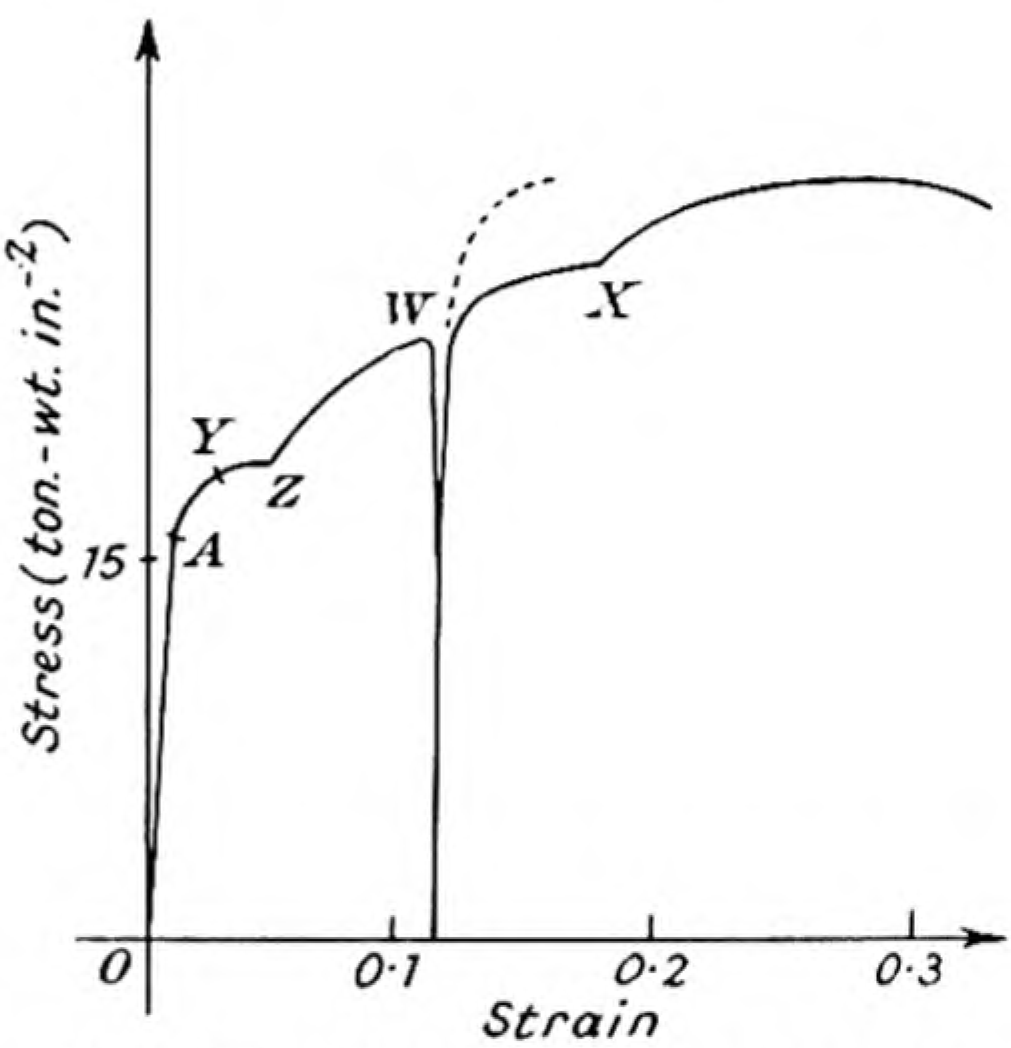


FIG. 7·11.—The cold-working of metals.

strain; the metal is said to have been cold worked. When a wire is drawn through a small hole in a so-called WURTEL plate it is cold-worked. Similar effects are produced by rolling it in a mill or by hammering it. In addition to the effect already noticed other effects of cold-working a metal are to lower its ductibility, to increase its hardness and to diminish its capacity for resisting mechanical shocks.

Suppose now that a specimen identical with that described above is taken to the point W and the stress removed and kept at zero for a considerable time. When a tensile stress is subsequently carried out the yield-point is raised considerably; thus a process of hardening has been going on during the interval of rest.

The hardening effect produced by cold-working may be removed by a process of annealing, that is of heating the material to a suitable temperature and then permitting it to cool slowly.

Tensile strength of materials. Factor of safety.—The stress causing a material to break measures the ultimate strength of that material for the particular type of stress used—for a metal rod stretched by a longitudinal pull, it is known as the ***tensile strength of the material.***

In designing an engineering structure care must be taken to see that the maximum stress in it shall not exceed a certain fraction of the ultimate strength of the material. This fraction is known as the ***working stress.*** The ratio

$$\frac{\text{ultimate strength}}{\text{working stress}}$$

is termed the ***factor of safety.*** In the British Commonwealth a factor of safety of 10 is usually allowed; in the U.S.A. it is often only 5.

Young's modulus.—Consider a wire subjected to a simple longitudinal pull, i.e. it is stretched by two equal and opposite forces F applied at its ends, the wire being free to contract in a direction perpendicular to that of the stretching forces. Let r be the radius of cross-section of the wire, l its original length, and Δl the increase in length. Then the longitudinal strain is $\frac{\Delta l}{l}$, the stress being $\frac{F}{\pi r^2}$. If the material has not been stretched beyond its limit of perfect elasticity,

$$\frac{F}{\pi r^2} = (\text{constant})\,\frac{\Delta l}{l},$$

by Hooke's law. The constant is termed ***Young's modulus,*** or the ***modulus of longitudinal extensibility*** for the material of the wire. If it is denoted by E,

$$\frac{F}{\pi r^2} = E.\frac{\Delta l}{l}.$$

Generally in laboratory determinations of E for the material of a wire, the force F at the lower end of the wire is the gravitational pull on a load of mass m carried by a suitable attachment fixed to the wire. Then $F = mg$, where g is the intensity of gravity, so that

$$E = \frac{mgl}{\pi r^2 \Delta l}.$$

E may be measured in dyne.cm.$^{-2}$, lb.-wt.in.$^{-2}$, ton.-wt.in.$^{-2}$, etc.

Searle's apparatus for determining Young's modulus for the material of a long wire.—Two wires of the same material are hung from the same rigid support, their lengths being about 2 metres. Each carries at its lower end a brass rectangular frame from the lower sides of which suitable loads may be supported. In Fig. 7·12, A and B are the wires while C and D represent an end-on view of these frames. E is one of two bars freely hinged to the frames so that one frame may be displaced relatively to the other. H is a metal strip,

carrying a spirit level S, and freely moving about a fulcrum M at one end. At the other end it rests upon the point N of a vertical screw R, operated by the divided head T. The pitch of the screw is 0·5 mm. and the periphery of T is divided into 50 equal divisions. When the head T is rotated through one division the screw moves 0·01 mm.

A load of mass 1 kgm. is carried by each wire so that they shall be straight and the reading of the screw observed when one end of the air bubble is at the centre of the level. [This permits the position of the bubble to be adjusted more precisely than if it is attempted to adjust the bubble to a central position.] The load on one wire is then increased by 1 kgm. so that the wire is stretched and the air bubble displaced. By rotating the screw this bubble may be brought back to its standard position. The amount by which the point of the screw is moved is equal to the extension of the wire. The load is then increased in stages up to a maximum, removed 1 kgm. at a time, and readings of the screw taken for each load. A graphical or other method is then used to determine the mean extension for an increase in the mass of the load of 1 kgm. If this fraction is θ, a value for Young's modulus for the material of the wire may be obtained from the formula [cf. p. 276]

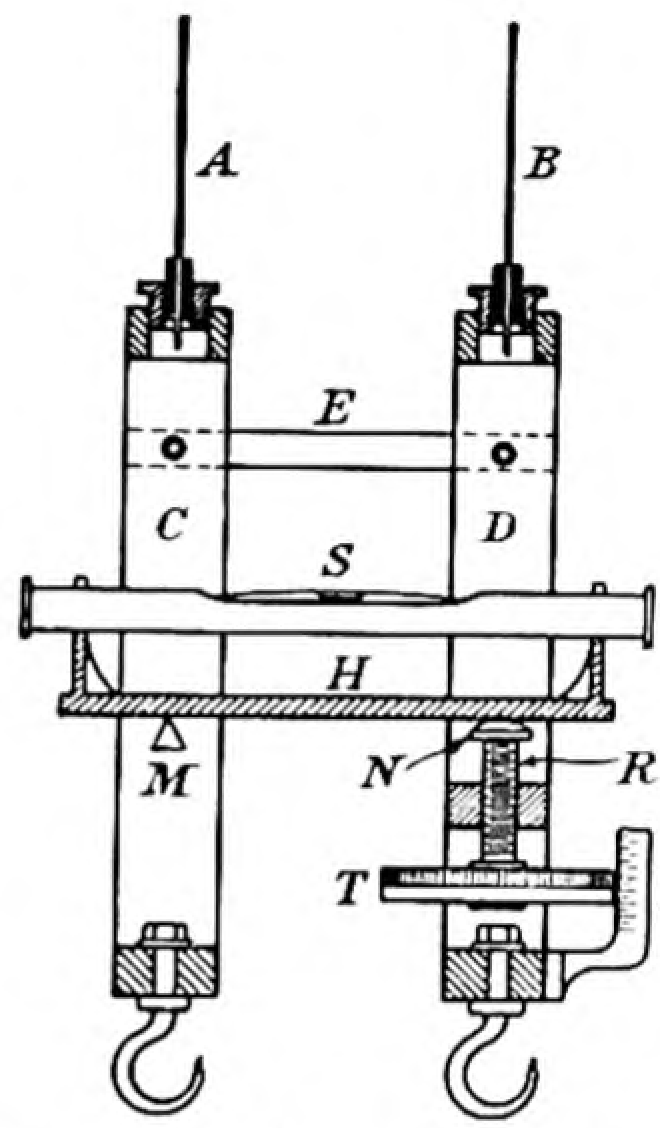


FIG. 7·12.—Searle's apparatus for investigating the stretching of wires.

$$E = \frac{gl}{\pi r^2}\left(\frac{1}{\theta}\right) \times 10^3.$$

Ewing's extensometer.—To determine the extension of a short bar of material subjected to a tensile stress, EWING devised the extensometer shown diagrammatically in Fig. 7·13. By its means the variation in length of the specimen can be watched continuously and it may be used in either vertical or horizontal testing machines.

The apparatus is clamped to the test piece A by two pairs of set screws, S_1 and S_2, attached to the clamping pieces B and C. The points of the screws are adjusted to a definite distance apart [10 cm. or 20 cm.] so that this distance determines the initial length of the specimen whose extension is to be measured. D is an upright clamp projecting from the lower clamp C and ending in a rounded point which engages with a conical hole in a screw attached to the upper

clamp B, thus forming a fulcrum about which the clamp B rotates when an extension of the test piece takes place. A point Q, equally distant from the axis of the test piece and on the opposite side of the clamp B, moves relatively to C through a distance equal to twice the extension of the test piece. This movement is measured by means of a microscope M, fixed in line with C and focused upon a mark on a rod pivoted at Q. This mark is a fine line ruled on a glass plate set in an aperture in the rod; it is illuminated by means of the light reflected from a small mirror. The microscope M is focused on the image of the line and any displacement of this image is measured directly on a micrometer scale in the eye-piece of the microscope. Extensions of the order $2{\cdot}5 \times 10^{-5}$ cm. are observable. The conical pocket for the fulcrum P is formed on the end of a micrometer screw L which may be adjusted to bring the image of the mark on the rod R to a convenient point on the eye-piece scale in M; it also enables the micrometer scale in the eye-piece to be calibrated. When making observations on the behaviour of a specimen after the elastic limit has been passed, the movement may be so large as to carry the image of the sighting mark out of the field of view of the microscope. The screw enables the image to be brought back on to the scale and the observations continued.

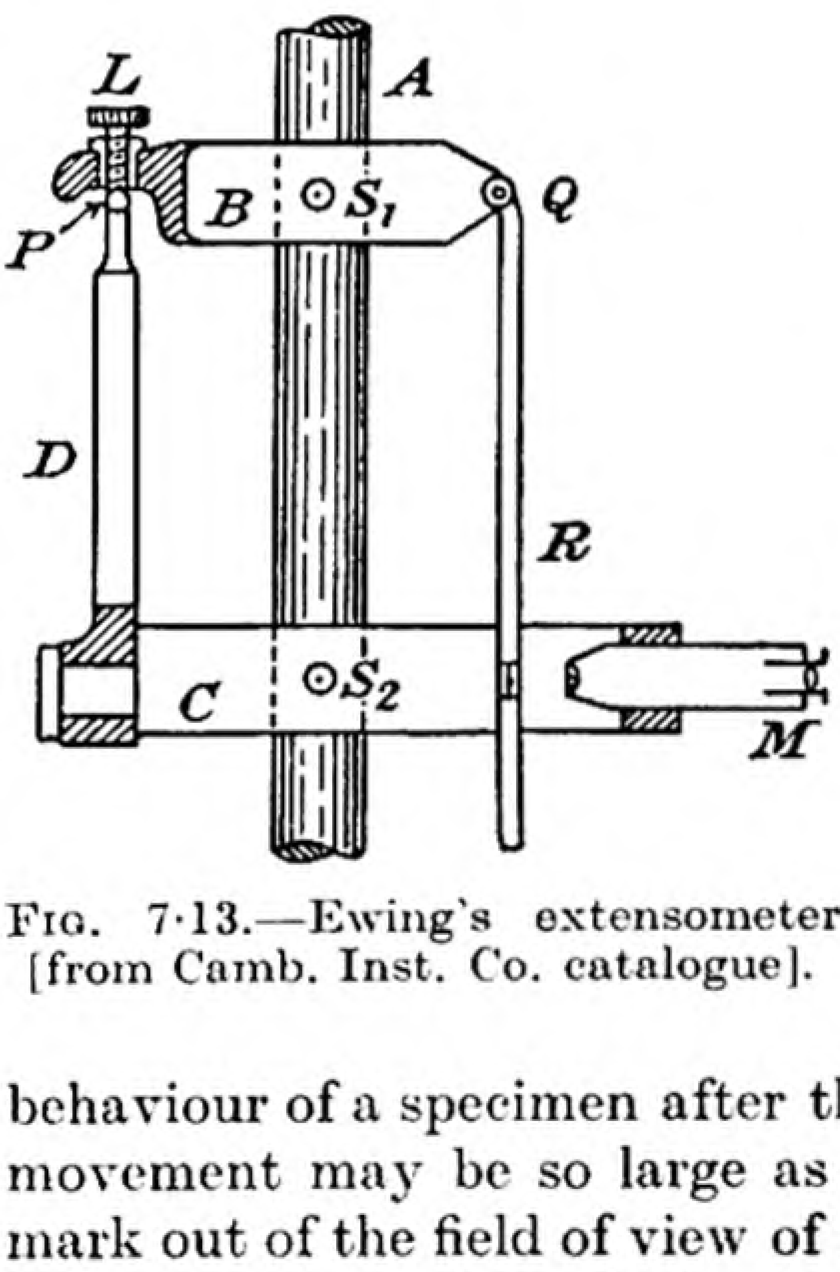


FIG. 7·13.—Ewing's extensometer [from Camb. Inst. Co. catalogue].

Determination of Young's modulus for the material of a wire by Gravesande's method.—For this experiment a nickel wire, diameter 0·056 cm., and nearly a metre long is suitable. It is held taut between two clamps, the wire itself being horizontal—cf. Fig. 7·14. At the centre of the wire there is suspended a pan, or hook, so that the wire may be loaded. Let $2l$ be the initial length of

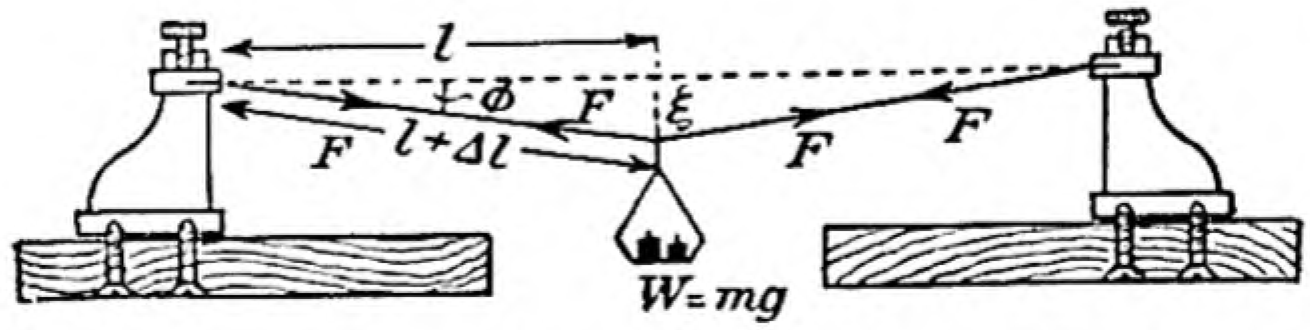


FIG. 7·14.—Gravesande's method for determining Young's modulus for the material of a wire.

the wire and suppose that the depression at the centre is ξ, when the weight suspended is W. Let F_0 be the initial tension in the wire, and suppose this is increased to F when the wire is loaded: let $l + \Delta l$ be the new semi-length of the wire. Let ϕ be the inclination (small) of each half of the wire to the horizontal, and E be Young's modulus for the material of the wire. Then

$$W = 2F \sin \phi, \qquad \text{and} \qquad E.\frac{\Delta l}{l} = \frac{F - F_0}{A},$$

where A is the cross-sectional area of the wire.

Now $$\sec \phi = 1 + \frac{\Delta l}{l} = 1 + \frac{F - F_0}{EA}.$$

$$\therefore\ F = F_0 + EA(\sec \phi - 1),$$

and $$W = 2 \sin \phi[F_0 + EA(\sec \phi - 1)].$$

But $$\sec \phi = (1 + \tan^2 \phi)^{\frac{1}{2}} \simeq 1 + \frac{1}{2}.\frac{\xi^2}{l^2},$$

and $$\sin \phi = \frac{\xi}{(\xi^2 + l^2)^{\frac{1}{2}}} \simeq \frac{\xi}{l}\left[1 - \frac{1}{2}.\frac{\xi^2}{l^2}\right].$$

$$\therefore\ W = \frac{2\xi}{l}\left(1 - \frac{\xi^2}{2l^2}\right)\left(F_0 + EA\,\frac{\xi^2}{2l^2}\right).$$

$$\therefore\ \frac{Wl}{2\xi} = \frac{EA\xi^2}{2l^2} + F_0\left(1 - \frac{\xi^2}{2l^2}\right),$$

neglecting $\frac{\xi^4}{l^4}$ as in the expressions for $\sec \phi$ and $\sin \phi$.

$$\therefore\ \frac{W}{\xi} = \xi^2\left(\frac{EA}{l^3} - \frac{F_0}{l^3}\right) + \frac{2F_0}{l}.$$

Put $\xi^2 = x$ and $\frac{W}{\xi} = y$, and then the above equation becomes

$$y = x\left(\frac{EA}{l^3} - \frac{F_0}{l^3}\right) + \frac{2F_0}{l}.$$

This is the equation to a straight line whose slope is $\left(\frac{EA - F_0}{l^3}\right)$ and which makes an intercept $\frac{2F_0}{l}$ on the y-axis. Thus F_0 and E may be determined.

This method for determining E for the material of a wire was suggested by Gravesande in the early eighteenth century but the difficulty of determining the initial tension always militated against its use. When the graphical method, first suggested by Ferguson,*

* *Sci. Prog.*, **22**, 461, 1928.

is used both E and F_0 may be determined. The method is then a reliable one.

The formula just obtained in connexion with Gravesande's method may be derived in a more elementary way as follows. Let A and B, Fig. 7·15(a), be the points to which the ends of a horizontal wire of length $2l_0$ are fixed; let O be the mid-point of AB and suppose that when the wire is loaded centrally with a weight W the mid-point of

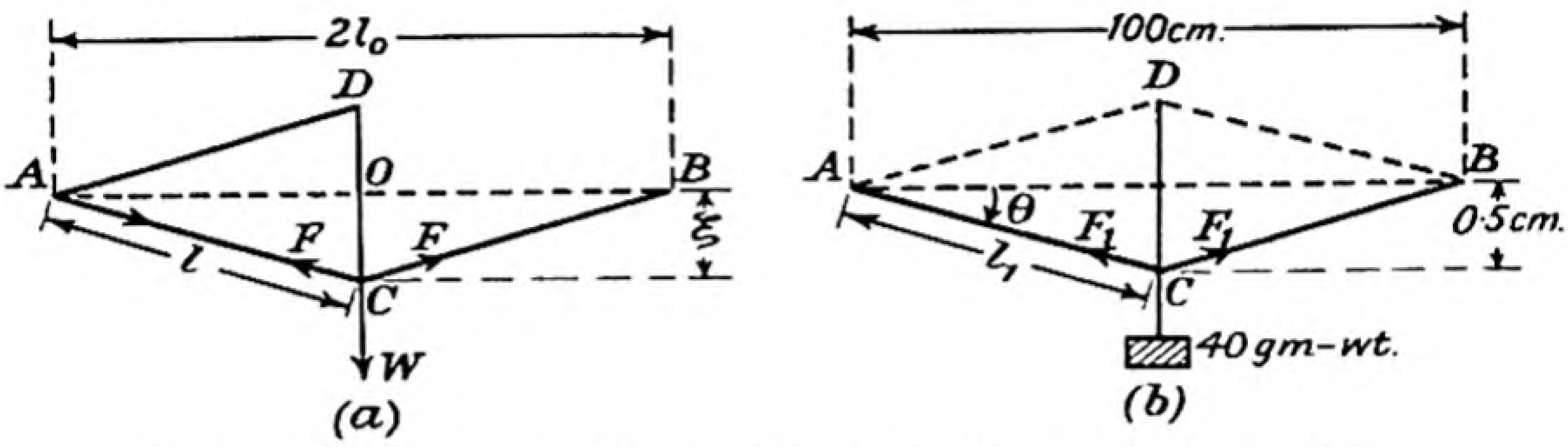


FIG. 7·15.—Gravesande's method for finding Young's modulus.

the wire descends to C, where OC $= \xi$. Let AC $= l$. If we make CD $= 2.$OC, then ΔACD is a triangle of forces giving the conditions that C shall be in equilibrium; we have

$$\frac{F}{l} = \frac{W}{2\xi}, \quad \text{or} \quad F = \tfrac{1}{2}W\,\frac{l}{\xi}.$$

Also
$$l = \sqrt{l_0^{\,2} + \xi^2} \doteqdot l_0\left(1 + \frac{\xi^2}{2l_0^{\,2}}\right).$$

$$\therefore\ l - l_0 = \frac{\xi^2}{2l_0}.$$

Now to calculate a value for Young's modulus, E, for the material of the wire, it is necessary to know the change in the stretching force, i.e. $F - F_0$, where F_0 is the initial tension in the wire. Unfortunately F_0 is unknown, but if the wire is loaded in turn with weights W_1 and W_2, and the corresponding depressions ξ_1 and ξ_2 observed, we have

$$F_1 = \tfrac{1}{2}W_1\,\frac{l_1}{\xi_1} \qquad F_2 = \tfrac{1}{2}W_2\,\frac{l_2}{\xi_2}$$

and
$$l_1 - l_0 = \frac{\xi_1^{\,2}}{2l_0} \qquad l_2 - l_0 = \frac{\xi_2^{\,2}}{2l_0}.$$

If A is the cross-sectional area of the wire

$$E = \frac{F_2 - F_1}{A} \div \frac{l_2 - l_1}{l_0}$$

$$\simeq \left[\left(\frac{W_2}{\xi_2} - \frac{W_1}{\xi_1}\right)\frac{1}{A}.l_0^{\,3}\right] \div [\xi_2^{\,2} - \xi_1^{\,2}].$$

Or, to use a graphical method, we may write

$$\mathrm{E} = \frac{\mathrm{F} - \mathrm{F}_0}{\mathrm{A}} \div \frac{l - l_0}{l_0}$$

$$= \frac{1}{\mathrm{A}}\left(\frac{l}{2\xi}\mathrm{W} - \mathrm{F}_0\right) \div \frac{\xi^2}{2l_0^{\,2}}.$$

$$\therefore\ \mathrm{EA}\xi^2 = l_0^{\,2}\left(\frac{l}{\xi}\,\mathrm{W} - 2\mathrm{F}_0\right),$$

or

$$\frac{\mathrm{W}}{\xi} \doteqdot \frac{\mathrm{EA}}{l_0^{\,3}}\,\xi^2 + \frac{2\mathrm{F}_0}{l_0}. \qquad \left[\because\ \frac{l}{l_0} \to 1\right]$$

Example.—A horizontal steel wire 1·0 metre long and 1·00 mm.^{2} in cross-section is stretched between two fixed supports and then loaded at its mid-point. A load of mass 40 gm. produces a sag of 5·0 mm. and a load of 600 gm. a sag of 15 mm. Deduce a value for Young's modulus for steel.

Let l_1 be the semi-length of the wire and F_1 the tension in it when the load has a mass of 40 gm. If CD, Fig. 7·15(b) is twice the sag, we have ACBD as a force parallelogram and from it obtain $\frac{\mathrm{F}_1}{l_1} = \frac{40}{1}$, or $\mathrm{F}_1 = 40 \times 50 = 2000$ gm.-wt. [$l_1 \doteqdot 50$ cm.]

If l_2 is the semi-length of the wire and F_2 the tension in it when the load of mass 600 gm. is used, we have, in similar way, $\mathrm{F}_2 = 10{,}000$ gm.-wt.

$$\text{Now } l_1 = \sqrt{50^2 + 0{\cdot}5^2} = 50\sqrt{1 + \frac{1}{10^4}} = 50\left[1 + \frac{5}{10^5}\right],$$

$$\text{and } l_2 = \sqrt{50^2 + 1{\cdot}5^2} = 50\sqrt{1 + \frac{2{\cdot}25 \times 4}{10^4}} = 50\left[1 + \frac{45}{10^5}\right].$$

$$\text{Change in strain} = \frac{l_2 - l_1}{l_1} \doteqdot \frac{l_2 - l_1}{50}.$$

$$\mathrm{E} = \frac{\text{change in stress}}{\text{change in strain}} = \left(\frac{8000 \times 980}{0{\cdot}010}\right) \div \left(\frac{50}{50} \cdot \frac{40}{10^5}\right)$$

$$= 1{\cdot}96 \times 10^{12}\ \text{dyne.cm.}^{-2}$$

The extension of a uniform rod hanging vertically under its own weight.—Let S be the cross-sectional area of the rod and l its unstretched length. When the rod is hanging vertically downwards, as shown in Fig. 7·16(a), let the coordinates at the side of the rod indicate the distances of horizontal sections in the rod from its uppermost end when the rod is not stretched. Consider, therefore, the mean stress p across a horizontal portion at A; its position is defined by x. Then since the force on the lower portion of the rod which its upper part exerts on it across the horizontal plane at A must balance the weight of the portion of the rod below that plane,

$$p\mathrm{S} = g\rho\mathrm{S}(l - x),$$

where g is gravity and ρ the density of the material of the rod.

Let ξ, Fig. 7·16(*b*), be the mean displacement downwards of all particles in the plane considered. Then

$$\xi = f(x),$$

and the displacement of particles in the plane $x + \delta x$ will be

$$\xi + \frac{\partial \xi}{\partial x}\,\delta x.$$

In Fig. 7·16(*b*) OA $= x$, AB $= \delta x$; when the rod extends under its

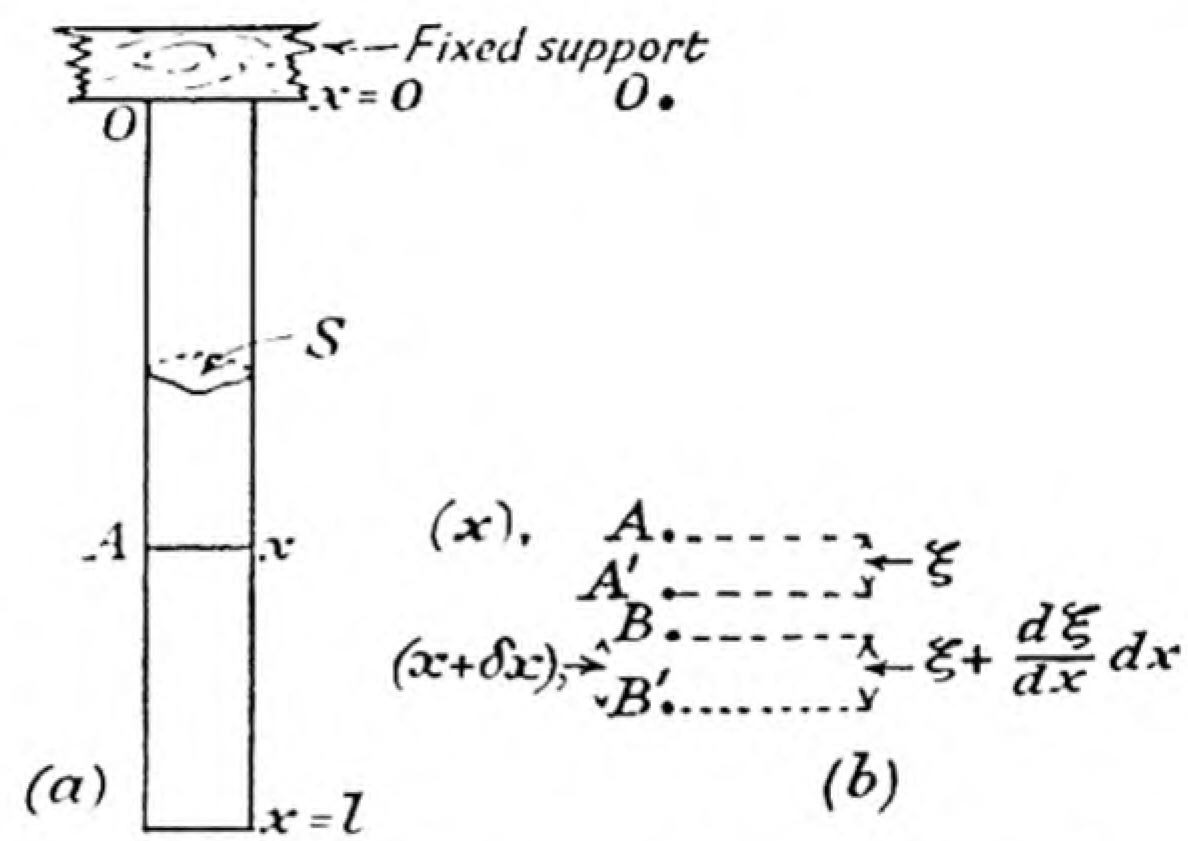


Fig. 7·16.—A rod stretched by its own weight.

own weight A moves to A′. i.e. ξ = AA′. At the same time B moves to B′ where BB′ $= \xi + \dfrac{\partial \xi}{\partial x}\,\delta x$. Hence the element AB becomes one of length A′B′, where

$$\xi + \mathrm{A'B'} = \delta x + \left(\xi + \frac{\partial \xi}{\partial x}\,\delta x\right).$$

$$\therefore\ \mathrm{A'B'} = \left(1 + \frac{\partial \xi}{\partial x}\right)\delta x.$$

Hence the extensional strain in the element considered is

$$\frac{\mathrm{A'B'} - \mathrm{AB}}{\mathrm{AB}} = \frac{\partial \xi}{\partial x} = \frac{d\xi}{dx}.$$

If E is Young's modulus for the material of the beam

$$\mathrm{E}\,\frac{d\xi}{dx} = p = g\rho(l - x).$$

$$\therefore\ \xi = \frac{g\rho}{\mathrm{E}}\,(lx - \tfrac{1}{2}x^2),$$

the integration constant being zero since at $x = 0$, $\xi = 0$. It follows at once that the increment in the length of the rod due to its own weight is

$$[\xi]_{x=l} = \frac{g\rho}{\mathrm{E}}(l^2 - \tfrac{1}{2}l^2) = \frac{1}{2}\frac{g\rho l^2}{\mathrm{E}}.$$

Poisson's ratio.—When a wire at constant temperature is under the action of two equal but oppositely directed forces a tensile strain is produced in the wire. The ratio of the increase in length, Δl, to the original length, l, is known as the tensile or longitudinal strain. Under such conditions careful measurements show that the increase in length of the wire is accompanied by a diminution in the diameter of the wire. The ratio of the decrease in the diameter of a stretched wire to its initial diameter is termed the lateral strain. For a material at constant temperature and not stressed beyond its elastic limit, the ratio of the lateral strain to the longitudinal strain is a constant known as Poisson's ratio, σ.

Thus if r and $(r + \Delta r)$ are the radii of the wire before and after straining, the lateral strain is $-\dfrac{\Delta r}{r}$.

$$\therefore\ \sigma = \left(-\frac{\Delta r}{r}\right) \div \left(\frac{\Delta l}{l}\right).$$

Later on, cf. pp. 314, 359, 375, 382 and 403, experimental methods for determining Poisson's ratio for a given material will be described.

The change in volume accompanying longitudinal and lateral strain.—Let l_0, r_0 and V_0 be the length, radius of cross-section and volume of a wire, Fig. 7·17, before it is subjected to a pair

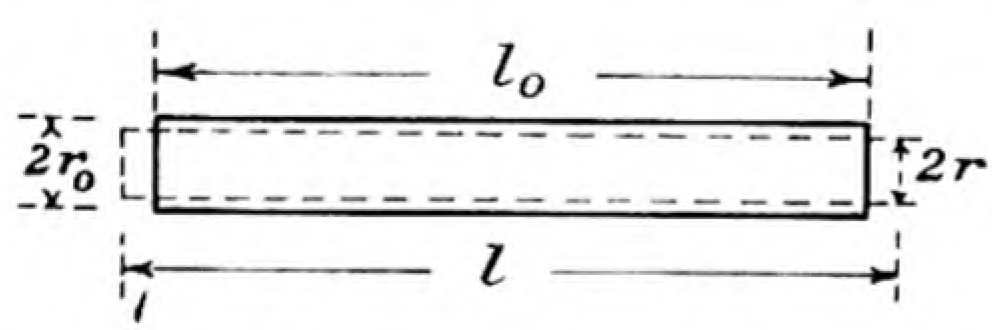


FIG. 7·17.—Change in volume accompanying longitudinal and lateral strain.

of stretching forces. Let l, r, and V be the corresponding values of the length, radius and volume when the wire has been stretched; it is assumed that the wire is free to contract in all directions normal to the axis of stretch and that the limit of perfect elasticity has not been exceeded. Then the fractional change in volume is

$$\frac{\mathrm{V} - \mathrm{V}_0}{\mathrm{V}_0} = \frac{r^2 l - r_0^2 l_0}{r_0^2 l_0}.$$

The longitudinal strain, ϵ, is $\frac{l - l_0}{l_0}$, while if σ is Poisson's ratio for the material of the wire,

$$\frac{r_0 - r}{r_0} = \sigma\left(\frac{l - l_0}{l_0}\right) = \sigma\epsilon.$$

Hence
$$\frac{r_0}{r} = (1 + \sigma\epsilon).$$

$$\therefore \quad \frac{V - V_0}{V_0} = \left[\frac{1}{(1 + \sigma\epsilon)^2}.(1 + \epsilon)\right] - 1,$$

and it will be observed that this strain is always positive except that it is zero if $\sigma = 0{\cdot}5$.

If $\sigma = 0{\cdot}25$, a typical value for many materials, we have

$$\left[\frac{V - V_0}{V_0}\right]_{\sigma = 0\cdot25} = \left(1 - \frac{\epsilon}{2} + \frac{3}{16}\epsilon^2\right)(1 + \epsilon) - 1$$
$$= \tfrac{1}{2}\epsilon - \tfrac{5}{16}\epsilon^2,$$

if terms in ϵ^3 etc. are neglected. Again, if ϵ^2 is neglected, the volume strain in this particular instance is $\frac{1}{2}\epsilon$, viz. one-half the longitudinal strain.

Example.—When an indiarubber cord is stretched the change in volume is almost negligible compared with the change in shape. Obtain a value for Poisson's ratio for indiarubber.

Let l and r be the initial length and radius of the cord which is assumed to be circular in cross-section. When the cord is stretched, but not beyond the elastic limit, let the length become $l(1 + \alpha)$ and the radius $r(1 - \beta)$. Since the volume of the cord is unchanged

$$(1 + \alpha)(1 - \beta)^2 = 1,$$

i.e.
$$(1 + \alpha)(1 - 2\beta) = 1, \quad \text{since } \beta^2 \to 0.$$

But Poisson's ratio is $(\beta \div \alpha)$, which from the above is $0{\cdot}5$.

The modulus of rigidity.—When the state of strain is one due to a simple shear it has been shown that the distortion in the material is such that an element, whose section was originally a square, becomes lengthened along one diagonal and shortened to an equal extent along the other. If the new section is rotated so that two sides are parallel to two original sides of the cube, then the angle between the other pairs of sides in their new and old positions is ϕ, the angle of shear [cf. Fig. 7·02(a), p. 264]. Since Hooke's law is valid if the elastic limit for this type of strain is not exceeded, we have, if p is the shearing stress,

$$p = n\phi,$$

where n is a constant for the particular material. This constant

is termed the modulus of rigidity for the material. [For metals it is approximately 0·4E, where E is Young's modulus for the particular metal.]

Theory of a twisted wire, or circular cylinder, or shaft.—The following theoretical discussion is necessary before the method of determining experimentally the modulus of rigidity of a material in

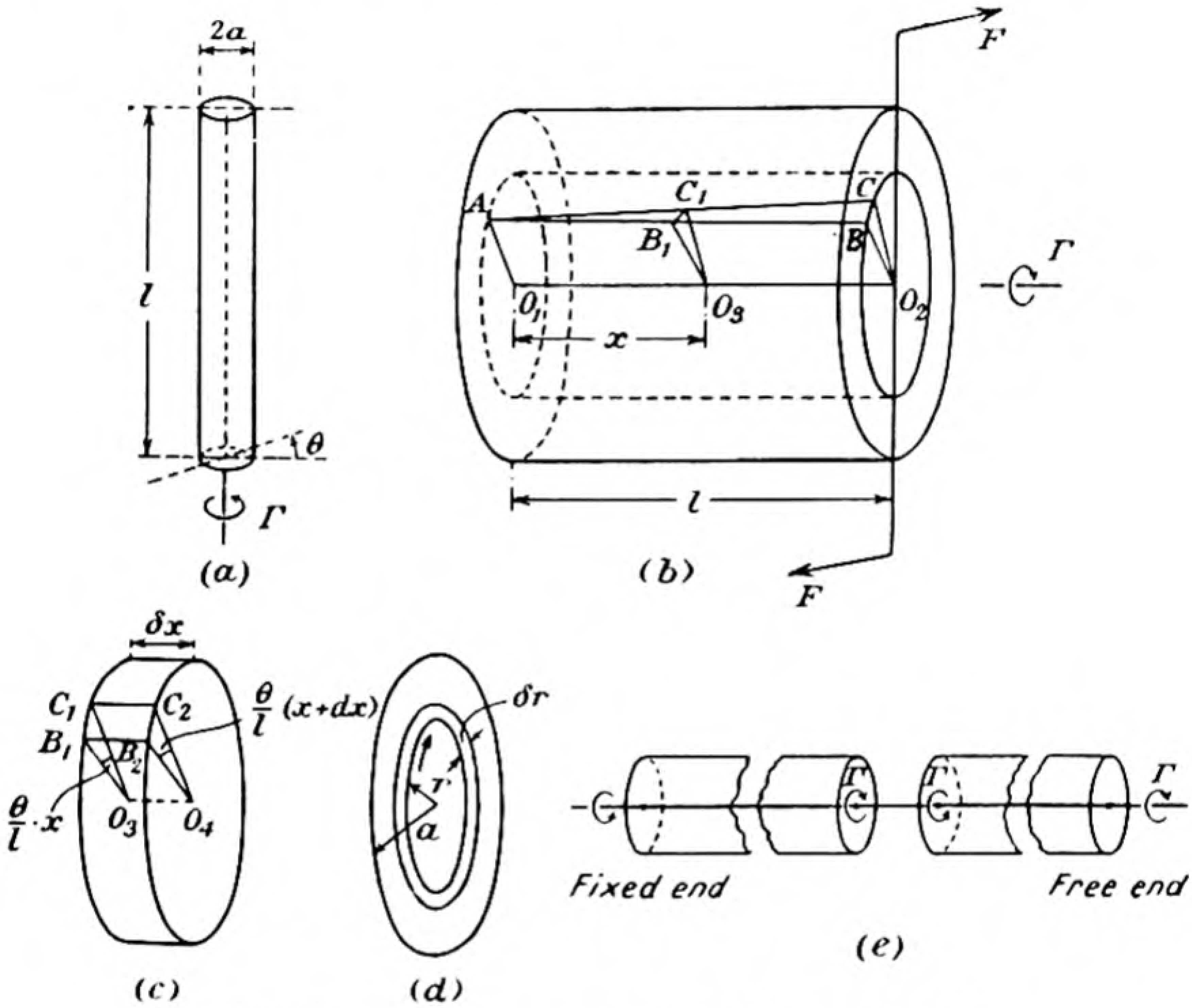


FIG. 7·18.—Theory of a twisted wire or cylinder.

the form of a wire can be fully appreciated. Let l be the length of a uniform wire [a right-circular cylinder] fixed at its upper end, cf. Fig. 7·18(*a*), and twisted at its lower end by a horizontal couple Γ through an angle θ. The material is then in a state of pure shear; there could be no change in the length of the wire or in its radius of cross-section a, for if there were the sign of the change would be reversed with that of the couple, and the action of the cylinder under the influence of the couple is essentially independent of the direction of the couple.

Let O_1O_2, Fig. 7·18(*b*), be the axis of the wire of radius a. Imagine that from this there is cut a cylinder of radius r, the axes of the two cylinders coinciding. Let AB be the unstrained position of a straight line parallel to O_1O_2 and on the surface of this cylinder of

radius r. When the couple is applied let B move to C, where $B\widehat{O_2}C = \theta$, the angle of twist at the end of the wire which is free. Now for B_1, a point in AB at a distance x from A, the shift will be B_1C_1, where

$$\frac{B_1C_1}{x} = \frac{BC}{l} = \frac{r\theta}{l}.$$

Hence the angle of twist at $B_1 = \dfrac{B_1C_1}{r} = \dfrac{\theta}{l}.x = \theta\left(\dfrac{x}{l}\right)$.

Consider now a slice of the material perpendicular to the axis of the cylinder and passing through B_1, the thickness of the slice being δx, where, cf. Fig. 7·18(c), $\delta x = B_1B_2 = C_1C_2$. Then the angle of twist at B_2 is $\dfrac{\theta}{l}(x + \delta x)$, i.e. the relative twist between the flat faces of the slice is $\dfrac{\theta}{l}.\delta x$.

$$\therefore \text{ Angle of shear} = \frac{(\text{relative twist}) \times \text{radius}}{\text{distance between plane faces}} = \frac{r\theta}{l}.$$

Hence the shearing stress, p, is given by

$$p = n.\frac{r\theta}{l},$$

where n is the torsional rigidity of the material of the wire.

Now imagine that the wire is cut into two portions by a plane through O_3, the projection of B_1 on the axis, and perpendicular to the axis O_1O_2. Then the portion to the left of O_3 must be acted upon by internal forces of such moment about the axis that they balance the couple Γ at the fixed end of the wire. To calculate this couple consider a small element of length δs on the ring on the end surface of the left hand portion defined by the concentric circles of radii r and $(r + \delta r)$. Its area is $\delta s.\delta r$, so that the force acting on it is $p.\delta s.\delta r$, and the moment of this force about the axis of the wire is $pr.\delta s.\delta r$. Hence the stress on the whole elementary ring considered gives rise to a torque of moment $\delta\Gamma$, where

$$\delta\Gamma = 2\pi r^2 p.\delta r. \qquad [\because \oint ds = 2\pi r].$$

$$= \frac{2\pi n\theta}{l}.r^3\,\delta r.$$

$$\therefore\ \Gamma = \frac{2\pi n\theta}{l}\int_0^a r^3.dr = \frac{\pi n\theta}{2l}.a^4,$$

so that

$$\theta = \frac{2l\Gamma}{\pi n a^4}.$$

The couple required to produce unit twist (i.e. one radian) in a given wire is termed the ***torsional constant*** of the wire and may be denoted by b; this is a constant involved in the theory of moving-coil galvanometers, cf. Vol. V., p. 342; its value is $\pi n a^4 \div 2l$.

Alternative method.—The relation between the twist produced in a wire by an external couple and the dimensions and rigidity of the material of the wire may be obtained in the following way, which is more elementary but not so instructive as that just given.

When equal and opposite couples are applied at the ends of a wire in planes normal to the axis of the wire, the wire is twisted, the particles in each layer undergoing slight relative displacements with respect to their neighbours in a continuous layer. The wire is in a state of shear and the relation between the angle of twist and the applied couple in terms of the dimensions of the wire and the rigidity of its material may be obtained as follows.

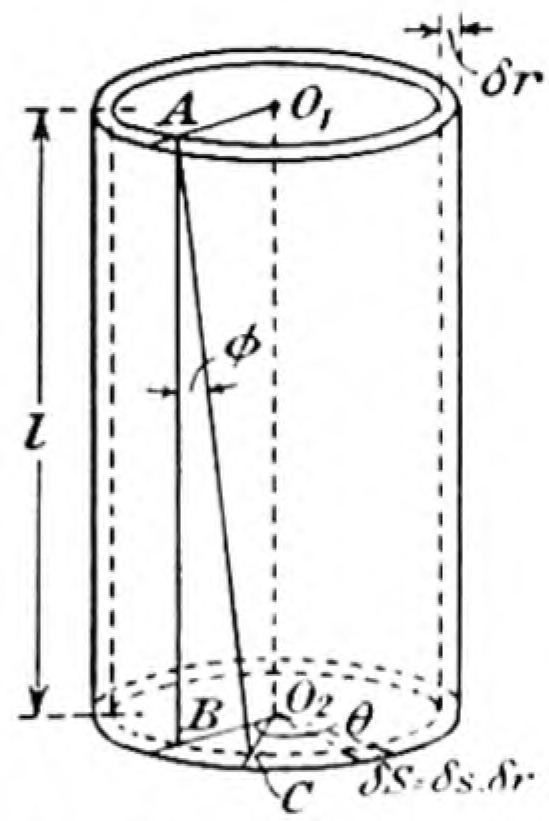


FIG. 7·19.—Theory of a twisted wire (alternative method).

Suppose that the wire is fixed at its upper end and that Γ is the applied couple at its lower end—cf. Fig. 7·19. Consider a thin cylindrical tube of length l, internal radius r and thickness δr coaxial with and forming part of the wire. Let a generating line AB on the inner surface of this thin cylinder be displaced into the position AC when the wire is twisted. If θ and hence ϕ, the angles of twist and of shear respectively, are small, we have $l\phi = r\theta$.

Let p be the tangential stress, i.e. the tangential force per unit area acting on the base of the tube. Then the force on a small element of length δs and in the base of the tube is

$$p\, \delta s \, . \, \delta r,$$

since $\delta s \, . \, \delta r$ is the area of the element. The moment of this force about the axis of the tube is

$$pr\, \delta s \, . \, \delta r = n\phi r\, \delta s \, . \, \delta r,$$

since $p = n\phi$, where n is the rigidity of the material of the tube. Hence the stress on the base of the tube gives rise to a couple $\delta\Gamma$, where

$$\delta\Gamma = 2\pi n\phi r^2 . \delta r, \qquad [\because \oint ds = 2\pi r.]$$

$$= \frac{2\pi n\theta r^3\, \delta r}{l}.$$

Now the couple Γ required to twist the wire of radius a and length l is the sum of the elementary couples which are necessary to twist all the elementary coaxial cylindrical tubes into which the wire may be considered to be divided. Thus

$$\Gamma = \int_0^a \frac{2\pi n\theta r^3\,dr}{l} = \frac{\pi n a^4}{2l}\,\theta,$$

as determined previously in a somewhat more detailed manner.

Variation of the state of stress in a twisted rod.—Let A_0B_0, Fig. 7·20(a) be a generator on the surface of the wire of radius a before twisting occurs. Let AB be the corresponding generator on a

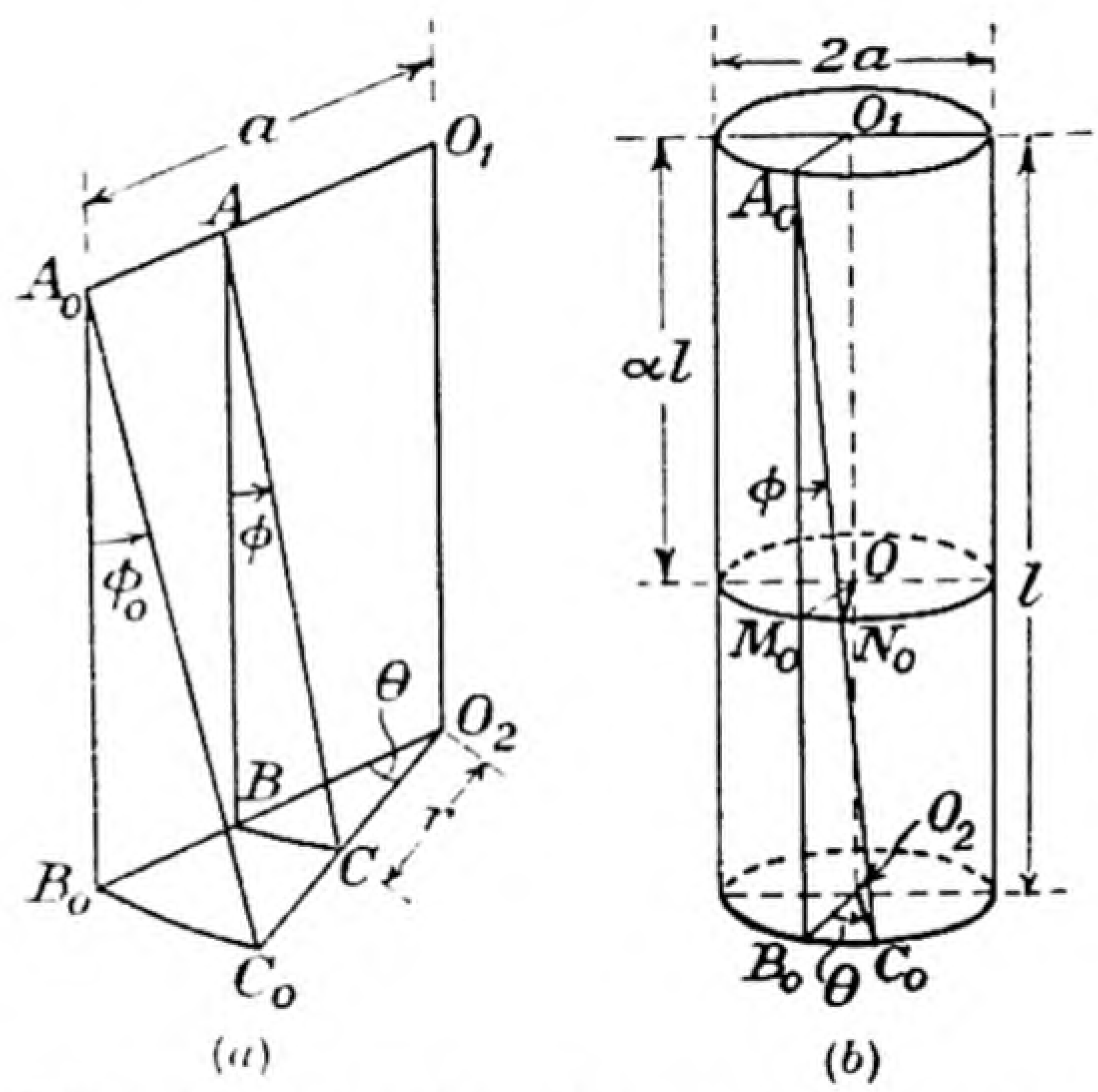


FIG. 7·20.—Variation of the state of stress in a twisted wire.

cylindrical surface of radius r and within the wire; then AB lies in the plane $A_0O_1O_2B_0$. When twisting has occurred let A_0B_0 move to A_0C_0 and AB to AC; moreover, let θ be the angle of twist at the lower end of the wire. Then

$$B_0C_0 = a\theta, \qquad \text{and} \qquad BC = r\theta.$$

Also, if ϕ_0 and ϕ are the angles indicated

$$\phi_0 = \frac{B_0C_0}{l} = \frac{a\theta}{l} = \frac{a}{l}\cdot\frac{BC}{r} = \frac{a}{r}\,\phi,$$

or

$$\phi = \frac{r}{a}\,\phi_0.$$

But the ϕ's are angles of shear, so that the stress p, at elements distance r from O_1O_2 is given by

$$p = n\phi = n \cdot \frac{r}{a}\phi_0 = \frac{r}{a}p_0,$$

where p_0 is the maximum stress in the wire.

Now let us consider the stresses in a plane M_0ON_0, Fig. 7·20(b) where $O_1O = \alpha l$, where $0 < \alpha < 1$. The two radii OM_0 and ON_0 are inclined to each other at an angle $\alpha\theta$, i.e. the shearing strain at points on the surface of the wire in the plane M_0ON_0 is given by

$$\frac{M_0N_0}{A_0M_0} = \frac{a \cdot \alpha\theta}{\alpha l} = \phi_0,$$

the strain at B_0C_0. Hence the shearing stress at a point depends only on r, its distance from the axis of the wire and not upon its vertical distance from the free end of the wire; its maximum value is p_0, the shear stress at the surface of the wire.

Determination of the rigidity of the material of a uniform wire by a statical method.—The wire W, Fig. 7·21(a), which should be from 2 mm. to 4 mm. in diameter and about 50 cm. long, is soldered into two small rectangular blocks of brass which are then

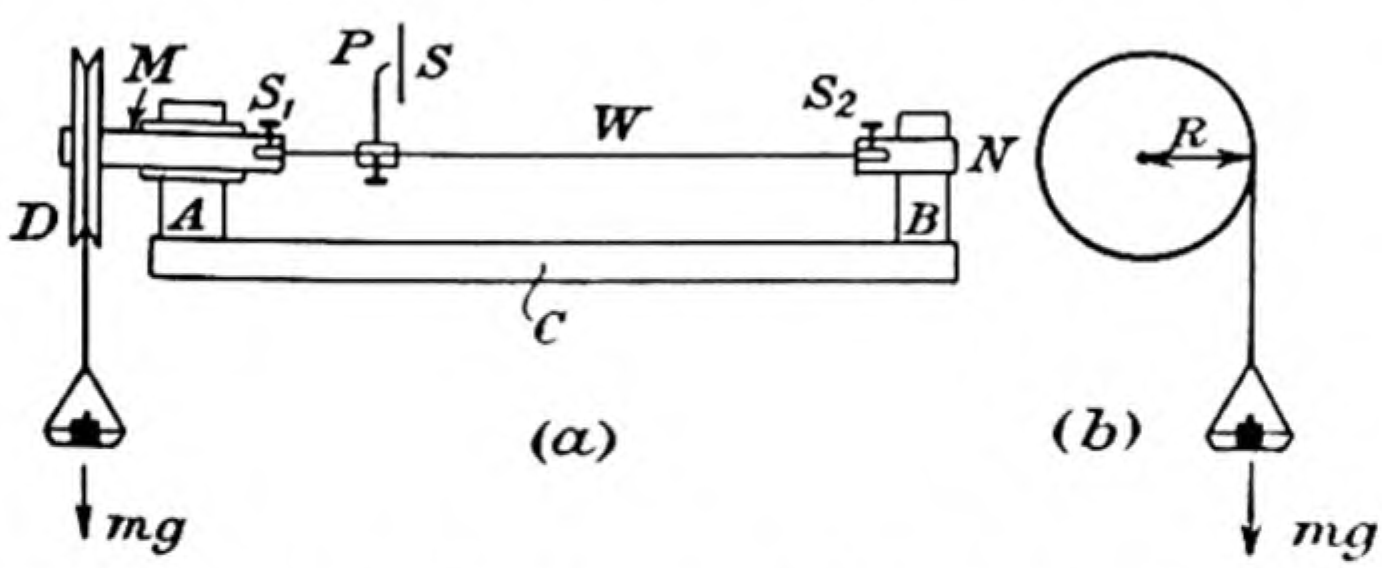


FIG. 7·21.—Determination of torsional rigidity by a statical method.

clamped by means of the screws S_1 and S_2 into short horizontal rods M and N supported in metal uprights A and B carried in a heavy bed C. The axes of the rods coincide with that of the wire. The rod N is fixed into B; the rod M carries a pulley wheel and is free to rotate in A. The wheel D carries a cord fastened to it at one end while the other end has a scale pan which may be loaded. A pointer P is rigidly clamped to the wire at right angles to its axis and moves over a portion of a circular scale S, graduated in degrees and placed at right angles to the plane of the diagram. The pointer may be attached to any point along the wire.

If m is the mass of the load in the pan, g the intensity of gravity,

and R the sum of the radii of the wheel and cord attached to it, the torque Γ applied to the wire is given by

$$\Gamma = mg\mathrm{R}.$$

If l is the distance of the pointer P from the fixed end of the wire of radius a, and θ the angular deflexion of the pointer P, the rigidity of the material of the wire is given by

$$n = \frac{2mg\mathrm{R}l}{\pi\theta a^4}.$$

In experiments with this apparatus any error arising from the fact that the axis of the wire may not pass through the centre of the pulley may be eliminated by twisting the wire in both directions using the same load in the pan, and finding the mean angle of twist.

Experiment.—Let us suppose that three wires suitable for use in the above apparatus are available and are such that two have equal radii but are of different materials, while two have unequal radii but are of the same material.

(*a*) Using one of the wires and the pointer near to the wheel D place different masses in the scale pan and observe the corresponding angular deflexions of the pointer. The above formula suggests that θ is directly proportional to m; verify this graphically.

(*b*) Keep the mass suspended from the cord constant, while a series of readings of the angular twist is taken with the scale and pointer in various positions along the wire, i.e. for various values of l. In order to obtain the twist for each value of l it is necessary to observe the scale reading when the wire is in its normal unstrained condition, and then when the fixed mass is in the scale pan. Thus the twist θ for various values of l is obtained. Use a graphical means to show that $\frac{\theta}{l}$ is constant for a given wire.

(*c*) Compare the moduli of rigidity of the wires of the same radius but different materials. Let the common radius be a, while n_1 and n_2 are the rigidity moduli. For each wire vary m and the position of the pointer along the wire. Now, in general, the equation given above may be written

$$\theta = \frac{c}{n}(ml),$$

where c is a constant. Hence if we plot θ against ml the slope of the line will be inversely proportional to the rigidity n. Thus, from the slopes of the two graphs which may be constructed the moduli of rigidity may be compared.

(*d*) In a similar way, using two wires of unequal radii but the same material, show that for a constant length of the wires and a constant mass in the scale pan the angular deflexion in inversely proportional to the fourth power of the radius of the wire.

(*e*) Using a series of different masses and different values of l for the same mass, show, by means of a graphical method, that $ml = \kappa\theta$, where κ is a constant. Deduce κ from the slope of the line, and hence determine n, the modulus of rigidity for the material of the wire under test.

To determine the torsional rigidity of the material of a wire, using torsional oscillations [and to compare the moments of inertia of rods about a specified axis].—The apparatus is shown in Fig. 7·22. The suspension wire PQ, Fig. 7·22(*a*) [preferably of phosphor bronze if two moments of inertia are to be compared], carries a metal cradle A to which is attached a plane mirror M, by means of which an image of a horizontal scale about one metre from the mirror is reflected into a horizontal telescope T, as indicated

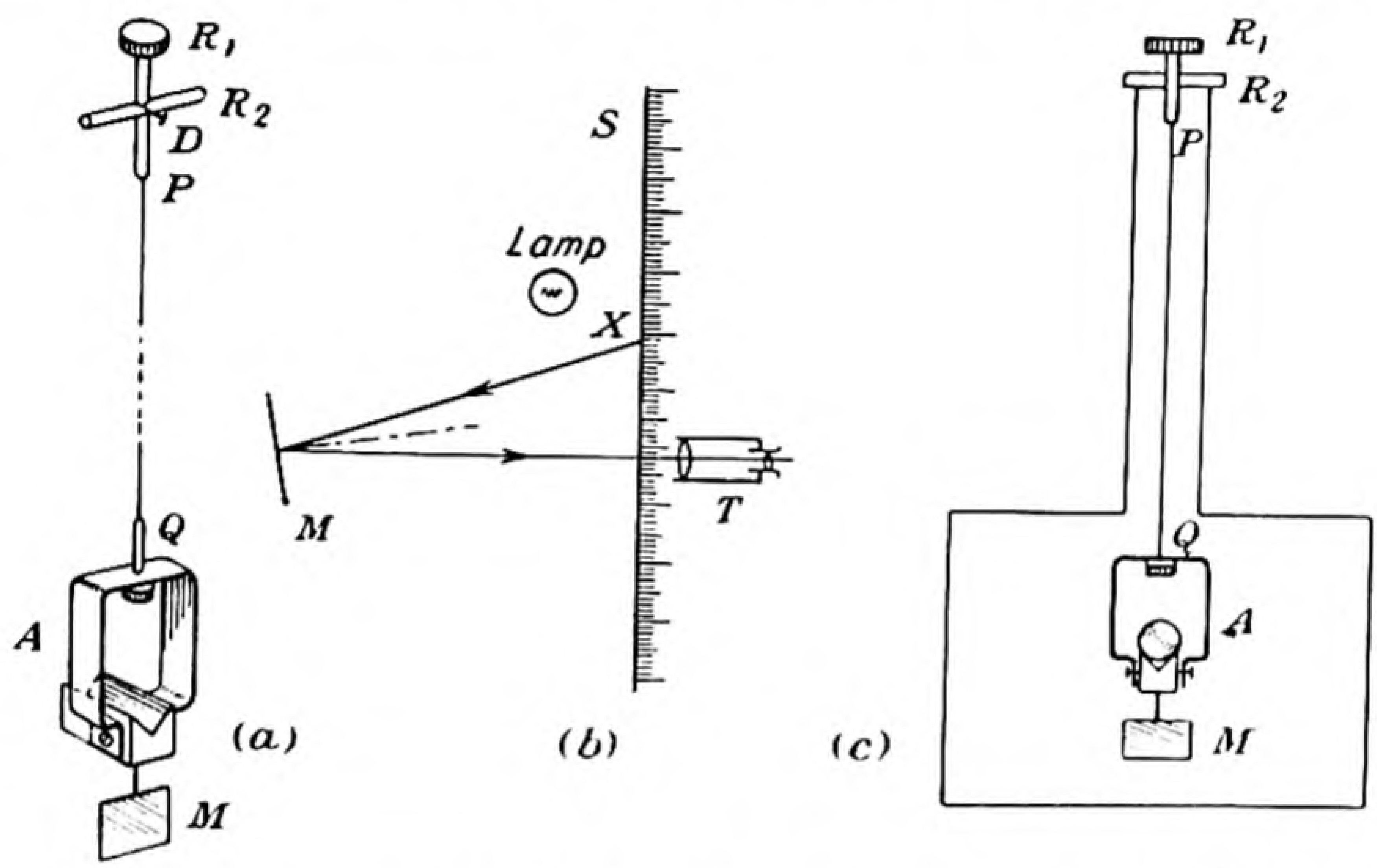


FIG. 7·22.—Dynamical method for determining the torsional rigidity of the material of a wire.

in Fig. 7·22(*b*). The wire is soldered to the rod R_1 passing at right angles through another rod R_2, which serves to support the suspension as in Fig. 7·22(*c*). If the screw *D* is loosened R_1 may be rotated and the mirror set parallel to the scale. A wooden box, provided with two glass sides, protects the apparatus from draughts.

The telescope and scale, the latter being illuminated by a lamp, are adjusted so that the image of a point X on the scale is on the cross-wire of the telescope. X is marked by a black arrow. The cradle is set into small torsional oscillations by a slight twist, care being taken not to produce pendulum oscillations. A chronometer is used for timing the oscillations. If two observers are working together, one observes the transit of the image of the mark X across the wires in the telescope, giving a sharp tap when this occurs so that the second observer may note the time of its occurrence. The mean period is deduced in the usual way.

A metal cylinder is then placed in the cradle so that the axis of suspension passes through the centre of gravity of the cylinder.

If the telescope has been adjusted so that the image of the point of the arrow is on the horizontal cross-wire, the cylinder is moved in the cradle until this view is restored. By this means it is ensured that the moment of inertia of the cradle about the suspension axis is unaltered, and that the appropriate moment of inertia is that referred to an axis through the suspension and through the centre of gravity of the cylinder.

The period of the cradle and cylinder is then found using one of the well-known methods.

The differential equation expressing the motion of the cylinder and its cradle is

$$\mathrm{I}\ddot{\theta} + b\theta = 0,$$

where θ is the angular deflexion, b the torsional constant for the suspension, and I is the moment of inertia of the system about its axis of rotation. But

$$b = \frac{\pi n a^4}{2l}, \qquad [\text{cf. p. 287}].$$

so that the period T is given by

$$\mathrm{T} = 2\pi\sqrt{\frac{\mathrm{I}}{b}} = 2\pi\sqrt{\frac{2l\mathrm{I}}{\pi n a^4}}.$$

Unfortunately I is not the moment of inertia about the axis of suspension of the cylinder alone, for the cradle has a definite, but as yet unknown moment of inertia about the above axis. It is for this reason that it is necessary to determine the period of the cradle alone.

Let T_0 be the period of the cradle, T_1 that of the cylinder and cradle together. Then

$$\mathrm{T}_0 = 2\pi\sqrt{\frac{\mathrm{I}_0}{b}}, \qquad \text{and} \qquad \mathrm{T}_1 = 2\pi\sqrt{\frac{\mathrm{I}_0 + \mathrm{I}_1}{b}},$$

where I_0 and I_1 are the moments of inertia of the cradle and of the cylinder about the axis of rotation. Now I_1 may be calculated from the dimensions of the cylinder [cf. p. 73]. Squaring the above equations and eliminating I_0, we obtain

$$b = \frac{4\pi^2\mathrm{I}_1}{(\mathrm{T}_1^2 - \mathrm{T}_0^2)}.$$

Hence

$$n = \frac{8\pi l\mathrm{I}_1}{(\mathrm{T}_1^2 - \mathrm{T}_0^2)a^4}.$$

It remains to measure the length and radius of the wire. Since

the fourth power of the radius occurs in this formula, it is necessary to measure the mean diameter of the wire with considerable accuracy. If it is permitted to remove the wire from the rod R_1 and the cradle, the mean radius may be determined from observations on the apparent loss in mass of the wire when it is suspended in water at a known temperature from one arm of a balance. Otherwise a micrometer screw gauge must be used.

If the object of the experiment is not to find the modulus of rigidity of the material of a wire but to compare the moments of inertia of two bodies about a specified axis it is not necessary to find b or to know the dimensions of the wire. For if T_2 is the period when the second cylinder is in position, and I_2 its moment of inertia about the axis of suspension,

$$T_2 = 2\pi\sqrt{\frac{I_0 + I_2}{b}}.$$

Hence
$$\frac{I_1}{I_2} = \frac{T_1^2 - T_0^2}{T_2^2 - T_0^2} = \frac{(T_1 - T_0)(T_1 + T_0)}{(T_2 - T_0)(T_2 + T_0)}.$$

The rigidity of a metal wire and its variation with temperature.—In these experiments by Horton† on the variation of the rigidity of the material of a wire, the wire under test and the vibrator, i.e. the body it carried at its lower end, were enclosed in a vapour-heated jacket A, 7·23(a). The vapour entered at the upper end and escaped from the lower end into a condenser from which it returned to the boiler. In this way there was no variation in the nature of the evaporating liquid. The wire W was held at both its ends in clamps specially designed so that the wire was tightly gripped all round its circumference across one right circular section, viz. at the ends of the jaws. To make such a clamp the steel rod B carrying the wire was turned down as in Fig. 7·23(b), and fitted with a steel collar C. A slot D parallel to the axis of B was then made in this rod near to its lower end and a fine slit S made. A very thin piece of metal was then inserted in the slit, the collar placed on and screwed up tightly by means of the screw H. A hole was then drilled down the axis of the rod of the same diameter as the wire to be used [No. 20 s.w.g.]. The wire under test could then be placed in position.

The vibrating body P was a circular disc of gun-metal, $\frac{3}{8}$ in. thick and 5 in. in diameter, to the under side of which a plane mirror was attached. To increase the moment of inertia of the vibrator P about its axis of rotation, a circular ring R could be attached

† Horton, *Phil. Trans.*, A, **204**, 1, 1905.

to P. It fitted into a circular groove formed by turning down the edge of the upper face of P. The ring fitted the groove exactly so that it was easily placed centrally on the plate.

To set the plate in vibration, without swinging it like a pendulum, two intermittent jets of air were directed on the underside of the plate in such a manner as to cause it to rotate about a vertical axis through its centre of gravity. By properly timing these jets, the amplitude of the vibration could be quickly increased.

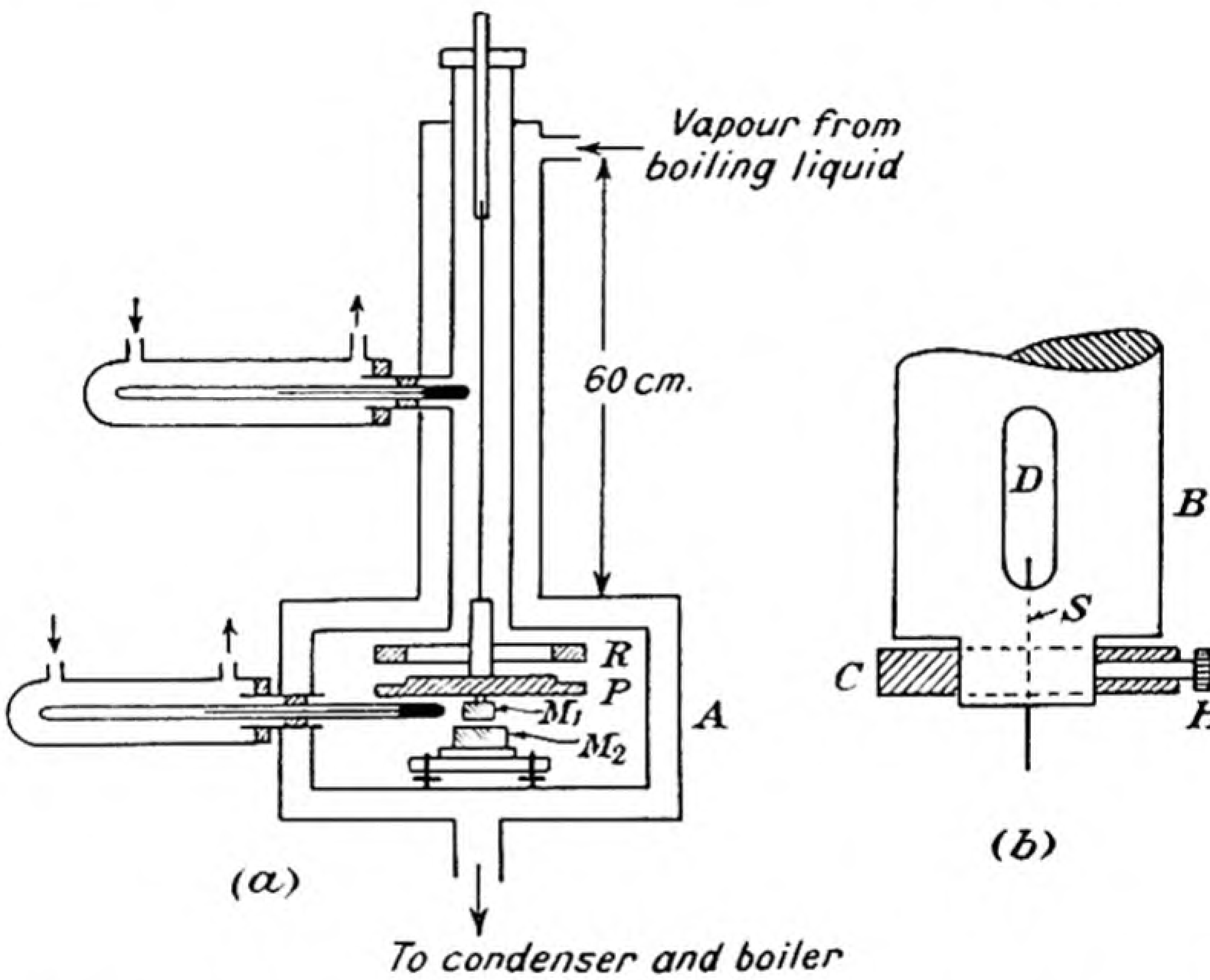


FIG. 7·23.—Experimental investigation of the variation of rigidity with temperature.

The temperature of the interior of the apparatus was given by two mercury-in-glass thermometers, specially standardized at the National Physical Labroatory. The stems were surrounded by water jackets maintained at a constant temperature so that the correction for stem exposure was more easily applied.

To time the vibrations accurately a second plane mirror M_2 was arranged below M_1. M_2 was fixed and parallel to M_1 when the latter mirror was at rest. A flash, of period 1 second, was observed, after reflexion from both mirrors, through a telescope, the field of view of which was so large that the two images of the flash were always seen, one occurring in the same position, and the other in a different part of the field according to the position of the moving mirror when the flash occurred. If a second signal happens exactly when the two mirrors are parallel, the two images of the flash

coincide; from these 'coincidences' the period of the vibrating system was deduced.

Usually the method of coincidences is applied to the comparison of times which are very nearly equal. Horton followed a suggestion made by Poynting, whereby the method could be applied to compare two periods even if they were quite different. In order to illustrate this, let us assume that the time of a complete vibration is 4·116 seconds. This has to be compared with the 1 second period of the flashes. Now suppose that the two images of the flash as seen in the telescope have just coincided, and let us call the second at which this coincidence occurred 0. Then after 4 seconds the two mirrors will not be exactly parallel again, and the flashes observed in the telescope will consequently be some distance apart. Since the period is greater than 4 seconds, the moving mirror will not yet have become parallel to the fixed one, and the moving flash will appear to have fallen short of its zero position. If, however, we wait for such a number of seconds, n, as is very nearly an integral multiple of the time of vibration, then at the nth flash the two mirrors will be very nearly parallel, and the flashes will very nearly coincide. Now $9 \times 4{\cdot}116 = 37{\cdot}044$; therefore after 37 seconds, the moving flash will appear very near to the fixed one, for it would only take the swinging mirror 0·044 second to become parallel to the fixed mirror. As before, since the multiple of the time of swing is greater than 37 seconds, the flash will appear to have fallen short of its position of rest. Now suppose that we go on counting the seconds, calling the flashes after the 37th, one, two, etc., and then starting at one again, and so on. Every 37th flash will appear to have lost on the position of the preceding 37th, i.e. to have moved further away from the central fixed flash. As this goes on the number 1 flash (i.e. the one next after the 37th) will be getting nearer and nearer to the central fixed flash, until after a time it coincides with it. Then it is evident that the moving mirror will have lagged behind one complete second, for it takes one second longer to become parallel to the fixed mirror. Suppose that when this happens N sets of 37 seconds have been counted.

The vibrator therefore makes 9N vibrations in 37N + 1 seconds, and if T is the period of vibration we have

$$9\mathrm{NT} = 37\mathrm{N} + 1, \quad \text{or} \quad \mathrm{T} = \left(\frac{37}{9} + \frac{1}{9\mathrm{N}}\right) \text{ seconds.}$$

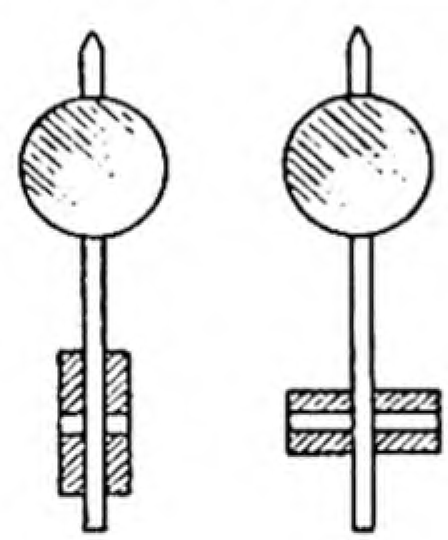
A slight departure from the above method of timing was made in the actual experiment owing to the fact that an exact coincidence was a very rare event—for details of this the original paper must be consulted.

Corrections were made for the change in the linear dimensions of the wire and vibrator with temperature, their coefficients of linear expansion being measured with the aid of a standard metre-comparator.

From the results obtained it was found that the modulus of rigidity, except in the case of pure copper and steel (pianoforte wire), was not constant for a given material at a fixed temperature, but increased with time. For pure copper and steel there was a diminution in rigidity with increase in temperature, the relation being a linear one. In the case of gold, iron, tin and lead, the relation is such that it suggests that if the 'time effect' referred to above could be avoided, the relation would be a linear one.

The moduli of elasticity were calculated from the equations already given in connexion with this method, the mean diameter of the wire being obtained by the use of the hydrostatic method there suggested.

The torsional rigidity of fused quartz.—A similar apparatus to that just described was used by HORTON† to determine the modulus of rigidity of fused quartz and its variation with temperature. BOYS had already shown that in dealing with fibres made from fused quartz it was necessary to keep the mass constant when the moment of inertia of the vibrator about its axis of rotation was altered [cf. p. 240]. In order to fulfil this requirement Horton used a vibrator made of two parts; it comprised a brass rod and a brass cylinder with two holes drilled in it, one along its axis, and the other perpendicular to its axis and passing through its centre of gravity. These holes were the same diameter as the rod, so that it could be attached as in Fig. 7·24(*a*) or (*b*). The other experimental details were not materially changed. At 15° C. the modulus of rigidity for fused quartz was found to be $3{\cdot}00 \times 10^{11}$ dyne.cm.$^{-2}$; the modulus increased with temperature.



(*a*) (*b*)

FIG. 7·24.—Vibrator for use in torsional experiments on quartz fibres.

Maxwell's needle used as a vibrator in experiments on torsional oscillations.—Maxwell's needle is the name of a very convenient vibrator for use in experiments designed to determine the rigidity of the material of a wire by a dynamical method. It consists of a brass tube into which four equal pieces of brass tube may be slipped, the length of each piece being exactly one fourth of the tube into which they fit. Two of these are hollow; the others are filled with lead [or they may be rods of solid brass of the appropriate length and diameter]. The long brass tube just fits

† Horton, *Phil. Trans.*, A, **204**, 407, 1905.

into a short brass tube A, Fig. 7·25(a), carrying a mirror M rigidly attached to a short thick brass wire B passing vertically through its centre. The wire W, whose rigidity is required, is attached at its lower end to B, its upper end being held in a suitable clamp.

By placing the tubes as shown in Figs. 7·25(a) and (b) the moment of inertia of the system about the axis of oscillation may be varied considerably. Let these be denoted by I_1 and I_2, I_1 being greater than I_2. The periods T_1 and T_2 are then determined by one of the usual methods. In order to determine the rigidity of the material of the wire it is necessary to find an expression for $(I_1 - I_2)$.

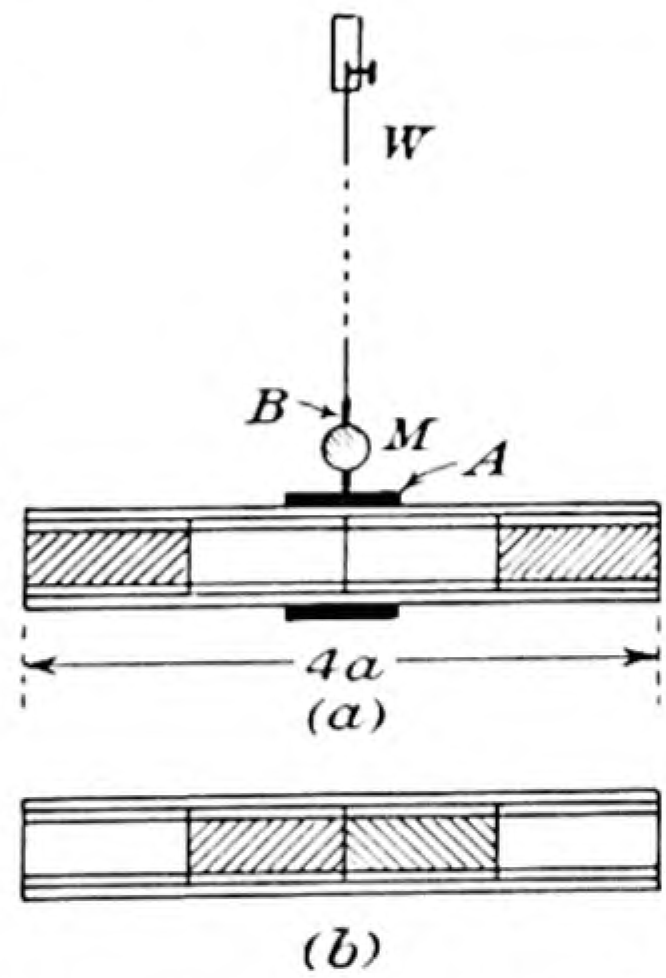


Fig. 7·25.—Maxwell's needle.

Let $4a$ be the length of the long tube, and m the mass of lead in each tube containing that metal—this mass is the difference between the mass of such a tube and that of an empty one. [If two solid brass rods are used, the difference in mass between one of them and that of an empty tube must be found.] Now the moment of inertia of a uniform rod of length a about a vertical axis passing through its centre of gravity [cf. p. 68] is $\frac{1}{12}ma^2$. Hence the moment of inertia of the lead in (a) is, by the theorem on p. 70,

$$2m[\tfrac{1}{12}a^2 + (\tfrac{3}{2}a)^2] = \tfrac{14}{3}ma^2.$$

In the second instance it is

$$2m[\tfrac{1}{12}a^2 + (\tfrac{1}{2}a)^2] = \tfrac{2}{3}ma^2.$$

Hence the change in the moment of inertia about the axis of rotation of the system is $4ma^2$, for this is the difference between the two expressions just obtained.

The above expression might have been written down at once, for the change in the moment of inertia of the system about the vertical axis through its centre of gravity has been brought about by displacing the mass-centres of the two lead portions through a horizontal distance equal to a, viz. $(\frac{3}{2}a - \frac{1}{2}a)$, so that the change in the square of the radius of gyration is

$$2[(\tfrac{3}{2}a)^2 - (\tfrac{1}{2}a)^2] = 4a^2.$$

Let T_1 and T_2 be the periods of oscillation corresponding to the

two arrangements (*a*) and (*b*). If I_0 is the moment of inertia of the tubes, mirror and rod B, about the axis of rotation, then

$$T_1 = 2\pi\sqrt{\frac{I_0 + I_1}{b}}, \quad \text{and} \quad T_2 = 2\pi\sqrt{\frac{I_0 + I_2}{b}}.$$

$$\therefore\ T_1^2 = \frac{4\pi^2}{b}(I_0 + I_1), \quad \text{and} \quad T_2^2 = \frac{4\pi^2}{b}(I_0 + I_2).$$

$$\therefore\ (T_1^2 - T_2^2)\frac{b}{4\pi^2} = (I_1 - I_2).$$

Thus the value of b, the torsional constant for the wire, may be calculated. When the diameter of the wire and its length have been found, the modulus of rigidity of the material of the wire may be deduced.

Example.—Prove that if a number of wires are joined end to end the torsional rigidity, i.e. $\Gamma \div \theta$, where the symbols have their usual meanings, is given by the same formula as that for a number of resistances in parallel. Hence find the torsional rigidity of a tapering wire of length l in terms of the rigidity modulus of its material and its end radii a and b, where $b > a$.

If you were provided with a metal wire of circular cross-section but whose diameter varied in a non-uniform manner along its length, indicate how the formula you have obtained would be modified so that it might be used in an experiment to determine an accurate value for the rigidity of the metal.

With the usual notation, we have

$$\Gamma = \frac{\pi n a^4 \theta}{2l}.$$

Hence for wires characterized by $l_1, a_1, n_1, \theta_1;\ l_2, a_2, n_2, \theta_2;\ \ldots$ we have

$$\Gamma = \frac{\pi n_1 a_1^4 \theta_1}{2l_1} = \frac{\pi n_2 a_2^4 \theta_2}{2l_2} = \ldots = \frac{\pi n_k a_k \theta_k}{2l_k} = \ldots$$

$$\therefore\ \Sigma(\theta) = \Gamma\left[\frac{2l_1}{\pi n_1 a_1^4} + \frac{2l_2}{\pi n_2 a_2^4} + \ldots + \frac{2l_k}{\pi n_k a_k^4} + \ldots\right]$$

$$\therefore\ \frac{1}{\left[\dfrac{\Gamma}{\Sigma(\theta)}\right]} = \sum_{k=1}^{n} \frac{1}{\left(\dfrac{\pi n_k a_k^4}{2l_k}\right)}$$

$$= \frac{1}{\left[\dfrac{\Gamma}{\theta_1}\right]} + \frac{1}{\left[\dfrac{\Gamma}{\theta_2}\right]} + \ldots + \frac{1}{\left[\dfrac{\Gamma}{\theta_k}\right]} + \ldots\Bigg],$$

which establishes the proposition.

Now consider a length δx of the wire shown in Fig. 7·26(*a*); let the mean radius of the cross-section at a distance x from the zero end of the wire be y, where

$$y = \left(\frac{b - a}{l}\right)x + a.$$

If $\delta\theta$ is the twist between the ends of this element, we have

$$\Gamma = \frac{\pi n y^4 \,\delta\theta}{2\delta x},$$

or

$$\frac{\Gamma}{\delta\theta} = \frac{\pi n}{2}\left(\frac{b-a}{l}x + a\right)^4 \frac{1}{\delta x}.$$

$$\therefore \quad \frac{1}{\left(\dfrac{\Gamma}{\theta}\right)} = \int_{x=0}^{x=l}\frac{d\theta}{\Gamma} = \frac{2}{\pi n}\int_0^l \frac{dx}{\left(\dfrac{b-a}{l}x + a\right)^4}$$

$$= -\frac{2}{3}\cdot\frac{1}{\pi n}\cdot\frac{l}{b-a}\left[\frac{1}{b^3} - \frac{1}{a^3}\right].$$

$$\therefore \quad \text{Torsional rigidity} = \frac{3}{2}\frac{\pi n a^3 b^3}{l}\cdot\frac{1}{a^2 + ab + b^2}.$$

A longitudinal section of the non-uniform wire is shown in Fig. 7·26(b).

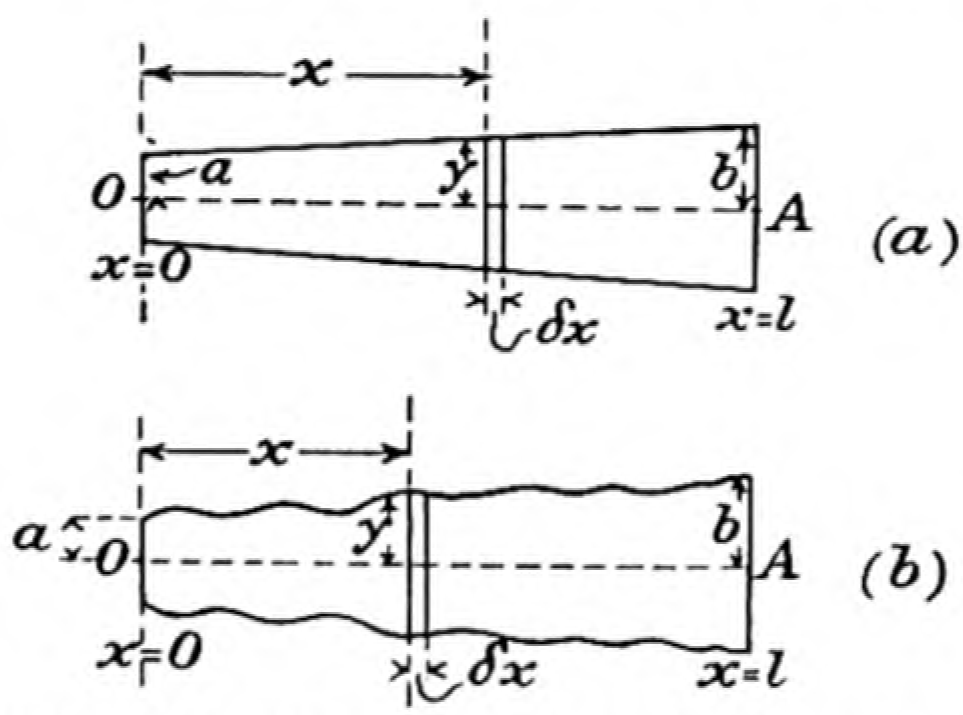


FIG. 7·26.—The torsional rigidity of wires of varying diameter.

As before, for an element of length δx, we have

$$\frac{\Gamma}{\delta\theta} = \frac{\pi n y^4}{2\delta x}, \quad \text{or} \quad 2\Gamma\frac{\delta x}{y^4} = \pi n\,\delta\theta.$$

Hence

$$2\Gamma\Sigma\left(\frac{\delta x}{y^4}\right) = \pi n \theta,$$

and this summation must be effected graphically after measuring the diameter of the wire at different sections and plotting the graph showing how y^{-4} varies with x.

A bifilar suspension; correction for rigidity.—(a) ***Parallel cords.*** When the suspending cords or wires are parallel, let Fig. 7·27(a) be a plan of the apparatus; the diameters of the circles which represent the cords are much enlarged. Let L be a line fixed in the section of a wire adjacent to the bar carried by the suspension. When the bar has rotated through an angle ψ about the vertical axis through its centre O, cf. Fig. 7·27(b), the line L will not have altered its position relative to the bar, i.e. the twist in the wire will be ψ. Hence if a_0 is the radius of cross-section of the wire and λ its length,

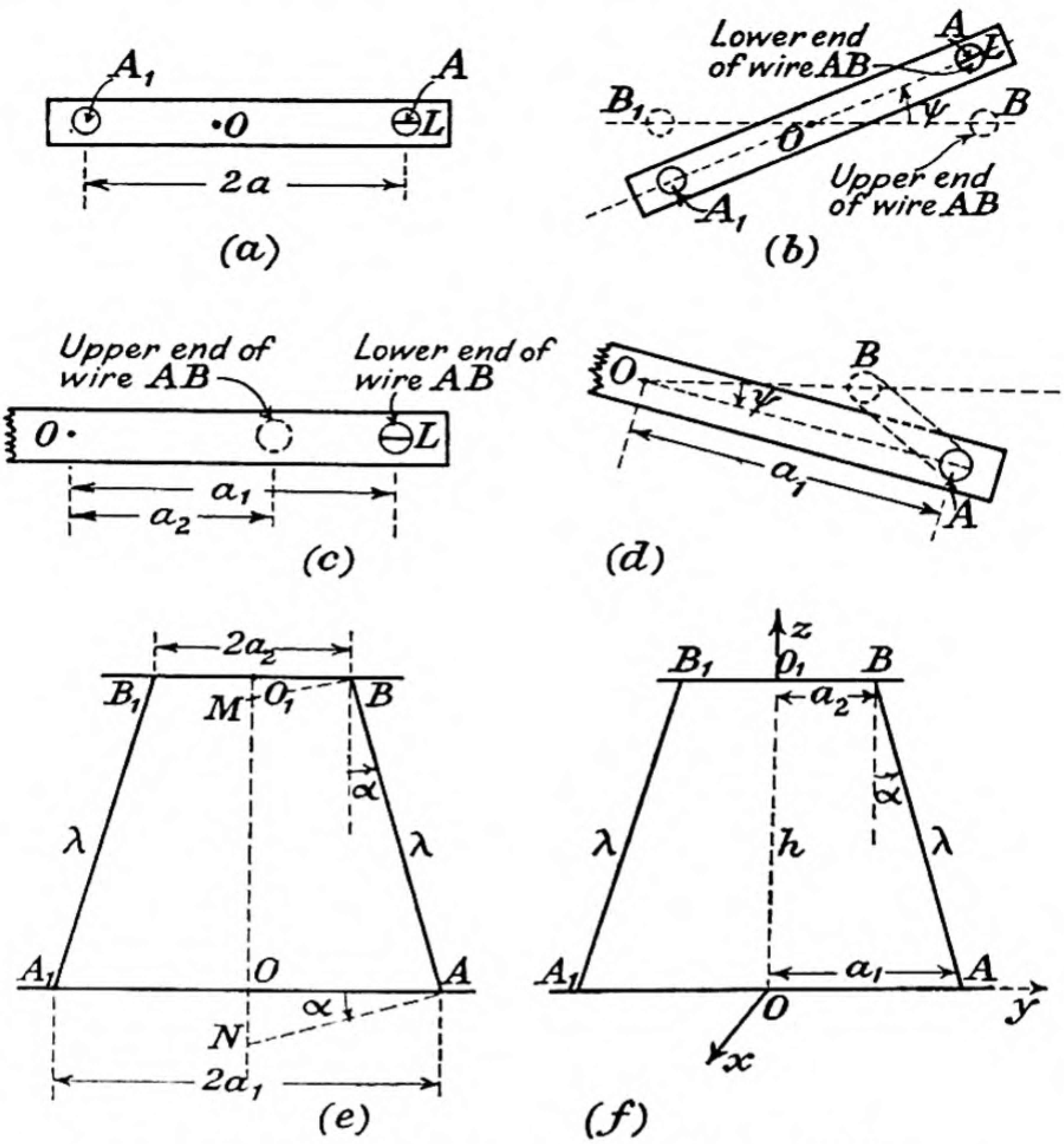


FIG. 7·27.—A bifilar suspension; correction for rigidity.

the additional restoring couple exerted on the bar by the wires and due to their rigidity will be

$$2\left(\frac{\pi n a_0{}^4 \psi}{2\lambda}\right) = \frac{\pi n a_0{}^4 \psi}{\lambda},$$

where n is the rigidity of the material of the wires. Hence, cf. p. 143, the equation of motion for small oscillations becomes

$$\mathrm{I}\ddot{\psi} + \left(\frac{mga^2}{\lambda} + \frac{\pi n a_0{}^4}{\lambda}\right)\psi = 0,$$

i.e. if $\mathrm{I} = m\kappa^2$, where m is the mass of the bar, etc.,

$$\mathrm{T} = 2\pi\kappa \sqrt{\frac{1}{a^2 + \dfrac{\pi n a_0{}^4}{mg}} \cdot \frac{\lambda}{g}}.$$

(*b*) ***Non-parallel cords.*** In this instance, using the same notation as on p. 142, let Fig. 7·27(*c*) be a plan of the apparatus in its position of static equilibrium. When the bar is displaced through an angle ψ as in Fig. 7·27(*d*), the fixed line L in the bar will have moved through an angle ψ but the twist in the wire, as measured in a plane normal to its axis, will be $\psi \cos \alpha$, where α is the inclination of a wire to the vertical. To prove this we have to note that when the suspended bar is displaced, the supporting wires do not only twist but bend slightly at their ends. If it is assumed that the energy associated with this bending is small compared with the total potential energy, then it may be supposed that each wire is attached to the bar AA_1, Fig. 7·27(*e*), by a hinge which allows the end of the wire AB to rotate freely about an axis AN normal to the length of the wire; similarly, the wires must be hinged at B and B_1, when the appropriate axis for AB will be BM. The angle of twist in the wire AB is then the angle between the directions AN and BM; in practice, BM will not rotate.

If $AA_1 = 2a_1$, then when the bar rotates through an angle ψ, A moves through a distance $a_1\psi$, normal to the plane of the paper, if ψ is small. The line AN will therefore move through an angle

$$\frac{a_1\psi}{AN} = \frac{a_1\psi}{a_1 \sec \alpha} = \psi \cos \alpha.$$

The couple exerted on the bar and due to the rigidity of the wires is

$$2\left[\frac{\pi n a_0{}^4 \psi \cos \alpha}{2\lambda}\right] = \frac{\pi n a_0{}^4 \psi \cos \alpha}{\lambda}.$$

To derive the equation of motion let us proceed from first principles and consider the energy of the system. Let the coordinates of A before the bar is displaced be $(a_1, 0, 0)$ and after displacement through an angle ψ be $(a_1 \cos \psi,\ a_1 \sin \psi,\ z)$, where z is the small vertical rise of any point in the bar. Since the length of each wire remains constant,

$$\lambda^2 = h^2 + (a_1 - a_2)^2 = (h - z)^2 + (a_1 \cos \psi - a_2)^2 + (a_1 \sin \psi)^2,$$

which gives

$$-2hz + z^2 - 2a_1a_2 \cos \psi + 2a_1a_2 = 0,$$

or if $z^2 \ll 2hz$,

$$hz = a_1a_2(1 - \cos \psi) = \tfrac{1}{2}a_1a_2\psi^2,$$

since ψ is small,

$$\therefore \text{ Increase in potential energy } = mgz = \frac{mga_1a_2}{2h}\psi^2$$

Now the elastic energy associated with the wires is

$$2[\tfrac{1}{2}(\text{final couple}) \times (\text{final twist})] = 2\left[\frac{1}{2}.\frac{\pi n a_0{}^4 \psi \cos\alpha}{2\lambda} \times \psi\cos\alpha\right]$$

$$= \frac{1}{2}.\frac{\pi n a_0{}^4 \cos^2\alpha . \psi^2}{\lambda}.$$

Hence the total energy of the system at the instant considered will be

$$\tfrac{1}{2}\mathrm{I}\dot{\psi}^2 + \left[\frac{mga_1a_2}{2h} + \frac{1}{2}\frac{\pi n a_0{}^4\cos^2\alpha}{\lambda}\right]\psi^2$$

and this is constant. Since $\lambda = h\sec\alpha$, by differentiating we obtain as the equation of motion,

$$m\kappa^2\ddot{\psi} + 2\left[\frac{mga_1a_2}{2h} + \frac{\pi n a_0{}^4\cos^3\alpha}{2h}\right]\psi = 0.$$

$$\therefore\ \mathrm{T} = 2\pi\sqrt{\frac{m\kappa^2}{\dfrac{mga_1a_2}{h} + \dfrac{\pi n a_0{}^4\cos^3\alpha}{h}}}$$

$$= 2\pi\kappa\sqrt{\frac{h}{ga_1a_2 + \dfrac{\pi}{m}\cdot n a_0{}^4\cos^3\alpha}}.$$

Bulk modulus and compressibility.—If a body of isotropic material is subjected to three equal compressive stresses mutually at right-angles, the stress on any plane in the body is wholly normal to that plane. Let V be the volume of a body of isotropic material when subjected to a pressure p. When the pressure is increased to $(p + \delta p)$, let the volume of the above body become $(\mathrm{V} + \delta\mathrm{V})$. Then the strain due to an increase in stress, δp, is $-\dfrac{\delta\mathrm{V}}{\mathrm{V}}$, the negative sign being necessary since an increase in the stress is accompanied by a decrease in the volume. In this instance, the appropriate elastic modulus is termed the ***bulk modulus*** of the material. If it is denoted by β, it is given by

$$\beta = \frac{\text{stress}}{\text{strain}} = \frac{\text{increase in stress}}{\text{increase in strain}}$$

$$= \lim_{\delta p \to 0} \frac{\delta p}{-\left(\dfrac{\delta\mathrm{V}}{\mathrm{V}}\right)} = -\mathrm{V}\left(\frac{\partial p}{\partial\mathrm{V}}\right).$$

[It is essential to use a partial differential coefficient here since, in general, p is a function of other variables than v, e.g. the temperature.]

Usually this type of strain is caused by subjecting a body to a uniform increase of pressure.

The reciprocal of the bulk modulus of a substance is known as its ***compressibility***, κ. It can be measured, not quite directly, for fluids, but for solids it is more usual to calculate it from their moduli of extensibility and rigidity by formulae we shall shortly proceed to establish.

In solids it is found that, whether the change in stress occurs adiabatically and reversibly, i.e. the change is isentropic, or slowly at constant temperature, i.e. isothermally, the bulk modulus does not vary appreciably; but in gases and liquids it is very important to specify exactly the conditions under which the change in stress occurs. [Cf. Vol. II.]

The bulk moduli of an ideal gas at constant temperature and at constant entropy.—(*a*) Let p and v denote respectively the pressure and volume of unit mass of gas. If the pressure becomes $(p + \delta p)$, let the volume be $(v + \delta v)$. If the gas is an ideal one maintained at constant temperature, the relation between the above changes will be given by Boyle's law, so that the product pv is constant. Differentiating, we obtain

$$p.\delta v + v.\delta p = 0.$$

Hence
$$\left(\frac{\partial p}{\partial v}\right)_{\mathrm{T}} = -\frac{p}{v}.$$

The bulk modulus is therefore given by

$$\beta_{\mathrm{T}} = -v.\frac{\partial p}{\partial v} = p,$$

which means that the bulk modulus of an ideal gas at constant temperature is equal to the pressure to which it is subjected. [N.B.—The pressure must be expressed in absolute or gravitational units.]

(*b*). For a reversible adiabatic (or isentropic) expansion of an ideal gas $pv^{\gamma} =$ constant, where γ is the ratio of its two principal specific heats. If β_{S} is the corresponding bulk modulus we have

$$\left(\frac{\partial p}{\partial v}\right)_{\mathrm{S}} = -\gamma\frac{p}{v},$$

i.e.

$$\beta_{\mathrm{S}} = -v\left(\frac{\partial p}{\partial v}\right)_{\mathrm{S}} = \gamma p.$$

On the relations between the elastic constants for an isotropic material.—The following relations between the elastic constants E, n, β and σ are very important in physics, and the applied sciences of engineering and metallurgy.

(*a*) ***Elementary treatment***: The elastic constants E, σ, n and β have already been defined; we shall now show that for an isotropic material only two of these constants are independent, i.e. when two are known the others may be calculated.

Consider a cube of the material. Let c be the natural length of each edge and suppose that the cube is subjected to three equal tensile stresses p, each normal to one pair of faces of the cube. Let ϵ be the strain. Then the new volume of the cube will be $c^3(1 + \epsilon)^3 = c^3(1 + 3\epsilon)$, if terms in ϵ^2 and higher are neglected. Hence the bulk strain is 3ϵ, and β, the bulk modulus, is given by

$$\beta = \frac{p}{3\epsilon}, \quad \text{or} \quad p = 3\beta\epsilon. \qquad \text{(i)}$$

If the applied stress acted only on one pair of opposite faces the extensional strain normal to them would be $\frac{p}{\mathrm{E}}$, where E is Young's modulus for the material of the cube. This, for example, is the strain along AB (or any line parallel to AB), Fig. 7·28(*a*), due to a stress parallel to AB. When, however, all three pairs of extensional stresses operate, part of the strain in AB will be due to the other pairs of stresses; each of these additional stresses will produce in AB a strain $-\sigma\frac{p}{\mathrm{E}}$, where σ is Poisson's ratio, cf. p. 283. Hence the total extensional strain along AB is

$$\frac{p}{\mathrm{E}} - 2\sigma\frac{p}{\mathrm{E}} = \frac{p}{\mathrm{E}}(1 - 2\sigma) = \epsilon, \qquad \text{(ii)}$$

Equating values of $\frac{p}{\epsilon}$ in equations (i) and (ii) we get

$$3\beta = \frac{\mathrm{E}}{1 - 2\sigma}.$$

Consider next the same cubical block with a uniform tensile stress p over one pair of opposite faces, an equal compressive stress over another pair of faces and no other stresses; these are shown in Fig. 7·27(*b*). Let L, M, N and Q be the mid-points of the sides AB, BC, CD and DA, respectively. Consider that portion of the cube for which ALQ, cf. Fig. 7·27(*c*) and (*d*), is a cross-section.

The external forces acting on this portion of the cube are (i) $\frac{1}{2}c^2p$, vertically upwards; (ii) $\frac{1}{2}c^2p$, sidewards to the right and (iii) the

force which the rest of the cube exerts on it. The resultant of the two forces $\frac{1}{2}c^2p$ is

$$2(\tfrac{1}{2}c^2p)\,\frac{\sqrt{2}}{2} = \tfrac{1}{2}\sqrt{2}c^2p,$$

and acts parallel to LQ. For equilibrium the force which the remainder of the cube exerts on the portion ALQ must therefore be

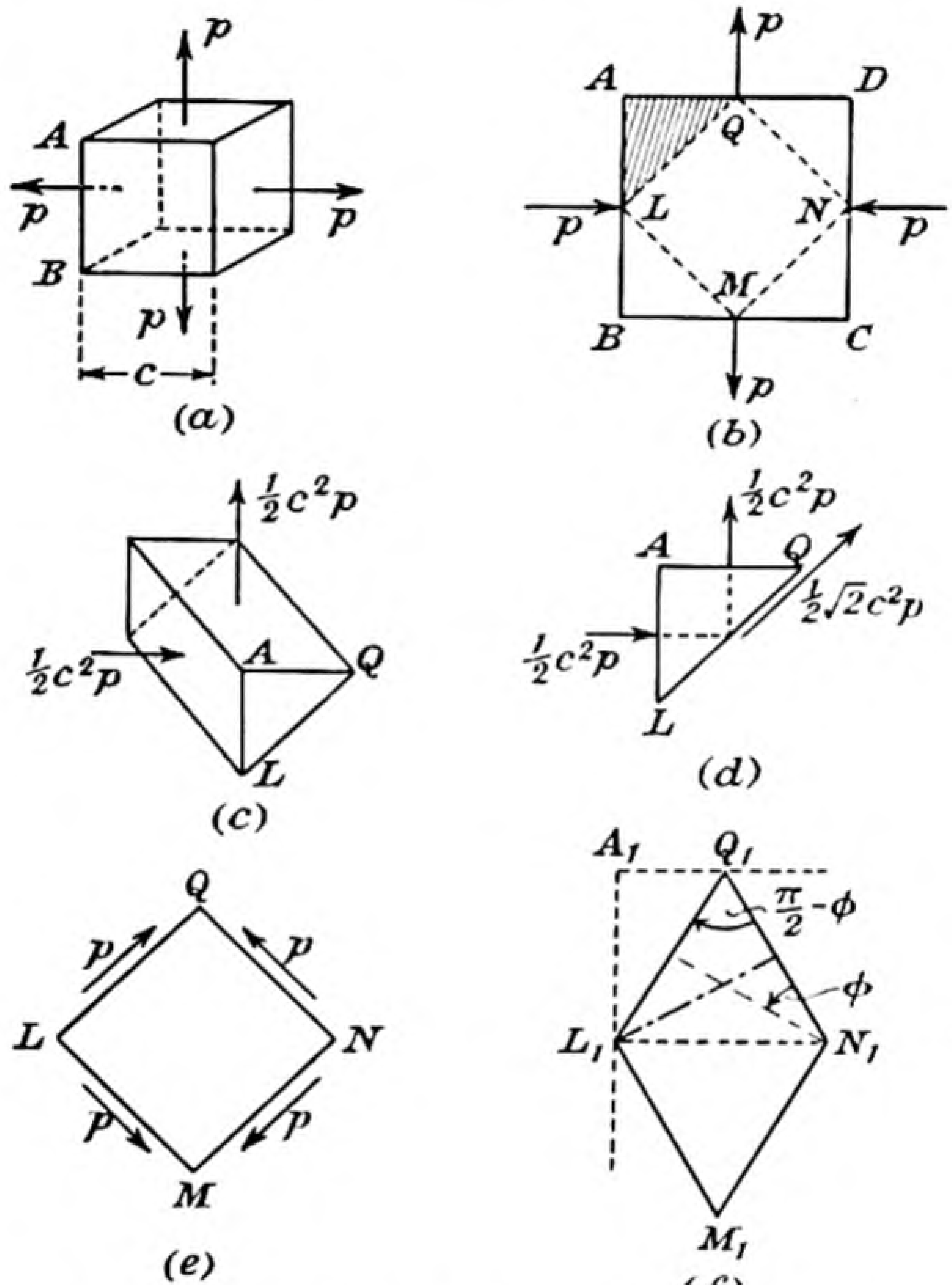


FIG. 7·28.—The relations between the elastic constants E, σ, n and β.

$\frac{1}{2}\sqrt{2}c^2p$ and it must act in a direction parallel to QL. But the area of the face QL is $\frac{1}{2}c\sqrt{2}.c = \frac{1}{2}\sqrt{2}c^2$. Hence across LQ there is a shearing stress p.

Fig. 7·28(*e*) shows the portion LMNQ of the cube and the system of stresses acting upon it. In consequence of these stresses the square LMNQ will become the rhombus $L_1M_1N_1Q_1$, as shown in Fig. 7·28(*f*).

Now $M_1Q_1 = c\left[1 + \frac{p}{E} + \sigma\frac{p}{E}\right]$, so that the extensional strain

along M_1Q_1 is $\frac{p}{E}(1 + \sigma)$; similarly the strain along L_1N_1 is a compressive one of amount $\frac{p}{E}(1 + \sigma)$.

Let $\epsilon = \frac{p}{E}(1 + \sigma)$; then the new lengths of AB and BC are $c(1 + \epsilon)$ and $c(1 - \epsilon)$, respectively. Let $L_1\widehat{Q_1}N_1 = \frac{\pi}{2} - \phi$, where ϕ is the angle of shear, and consider the $\Delta\, L_1N_1Q_1$.

$$L_1N_1^2 = (N_1Q_1 - L_1Q_1 \sin\phi)^2 + (L_1Q_1 \cos\phi)^2$$
$$= N_1Q_1^2 + L_1Q_1^2 - 2N_1Q_1 . L_1Q_1 \sin\phi.$$
$$\therefore \sin\phi = \frac{L_1Q_1^2 + N_1Q_1^2 - L_1N_1^2}{2L_1Q_1 . N_1Q_1} = \frac{2L_1Q_1^2 - L_1N_1^2}{2L_1Q_1^2}.$$

But $L_1Q_1^2 = A_1L_1^2 + A_1Q_1^2$, and $L_1N_1^2 = 4A_1Q_1^2$.

$$\therefore\ 2L_1Q_1^2 - L_1N_1^2 = 2[A_1L_1^2 - A_1Q_1^2]$$
$$= 2(A_1L_1 - A_1Q_1)(A_1L_1 + A_1Q_1)$$
$$= [c(1 + \epsilon) - c(1 - \epsilon)][\tfrac{1}{2}c(1 + \epsilon) + \tfrac{1}{2}c(1 - \epsilon)]$$
$$= 2c^2\epsilon.$$

Since $L_1Q_1^2 \doteqdot \frac{1}{2}c^2$,

$$\phi \doteqdot \sin\phi = \frac{2c^2\epsilon}{c^2} = 2\epsilon,$$
$$\therefore\ p = n\phi = 2n\epsilon,$$

i.e.
$$\epsilon = \frac{p}{2n}.$$

But $\epsilon = \frac{p}{E}(1 + \sigma)$; hence $2n = \frac{E}{1 + \sigma}$.

Since $3\beta = \frac{E}{1 - 2\sigma}$ and $2n = \frac{E}{1 + \sigma}$, we may eliminate σ from these two equations and get

$$\frac{3}{n} + \frac{1}{\beta} = \frac{9}{E}.$$

Similarly, it may be proved that

$$\sigma = \frac{E}{2n} - 1 = \frac{3\beta - 2n}{6\beta + 2n}.$$

(*b*) ***Analytical treatment:*** It has been shown that when a wire, whose material is isotropic, is stretched, there is an elongation in the direction of the applied stretching forces, and a contraction in all

directions normal to the above. Now consider a unit cube of an isotropic material subjected to three stresses, the directions along which they act being parallel to the edges of the cube. Let p_1, p_2 and p_3 be the stresses and ε_1, ϵ_2 and ϵ_3 the corresponding strains. A stress is considered positive if it tends to elongate the body along the direction in which it is applied, i.e. a tensile stress is a positive one. Then ε_1 will depend upon p_1 and also involve p_2 and p_3 symmetrically. Thus we may write

$$\epsilon_1 = \bar{\lambda} p_1 + \bar{\mu}(p_2 + p_3),$$
$$\epsilon_2 = \bar{\lambda} p_2 + \bar{\mu}(p_3 + p_1),$$
$$\epsilon_3 = \bar{\lambda} p_3 + \bar{\mu}(p_1 + p_2),$$

where $\bar{\lambda}$ and $\bar{\mu}$ are constants. These equations may be solved giving the p's in terms of the ϵ's. Thus we may write,

$$p_1 = \lambda\epsilon_1 + \mu(\epsilon_2 + \epsilon_3),$$
$$p_2 = \lambda\epsilon_2 + \mu(\epsilon_3 + \epsilon_1),$$
$$p_3 = \lambda\epsilon_3 + \mu(\epsilon_1 + \epsilon_2).$$

λ and μ depend on the nature of the material of the body under consideration.

(i) Let the pressure on a body in the form of a unit cube be reduced by an amount p, so that it expands: let Δ be the dilatation, the ratio of the increase in volume to the original volume. Then

$$p_1 = p_2 = p_3 = -p,$$

and

$$\Delta = \epsilon_1 + \epsilon_2 + \epsilon_3$$
$$= 3\epsilon,$$

since the strains are all equal to one another, say ϵ.
Hence

$$-p = (\lambda + 2\mu)\epsilon = \tfrac{1}{3}(\lambda + 2\mu)\Delta,$$

so that

$$\beta = -\frac{p}{\Delta} = \tfrac{1}{3}(\lambda + 2\mu) \quad . \quad . \quad . \quad \text{(i)}$$

(ii) Now let the applied stress, p, be directed along an axis parallel to one of the edges of the cube, i.e. $p_1 = p$, and $p_2 = p_3 = 0$. This is an instance of a simple longitudinal pull, a longitudinal extension in the direction of the pull being accompanied by a lateral contraction in directions normal to that of the pull.

Then

$$p_1 = p = \lambda\epsilon_1 + \mu(\epsilon_2 + \epsilon_3),$$
$$0 = \lambda\epsilon_2 + \mu(\epsilon_3 + \epsilon_1),$$

and

$$0 = \lambda\epsilon_3 + \mu(\epsilon_1 + \epsilon_2).$$

In consequence of the last two equations, $\epsilon_2 = \epsilon_3$. Hence

$$(\lambda + \mu)\epsilon_2 + \mu\epsilon_1 = 0,$$

so that σ, Poisson's ratio, is given by

$$\sigma = -\frac{\epsilon_2}{\epsilon_1} = \frac{\mu}{\lambda + \mu}.$$

Now $$p = \lambda\epsilon_1 + 2\mu\epsilon_2.$$

and $$\epsilon_2 = -\sigma\epsilon_1.$$

Hence $$p = (\lambda - 2\sigma\mu)\varepsilon_1.$$

so that E, Young's modulus, is given by

$$E = \frac{p}{\epsilon_1} = \lambda - 2\sigma\mu \quad . \quad . \quad . \quad \text{(ii)}$$

(iii) Consider now a unit cube which is sheared, the shearing stress being given by $p_1 = -p_2 = p$, and $p_3 = 0$. Then

$$p = \lambda\epsilon_1 + \mu(\epsilon_2 + \epsilon_3),$$

$$-p = \lambda\epsilon_2 + \mu(\epsilon_3 + \epsilon_1),$$

and $$0 = \lambda\epsilon_3 + \mu(\epsilon_1 + \epsilon_2).$$

Adding the first two of these equations we find

$$0 = (\lambda + \mu)(\epsilon_1 + \epsilon_2) + 2\mu\epsilon_3,$$

and, if we use the last of the above equations to eliminate $(\epsilon_1 + \epsilon_2)$ from that just obtained, we have

$$0 = -(\lambda + \mu)\lambda\epsilon_3 + 2\mu^2\epsilon_3.$$

Hence

$$\epsilon_3 = 0, \text{ and therefore } \epsilon_1 + \epsilon_2 = 0,$$

unless $$\lambda^2 + \mu\lambda - 2\mu^2 = 0,$$

i.e. $$\lambda = \mu, \quad \text{or} \quad \lambda = -2\mu,$$

and this will not occur in practice unless either, as the immediate sequel shows, $n = 0$, or $\beta = 0$. If we write $\epsilon = \epsilon_1 = -\epsilon_2$, we have

$$p = (\lambda - \mu)\epsilon.$$

But $\epsilon = \frac{1}{2}\phi$, where ϕ is the shear strain [cf. p. 267].

$$\therefore n = \frac{p}{\phi} = \frac{1}{2}(\lambda - \mu) \quad . \quad . \quad . \quad \text{(iii)}$$

It has just been shown that

$$3\beta = \lambda + 2\mu,$$

and

$$2n = \lambda - \mu.$$

$$\therefore\ \mu = \beta - \tfrac{2}{3}n, \qquad \text{and} \qquad \lambda = \beta + \tfrac{4}{3}n.$$

$$\therefore\ \sigma = \frac{\mu}{\lambda + \mu} = \frac{\beta - \frac{2}{3}n}{2\beta + \frac{2}{3}n} = \frac{3\beta - 2n}{6\beta + 2n} \quad . \quad \text{(iv)}$$

Also

$$E = (\lambda - 2\sigma\mu) = \frac{9\beta n}{3\beta + n} \quad . \quad . \quad . \quad \text{(v)}$$

after substituting the values for λ, μ and σ already found.

The above relations for E and for σ have been established because they are obtainable at once from the relations between λ and μ, and the elastic constants β and n. Now, in practice, it is found that usually E and n can be determined more readily experimentally than can either β or σ; it is therefore necessary to express σ and β in terms of E and n.

Since

$$E = \lambda - 2\sigma\mu = \lambda - \frac{2\mu^2}{\lambda + \mu},$$

and

$$2n = \lambda - \mu,$$

$$E - 2n = \mu\left[1 - \frac{2\mu}{\lambda + \mu}\right] = \mu . \frac{\lambda - \mu}{\lambda + \mu},$$

$$\therefore\ \frac{E - 2n}{2n} = \frac{\mu}{\lambda + \mu} = \sigma,$$

i.e.

$$\sigma = \left(\frac{E}{2n} - 1\right) \quad . \quad . \quad . \quad . \quad \text{(vi)}$$

Solving the equations $E = \lambda - 2\sigma\mu$, and $2n = \lambda - \mu$, we have

$$\mu = \frac{E - 2n}{1 - 2\sigma} = \frac{(E - 2n)n}{3n - E}. \qquad \left[\because\ \sigma = \frac{E}{2n} - 1\right]$$

$$\therefore\ \lambda = 2n + \mu = \frac{4n^2 - nE}{3n - E},$$

and

$$\beta = \tfrac{1}{3}(\lambda + 2\mu) = \frac{En}{9n - 3E} \quad . \quad . \quad \text{(vii)}$$

On the limiting values for Poisson's ratio.—It has just been shown that

$$\sigma = \frac{3\beta - 2n}{6\beta + 2n}.$$

Now n is essentially a positive quantity so that its least value is zero. The maximum value for Poisson's ratio, which is the value of σ when $n = 0$, is therefore 0·5. On the other hand, when $n = \infty$, we have

$$[\sigma]_{n\to\infty} = \left[\frac{3\frac{\beta}{n} - 2}{6\frac{\beta}{n} + 2}\right]_{n\to\infty} = -1.$$

Hence $$-1 < \sigma < 0{\cdot}5.$$

In practice, however, σ must be a positive quantity for a negative value would imply that if a wire, for example, of such a material were subjected to stretching forces it would expand in a direction normal to that along which it is stretched. No known metal (or other substance) behaves in this way. Hence, in practice,

$$0 < \sigma < 0{\cdot}5.$$

Some other important relations between the elastic constants.—Consider an isotropic material, the three principal stresses within it being p_1, p_2 and p_3. Each will produce the same strain as if it acted alone. Now a tensile stress p_1 will be accompanied by a strain ϵ_1, in its own direction, given by

$$\epsilon_1 = \frac{p_1}{\mathrm{E}},$$

where E is Young's modulus of elasticity for the given material. The strains along the directions of the two other principal stresses due to the principal stress p_1, are given by

$$\epsilon_2 = \epsilon_3 = -\sigma\epsilon_1 = -\frac{\sigma}{\mathrm{E}}\cdot p_1.$$

When all three stresses are operative

$$\epsilon_1 = \frac{p_1}{\mathrm{E}} - \frac{\sigma}{\mathrm{E}}(p_2 + p_3), \quad \epsilon_2 = \frac{p_2}{\mathrm{E}} - \frac{\sigma}{\mathrm{E}}(p_3 + p_1),$$

and $$\epsilon_3 = \frac{p_3}{\mathrm{E}} - \frac{\sigma}{\mathrm{E}}(p_1 + p_2).$$

If $p_1 = p_2 = p_3$, the strain is constant and given by

$$\epsilon = \frac{p}{\mathrm{E}}(1 - 2\sigma).$$

$$\therefore\ \beta = \frac{p}{\Delta} = \frac{p}{3\epsilon} = \frac{1}{3}\left[\frac{\mathrm{E}}{1 - 2\sigma}\right].$$

Example.—Calculate the strains in a rod of isotropic material stretched in such a way that all lateral strain is prevented.

From the relations just established, we have,

$$\epsilon_1 = \frac{p_1}{E} - \frac{\sigma}{E}(p_2 + p_3), \quad \epsilon_2 = 0 = \frac{p_2}{E} - \frac{\sigma}{E}(p_3 + p_1),$$

and $$\epsilon_3 = 0 = \frac{p_3}{E} - \frac{\sigma}{E}(p_1 + p_2).$$

$$\therefore \; p_2 = p_3.$$

Hence $$\epsilon_1 = \frac{p_1}{E} - \frac{2\sigma}{E} \cdot p_2,$$

and $$0 = \frac{p_2}{E} - \frac{\sigma}{E}(p_2 + p_1).$$

$$\therefore \; p_2\left[\frac{1}{E} - \frac{\sigma}{E}\right] = \frac{\sigma}{E} \cdot p_1.$$

$$\therefore \; p_2 = p_1\left(\frac{\sigma}{1 - \sigma}\right).$$

$$\therefore \; \epsilon_1 = \frac{p_1}{E} - \frac{2\sigma}{E}\left(\frac{\sigma}{1 - \sigma}\right)p_1 = \frac{p_1}{E}\left[\frac{1 - \sigma - 2\sigma^2}{1 - \sigma}\right].$$

If $\sigma = 0{\cdot}25$, $$\epsilon_1 = \frac{5}{6} \cdot \frac{p_1}{E},$$

i.e. the stretch is one-sixth less than in the case of the same rod stretched by an equal force but which is free to contract in directions normal to that of the applied pull.

Stresses in a thin spherical shell subjected to internal pressure.—When a spherical shell contains fluid under pressure there is, if we neglect the weight of the fluid itself, a uniform excess

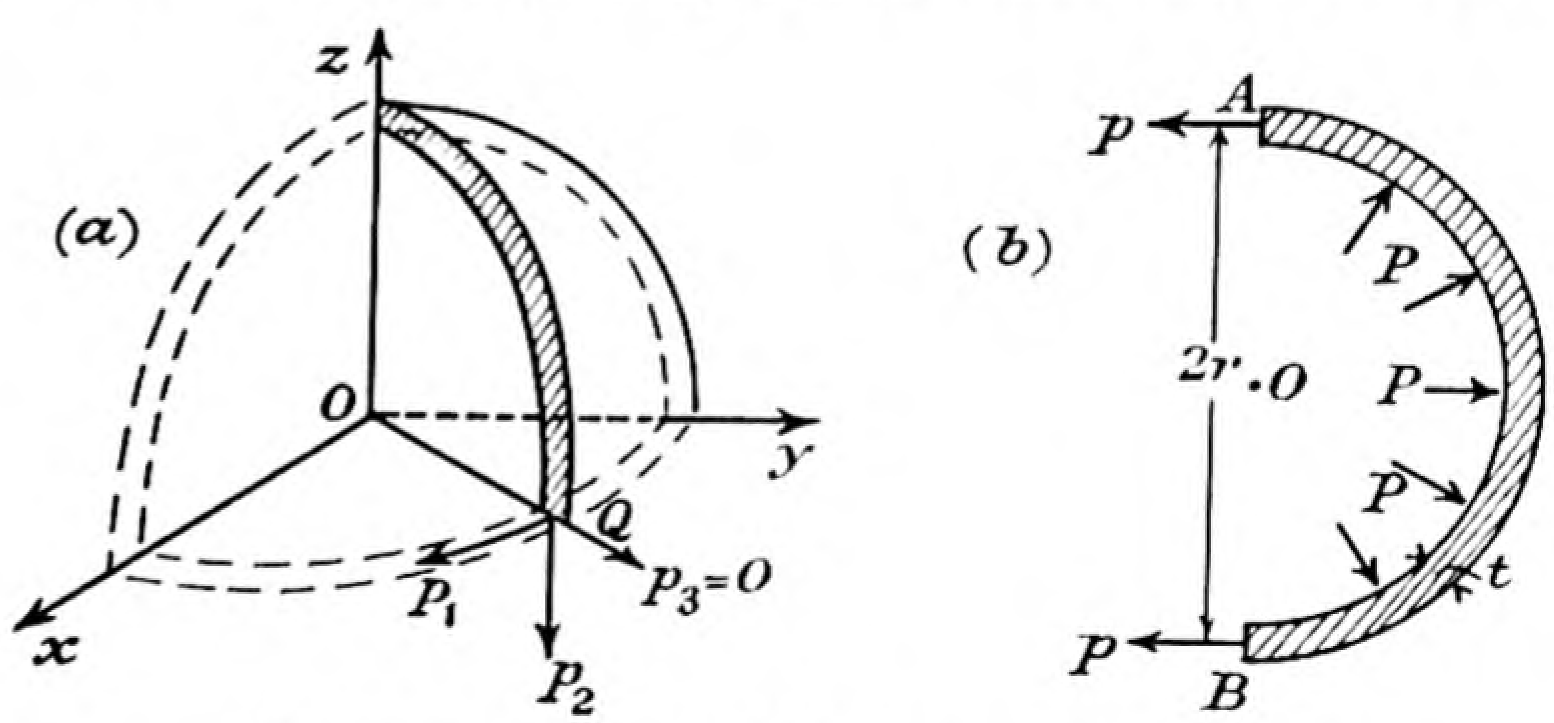


FIG. 7·29.—Stresses within a thin spherical shell.

pressure normal to the walls at every point. Let this excess pressure be P. Consequently the material of the shell is stressed. Consider Fig. 7·29(a) which shows a one-eighth part of a spherical shell less a portion which has been removed. We may obtain all

the information required about the stresses within the shell if we consider the three principal stresses at a point Q in the shell and in the xOy plane. These stresses are p_1, p_2 and p_3 and act respectively along a normal to an element of the section at Q, parallel to zO and along OQ produced. Now for thin shells the radial stress p_3 may be taken as zero and, on account of symmetry $p_1 = p_2 = p$, say; p is known as a ***tangential stress.*** To evaluate p let us imagine the whole shell to be divided into two equal portions by a diametral plane AOB, Fig. 7·29(b), and the shaded portion removed but the forces on the remaining hemisphere unaltered. Consider the equilibrium of this hemisphere. If t is the thickness of the shell, small compared with its radius r, the forces exerted on the part ACB by the portion which has been removed have a resultant $2\pi rt.p$, and this must be equal to $\mathrm{P}.\pi r^2$, which is the resultant force on the hemisphere due to the excess pressure within. Hence

$$2\pi rt.p = \pi r^2.\mathrm{P},$$

so that

$$p = \frac{\mathrm{P}r}{2t}.$$

Since $p_1 = p_2$ and p_3 may be taken as zero, the tangential strain is constant at all points in the shell and is given by

$$\epsilon = \frac{p_1}{\mathrm{E}} - \frac{\sigma}{\mathrm{E}}.p_2, \qquad \text{[cf. p. 310]}$$

$$= \frac{\mathrm{P}r}{2t\mathrm{E}}(1 - \sigma),$$

where E is Young's modulus and σ is Poisson's ratio for the material of the shell.

Stresses in a thin cylindrical shell subjected to internal pressure.—Let P be the excess pressure within a thin cylindrical shell, such as a pipe or boiler containing fluid under pressure. Then, neglecting the weight of the fluid, the pressure is everywhere constant and normal to the walls. The material will be stressed in such a way that the principal stresses are (a) in the direction tangential to the perimeter of a transverse section, (b) along a line parallel to the axis of the cylinder and (c) along the radius through the point in question. These are shown in Fig. 7·30(a) and known respectively as the ***circumferential*** or ***hoop stress*** p_1, the ***longitudinal stress*** p_2 and the ***radial stress*** p_3; they are the three principal stresses. The radial stress, in thin shells, is always negligible in comparison with the other two principal stresses.

Consider the forces on one half of that portion of the shell which lies between two planes at unit distance apart and normal to the

axis of the shell. The stress $p_3 \to 0$ and is neglected. To find a value for p_1 we use the fact that the forces at A and B exerted on ACB by the portion of the section which has been removed, are balanced by the resultant force on ACB which is assumed unaltered. The force at A (or B) is $p_1 t$ while the force on the curved surface due to the excess pressure of the fluid is $(2r \times 1)\mathrm{P}$. For equilibrium

$$2p_1 t = 2r\mathrm{P}, \quad \text{or} \quad p_1 = \frac{\mathrm{P}r}{t}.$$

To calculate the longitudinal stress, p_2, imagine that a portion of the cylinder has been removed as indicated in Fig. 7·30(*b*), and

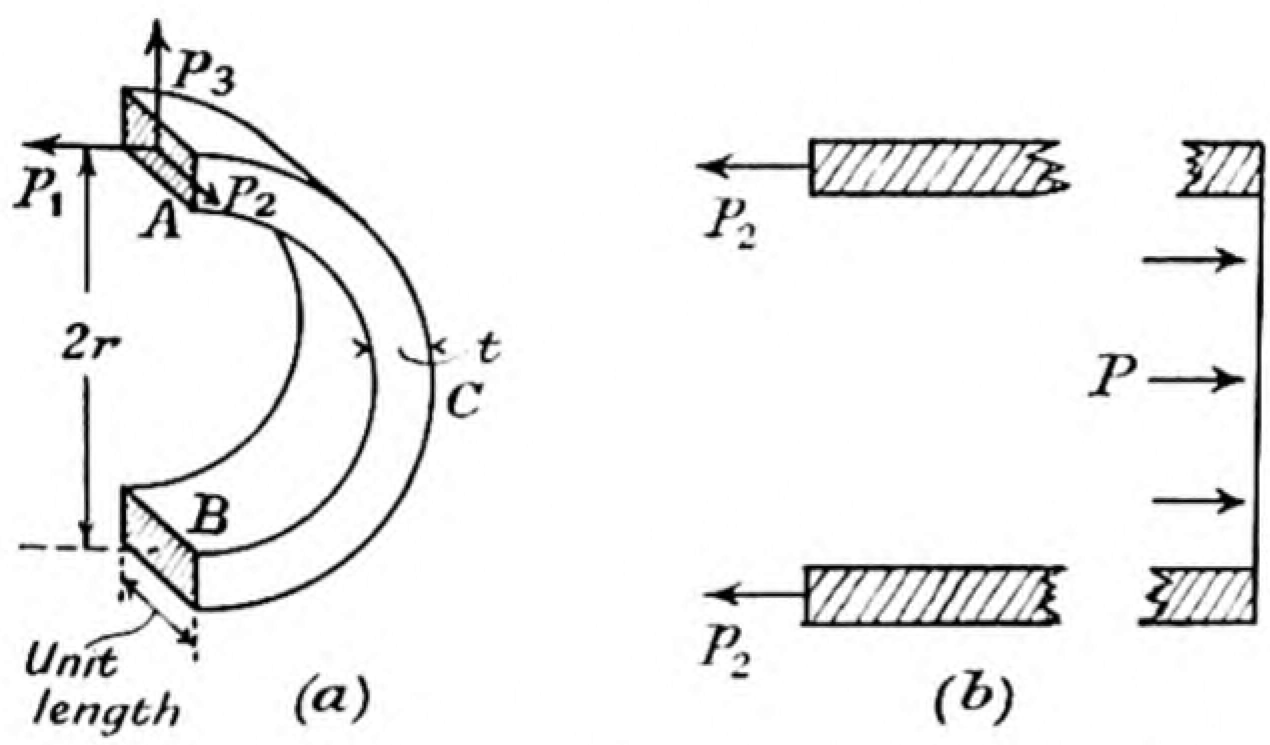


Fig. 7·30.—Stresses within a thin cylindrical shell.

consider the equilibrium of the remaining portion. Then $2\pi rt \,.\, p_2$ is the force exerted in the material due to the portion which has been removed. This must be equal to $\pi r^2 . \mathrm{P}$, the force on the end of the cylinder, if the whole is in equilibrium. Hence

$$p_2 = \frac{\mathrm{P}r}{2t}.$$

Thus the circumferential stress is twice the longitudinal stress. Let ϵ_1 and ϵ_2 be the circumferential strain and the longitudinal strain respectively. Then

$$\epsilon_1 = \frac{p_1}{\mathrm{E}} - \frac{\sigma}{\mathrm{E}} \cdot p_2 = \frac{\mathrm{P}r}{\mathrm{E}t}(1 - \tfrac{1}{2}\sigma),$$

and

$$\epsilon_2 = \frac{p_2}{\mathrm{E}} - \frac{\sigma}{\mathrm{E}} \cdot p_1 = \frac{\mathrm{P}r}{\mathrm{E}t}(\tfrac{1}{2} - \sigma).$$

A thin oval cylinder.—In thin cylinders of any oval section the hoop stress varies from point to point along the periphery and, in addition, the oval section tends to become more circular. Thus if

we consider the section of an elliptical cylinder, as shown in Fig. 7·31 the bending moment will tend to increase the curvature in the region of points of minimum curvature such as B and D; at A and C, where the curvature is a maximum, the bending moments will tend to decrease the curvature. The hoop stress at A or C may be calculated as in the previous paragraphs; it is given by

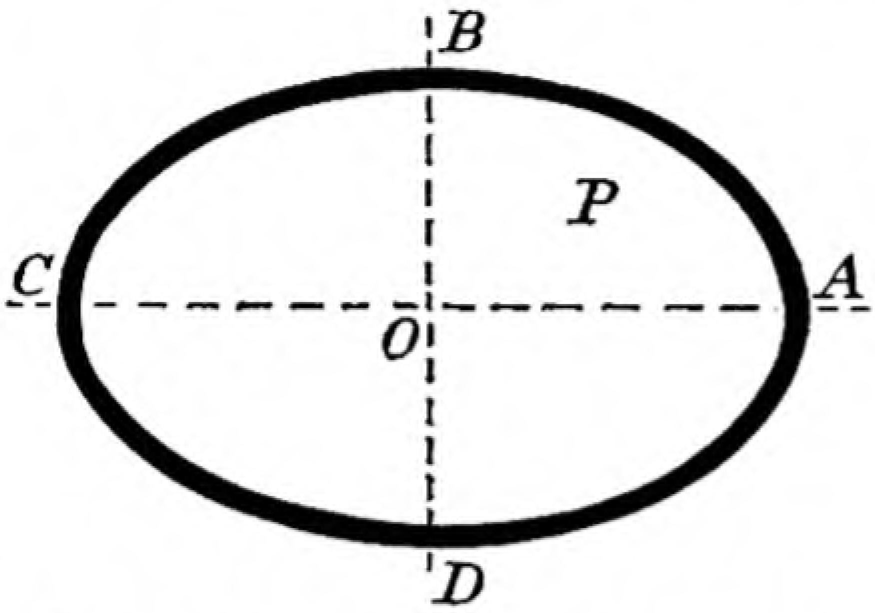


FIG. 7·31.—Stresses within a thin oval cylinder.

$$(p_1)_A = \frac{\mathrm{P.OA}}{t}.$$

At B or D the hoop stress is $\mathrm{P.OB} \div t$.

Similarly, the longitudinal stress is given by

$$p_2 = \frac{\begin{Bmatrix}\text{P} \times \text{internal cross-} \\ \text{sectional area of pipe}\end{Bmatrix}}{t \times \text{perimeter of pipe}}.$$

Experimental determination of Poisson's ratio for indiarubber.—AB, Fig. 7·32(a), is a piece of thin-walled indiarubber tubing about one metre long and 2 cm. in diameter. The lower end is closed and carries a hook from which various loads may be suspended. The upper end A is closed by a rubber bung which carries a capillary tube C. The rubber tube and part of C are filled with air-free water and the apparatus is supported on a rigid stand. When a load is applied to the rubber tube it is stretched and at the same time the internal volume increases and the meniscus in C is observed to fall. The change in internal volume is deduced from the fall of the meniscus and, if the extension of the tube AB is observed, a value for Poisson's ratio for indiarubber may be found.

Let v be the internal volume of the rubber tube, l its length and r its internal radius of cross-section. Let us suppose that when a load of mass m is applied l and r become $l + \Delta l$ and $r + \Delta r$, respectively. At the same time let the internal volume, v, become $v + \Delta v$. We have

$$v = \pi r^2 l, \quad \text{so that} \quad \Delta v = \pi r^2 \Delta l + 2\pi r l \Delta r.$$

To discover how these changes are connected with σ, Poisson's ratio for indiarubber, let us consider the three principal stresses at a point P, Fig. 7·32(b), in the walls of the tube. They will be p_1, p_2 and p_3 as shown; p_1 is the axial stress, p_2 the circumferential stress and p_3 the radial stress, which may be taken to be zero since the walls of the tube are thin. Now

$$p_1 = \frac{mg}{\pi(\mathrm{R}^2 - r^2)},$$

where R is the outer radius of the cross-section of the tube. To determine p_2 consider unit length of the tube and let this be divided

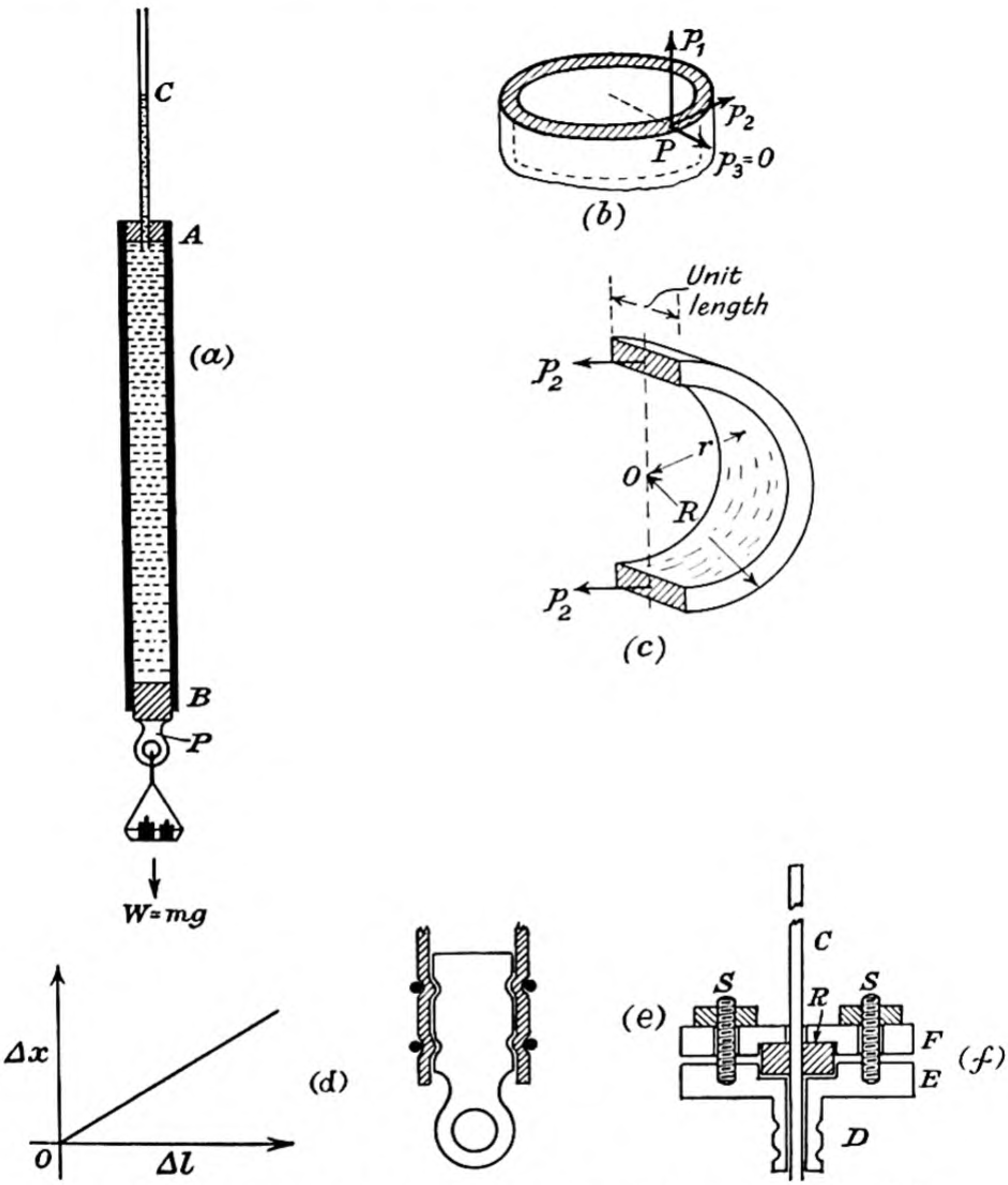


FIG. 7·32.—Experimental determination of Poisson's ratio for indiarubber.

longitudinally into two equal parts; one of these is shown in Fig. 7·32(c). The forces on the plane rectangular ends of the section, and due of course to the portion of the tube which has been removed, amount to

$$2p_2[1 \times (R - r)],$$

and this must be zero since the resultant force on the curved surfaces of the section is zero, if we assume that the pressure inside the tube is atmospheric at all points; this is tantamount to neglecting the pressure due to the water in the tube. Thus

$$p_2 = 0.$$

If ϵ_1 is the axial or longitudinal strain and E is Young's modulus for indiarubber, we have, cf. p. 310,

$$\epsilon_1 = \frac{p_1}{\mathrm{E}} - \frac{\sigma}{\mathrm{E}} p_2 = \frac{p_1}{\mathrm{E}}, \qquad [\because p_2 = 0]$$

and this is $\frac{\Delta l}{l}$, so that a value for E may be found.

Now $$\epsilon_2 = \frac{p_2}{\mathrm{E}} - \frac{\sigma}{\mathrm{E}} p_1 = -\frac{\sigma}{\mathrm{E}} p_1 = -\sigma\epsilon_1$$

But $$\epsilon_2 = \frac{2\pi(r + \Delta r) - 2\pi r}{2\pi r} = \frac{\Delta r}{r}.$$

Hence $$\sigma = -\frac{\epsilon_2}{\epsilon_1} = -\frac{\Delta r}{r} \div \frac{\Delta l}{l}$$

But $$\frac{\Delta v}{\Delta l} = \pi r^2 + 2\pi r l \frac{\Delta r}{\Delta l},$$

whence $$\sigma = -\frac{l}{r}\left(\frac{\Delta v}{\Delta l} - \pi r^2\right)\frac{1}{2\pi r l} = \frac{1}{2}\left[1 - \frac{1}{\pi r^2}\cdot\frac{\Delta v}{\Delta l}\right].$$

If a is the radius of the capillary tube,

$$\Delta v = \pi a^2 . \Delta x,$$

where Δx is the fall of the level of the meniscus in C, since a fall in the water level here corresponds to an increase in v.

$$\therefore \sigma = \frac{1}{2}\left[1 - \frac{a^2}{r^2}\cdot\frac{\Delta x}{\Delta l}\right].$$

If therefore a series of loads is used and corresponding values of Δx and Δl are plotted, the slope of the straight line so obtained, cf. Fig. 7·32(d), enables a value for σ to be obtained.

The changes in x and in l should be measured with the aid of a travelling microscope; a small pin, P, attached to the plug at P enables Δl to be observed.

In order to make the joints between the rubber tube and the rest of the apparatus free from leak the following devices may be used. The end B is plugged with a piece of solid brass with two grooves round its curved surface and a brass ring attached to its lower end

so that a known load may be suspended from it. This plug is slipped into the lower end of the rubber tube and secured with iron wire, cf. Fig. 7·32(*e*).

Since the liquid used is water it is essential that the capillary tube should be clean so that the meniscus may move freely in it. To enable this to be done easily and at the same time obtain a leak-free joint, a solid brass plug D is soldered to a brass plate E, Fig. 7·32(*f*) This plate carries two screws, S, and a second brass plate slips over these screws. Both E and F are recessed to take nearly one half of a cylindrical rubber bung through which the glass tube C passes, the plates and plug having been drilled, after assembling them, to take the glass tube C. Nuts on S hold F in position and enable the plates so to grip the rubber bung that the joint is leak-free. The whole is readily dismantled for cleaning, etc.

Strain energy.—Since an elastic body yields in the direction of the forces straining it, it follows that such forces do work during the process of deforming the body. To find the work done on each element of the body we may regard each such element as a separate body under the action of forces at its surface. For an internal element the forces at the surface are those which arise from the internal stresses, but for an element at the surface of the body the straining forces are the actions of the contiguous parts of the body together with the action of the forces applied at the external boundary. The total work done on all elements of the body is the work done in straining the body. The total work calculated in this way is exactly equal to the work done by the external forces since the internal forces always occur in pairs whose constituents are equal and opposite.

If a strained body is permitted slowly to regain its unstrained state it can, if it is perfectly elastic, react on the objects maintaining the strain at any instant with exactly the same forces as when the strain was increasing. Thus the strained body can do the same amount of work on external bodies in regaining its natural state as the external bodies did on the strained body in producing the strain. From these considerations the fact emerges that a perfectly elastic body is capable of giving back all the work that has been put into it and it is therefore convenient to regard the work done on such a body as energy stored within the body; it is termed ***elastic-energy*** or ***strain-energy***.

The strain energy in a stretched wire.—Let consider a wire which is fixed at one end while the wire itself may be elongated by a steadily increasing force F. [In reality the wire is under the action of a pair of opposing forces, one at either end, but since one end of the wire is assumed fixed only one force does work.] Let l denote the

natural length of the wire and S its cross-sectional area. Let ϵ be the strain when the extensional stress is p. The extension of the rod is then $l\epsilon$ and the work done when this extension of the rod increases by $\delta(l\varepsilon)$ is

$$\delta W = (pS)\,\delta(l\epsilon) = pSl(\delta\epsilon).$$

$\therefore$ Total work done as the strain is increased from 0 to ϵ_0 is given by

$$W = \int_0^{\epsilon_0} pSl\,d\epsilon = SlE\int_0^{\epsilon_0} \epsilon\,d\epsilon \qquad \left[\because\ E = \frac{p}{\epsilon}\right]$$

$$= \tfrac{1}{2}SlE\epsilon_0^{\,2}.$$

If F_0 is the maximum force applied, $E = \dfrac{F_0}{S} \div \varepsilon_0$.

$$\therefore\ W = \tfrac{1}{2}F_0 \times (l\epsilon_0)$$

$$= \tfrac{1}{2}F \times \text{the extension of the wire.}$$

The strain-energy per unit volume of the wire is therefore

$$[\tfrac{1}{2}F_0 \times l\epsilon_0] \div Sl = \tfrac{1}{2}p_0\epsilon_0,$$

where p_0 is the maximum tensile stress.

The above relation may be obtained in a less formal manner as follows. Suppose that the straight line OA, Fig. 7·33, represents the relation between the stretching force F and the extension of the wire; since OA is linear the limit of perfect elasticity cannot have been passed. The potential energy gained by the wire as the extension increases, say from m to n, is $Mm.mn$; hence the strain energy in the wire, when the state represented by the point A is reached, is

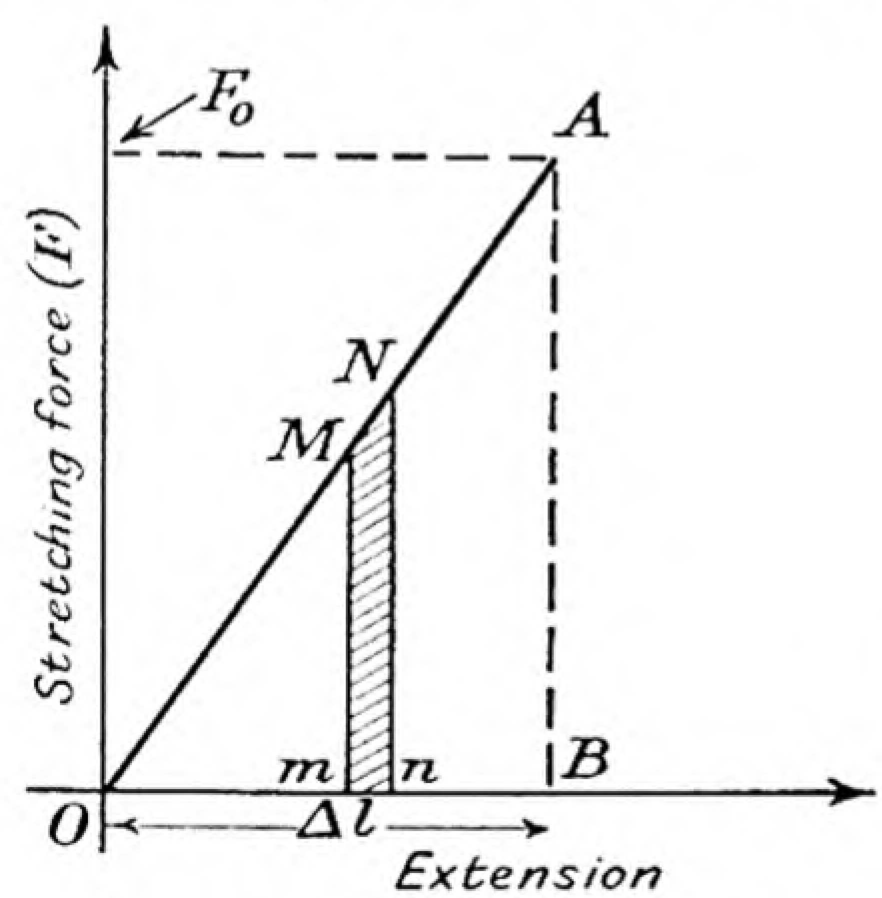


FIG. 7·33.—Strain energy in a stretched wire.

$$\tfrac{1}{2}OB.BA = \tfrac{1}{2}\Delta l.F_0,$$

where Δl is the extension OB and F_0 the corresponding force.

If l is the original length of the wire, r its radius of cross-section, the strain energy per unit volume is

$$\frac{\tfrac{1}{2}\Delta l.F_0}{\pi r^2.l} = \tfrac{1}{2}(\text{stress} \times \text{strain}),$$

where the stress and strain are the values appropriate to the point A.

Strain energy of a rod under variable tension.—Let us now consider the strain-energy associated with a uniform rod, length l and cross-sectional area S, hanging vertically under its own weight as shown in Fig. 7·34. The expression $\frac{1}{2}p_0\epsilon_0$ for the density of the strain energy has been obtained for a wire subjected to a constant stress along its whole length. When the stress is variable, as in the problem now being considered, the above expression is still valid for each element into which the rod may be divided. Thus for the element of the rod of length δx, we have

$$\delta W = \tfrac{1}{2}(p\epsilon)S\,\delta x$$
$$= \frac{1}{2}\frac{p^2 S}{E}\,.\,\delta x, \qquad \left[\because E = \frac{p}{\epsilon}.\right]$$

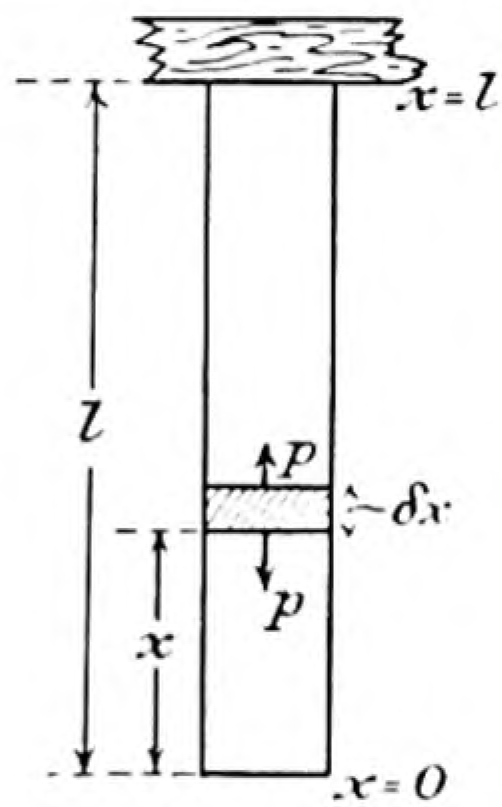


FIG. 7·34.—Strain energy in a rod under variable tension.

But p is caused by the weight of the rod below the element considered. Thus if ρ is the density of the material of the rod, g the intensity of gravity, we have

$$p = \frac{(xS\rho)}{S}g = x\rho g.$$

$$\therefore\ \delta W = \tfrac{1}{2}x^2\rho^2 g^2\,.\,\frac{S}{E}\,.\,\delta x.$$

$$\therefore\ W = \tfrac{1}{2}\int_0^l x^2\rho^2 g^2\frac{S}{E}\,dx = \frac{1}{6}\frac{S}{E}\,.\,\rho^2 g^2 l^3.$$

Strain-energy associated with a pure shear strain.—Let a naturally rectangular block of isotropic and homogeneous material be subjected to a pair of shearing stresses across those faces which are normal to the plane of the diagram. The system is shown in Fig. 7·35. Let the face of which CD is the trace be fixed and let the upper face move from a position defined by A_0B_0 to AB; ϕ is the shear strain.

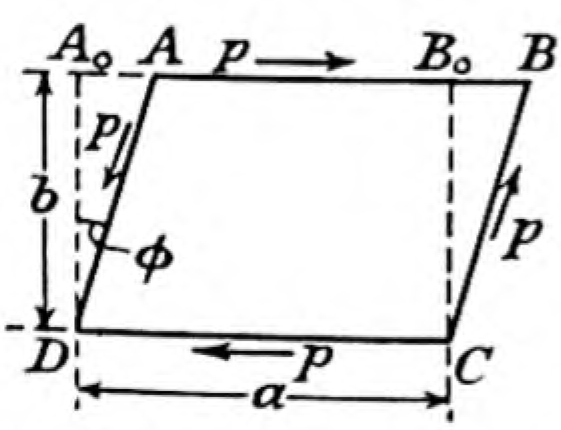


FIG 7·35.—Strain energy associated with a pure shear strain.

If S is the area of the face of which AB is the trace, the work done in effecting the above displacement is

$$\tfrac{1}{2}(pS)A_0A = \tfrac{1}{2}pS(b\phi), \qquad [\text{if } b = A_0D.]$$

But if n is the rigidity of the material of the block, $p = n\phi$.

$$\therefore\ \text{Work done} = \tfrac{1}{2}pS\,.\,b\,.\,\frac{p}{n}.$$

$$\therefore\ \text{Energy stored per unit volume} = \frac{1}{2}\frac{p^2}{n} = \tfrac{1}{2}p\phi.$$

Strain energy in a twisted wire.—Now consider a wire twisted by means of a couple Γ, applied to one end, the other end being fixed. The relation between Γ and θ, the angle of twist at the movable end of the wire, is a linear one provided the limit of perfect elasticity for the material of the wire has not been exceeded; this relation is shown graphically in Fig. 7·36. If the couple is increased from Γ to $\Gamma + \delta\Gamma$, the angle of twist increasing by $\delta\theta$, the work done on the specimen is $\Gamma\,\delta\theta$, for Γ is the average value of the couple. This energy goes to increase the strain energy of the wire. It is represented on the diagram by the element of area MNnm. Hence the area below the line OA represents the energy stored in the specimen due to the fact that it is strained. If θ_0 is the angle of twist when the couple has reached its maximum value, the elastic limit not having been exceeded, the strain energy, W, of the specimen is given by

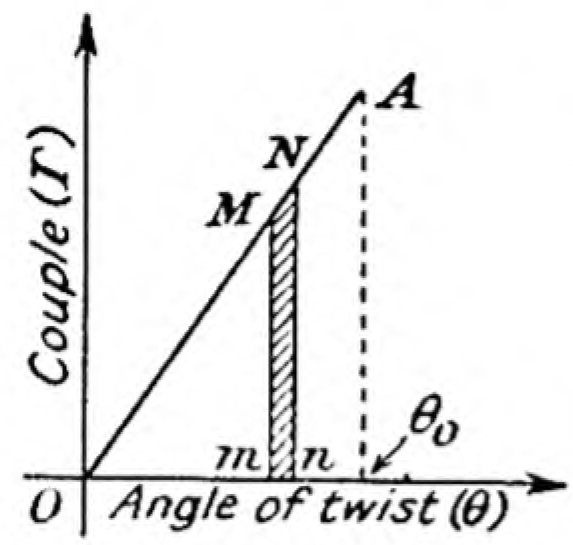


FIG. 7·36.—Strain energy in a twisted wire.

$$W = \int_0^{\theta_0} \Gamma\, d\theta.$$

But, when the twist in the wire is θ,

$$\Gamma = \frac{\pi n \theta}{2l} \,.\, a^4. \qquad \text{[cf. p. 286]}$$

Hence
$$W = \frac{\frac{1}{4}\pi n \theta_0^{\,2}}{l} \,.\, a^4 = \tfrac{1}{2}\Gamma_0\theta_0 = \frac{\Gamma_0^{\,2} l}{\pi n a^4},$$

where Γ_0 is the maximum external couple applied to the wire.

To obtain another expression for the strain energy in a twisted wire, we may use the fact that the strain-energy per unit volume is

$$\tfrac{1}{2}p\phi = \frac{1}{2}\frac{p^2}{n},$$

where the symbols have their usual meanings. Now at all points in a cylindrical section of the wire, defined by radii r and $r + \delta r$, the stress is constant, cf. p. 289, its value being

$$\frac{r}{a}\,p_0,$$

where p_0 is the maximum stress in the wire. Since the volume of the cylindrical element being considered is $2\pi rl.\delta r$, the energy associated with it is given by

$$\delta W = \frac{1}{2} \cdot \frac{r^2}{a^2} \cdot p_0{}^2 \cdot \frac{1}{n} \cdot 2\pi rl.\delta r$$

$$= \frac{\pi l p_0{}^2}{na^2} \cdot r^3\, \delta r.$$

$$\therefore\ W = \frac{\pi l p_0{}^2}{na^2} \int_0^a r^3\, dr = \frac{\pi a^2 l}{4n} \cdot p_0{}^2,$$

which is one-half the energy which would be stored in the wire if all elements were stressed to the maximum value p_0.

Example.—A mass m is supported by an inextensible light cord wound round the rim of a horizontal 'wheel and axle'. The system is kept in equilibrium by a cord round the axle; one end of this cord is tied to the lower end of a light helical spring whose upper end is anchored as shown in Fig. 7·37. If a is the radius of the axle, b that of the wheel and I its moment of inertia about its axis of revolution, find the period of small oscillations of the system.

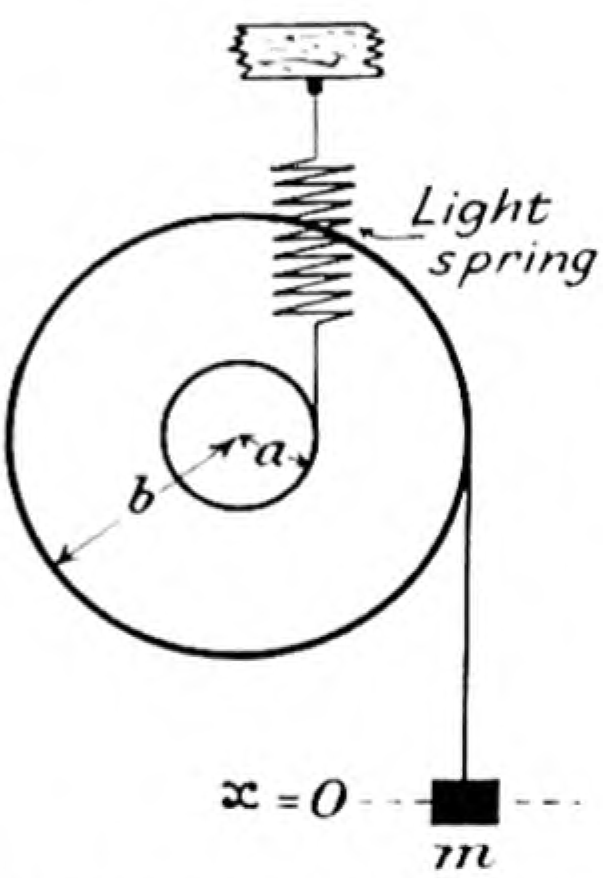


Fig. 7·37.—A loaded wheel and axle maintained in equilibrium by means of a light helical spring.

Let z be the extension of the spring when the system is in static equilibrium and k the constant of proportionality between the stretching force in the spring and its extension. Since the force is kz, for equilibrium

$$mgb = kza,$$

where g is gravity. The energy stored in the spring is

$$\tfrac{1}{2}(kz)z = \tfrac{1}{2}kz^2,$$

and this is the potential energy of the system if the potential energy of the mass m is considered to be zero when it is in this position.

Now suppose that the system is disturbed so that m moves along a vertical line. When it has risen a distance x the extension in the spring will be reduced by $\left(\frac{a}{b}\right)x$, so that the total energy of the system will be

$$\tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}I\left(\frac{\dot{x}}{b}\right)^2 + \tfrac{1}{2}k(z - \frac{a}{b}x)^2 + mgx,$$

and this must be equal to the energy stored in the spring before the

system m is set in motion, $\frac{1}{2}kz^2$. Hence

$$\tfrac{1}{2}\dot{x}^2\left\{m + \frac{I}{b^2}\right\} + x\left\{mg - \frac{a}{b}.kz\right\} + \tfrac{1}{2}x^2k\left(\frac{a}{b}\right)^2 = 0.$$

Since the second bracketed term in this expression is zero, it follows that the motion is simple harmonic and its period, T, given by

$$T = 2\pi\sqrt{\left(m + \frac{I}{b^2}\right) \div k\left(\frac{a}{b}\right)^2}.$$

$$= 2\pi\frac{b}{a}\sqrt{\frac{1}{k}\left(m + \frac{I}{b^2}\right)}.$$

Resilience and shock.—Colloquially, the term resilience is understood to indicate the capacity of a strained body to spring back when the applied forces are removed; in technical works the term ***resilience*** is generally used to denote the amount of work done in straining a body, the elastic limit not being exceeded.

Consider, therefore, a uniform rod of length l; when it is subjected to a pair of stretching forces, F, let Δl be the extension and p the stress within the bar. If E is Young's modulus for the material of the bar,

$$E = p \div \left(\frac{\Delta l}{l}\right) = \frac{F}{S}.\frac{l}{\Delta l},$$

where S is the cross-sectional area of the bar. The strain energy, U, or the work done in producing the above extension is given by, cf. p. 318,

$$U = \tfrac{1}{2}(\text{stress} \times \text{strain}) \times \text{volume}$$

$$= \frac{1}{2}\left(p.\frac{p}{E}\right) \times \text{volume}$$

$$= \tfrac{1}{2}(\text{stress})^2 \times \text{volume} \div \text{Young's modulus}.$$

Hence the resilience per unit volume is $\frac{1}{2}\frac{p^2}{E}$.

The resilience of an elastic body is a measure of its capacity to resist a blow or a mechanical shock without acquiring a permanent set.

Proof resilience.—The greatest strain energy per unit volume which can be stored in a piece of material without permanent strain is called its ***proof resilience***. Thus if p_0 is the elastic limit for a material in the form of a wire, its proof resilience is $\frac{1}{2} . \frac{{p_0}^2}{E}$.

Example.—Suppose a uniform rod AB Fig. 7·38, of length l and mean radius r is suspended vertically downwards as shown. Let it be provided with a collar at its lower end to receive a falling load of mass m—this is represented by the ring R. Suppose that this ring falls a height h before hitting the collar, the bar then lengthening by an amount x.

The potential energy lost by the falling ring, viz. $mg(h + x)$, is utilized in stretching the rod. If p_0 is the stress in the rod when the extension is a maximum, the restoring force will be $p_0(\pi r^2)$, so that the work done in stretching the rod is $\frac{1}{2}(p_0 \pi r^2)x$. Hence

$$mg(h + x) = \tfrac{1}{2}p_0\pi r^2 x.$$

But

$$x = \frac{p_0 l}{\mathrm{E}}.$$

$$\therefore\ 2mg\left(h + \frac{p_0 l}{\mathrm{E}}\right) = \frac{\pi r^2 p_0^2 l}{\mathrm{E}}.$$

$$\therefore\ p_0 = \frac{2mg\,.\,\dfrac{l}{\mathrm{E}} \pm \sqrt{4m^2g^2\,.\,\dfrac{l^2}{\mathrm{E}^2} + 8\,\dfrac{\pi r^2 l}{\mathrm{E}}\,.\,mgh}}{\dfrac{2\pi r^2 l}{\mathrm{E}}}$$

$$= \frac{mg}{\pi r^2} \pm \frac{\sqrt{m^2g^2l^2 + 2\pi r^2 l\,.\,mgh\,.\,\mathrm{E}}}{\pi r^2 l}.$$

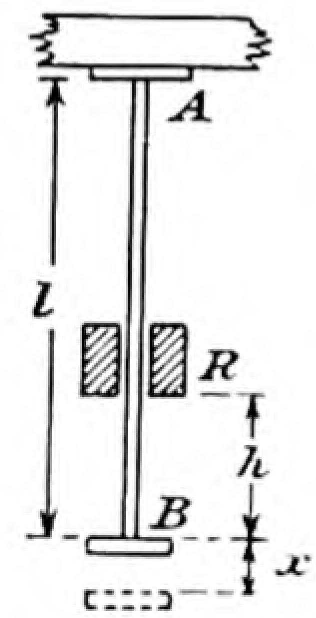


Fig. 7·38.

If $h \to 0$, $\quad p_0 \to \dfrac{mg}{\pi r^2} \pm \dfrac{mg}{\pi r^2}$, i.e. $\dfrac{2mg}{\pi r^2}$ or zero.

The zero value has no physical meaning. The other shows that if a load is applied so that a rod is suddenly stretched, the stress, when the extension of the rod is a maximum, is twice that produced when the load is applied gradually.

Influence of temperature on the elastic properties of iron.—Curves illustrating how the elastic properties of iron (almost pure) vary with temperature are shown in Fig. 7·39. Young's modulus

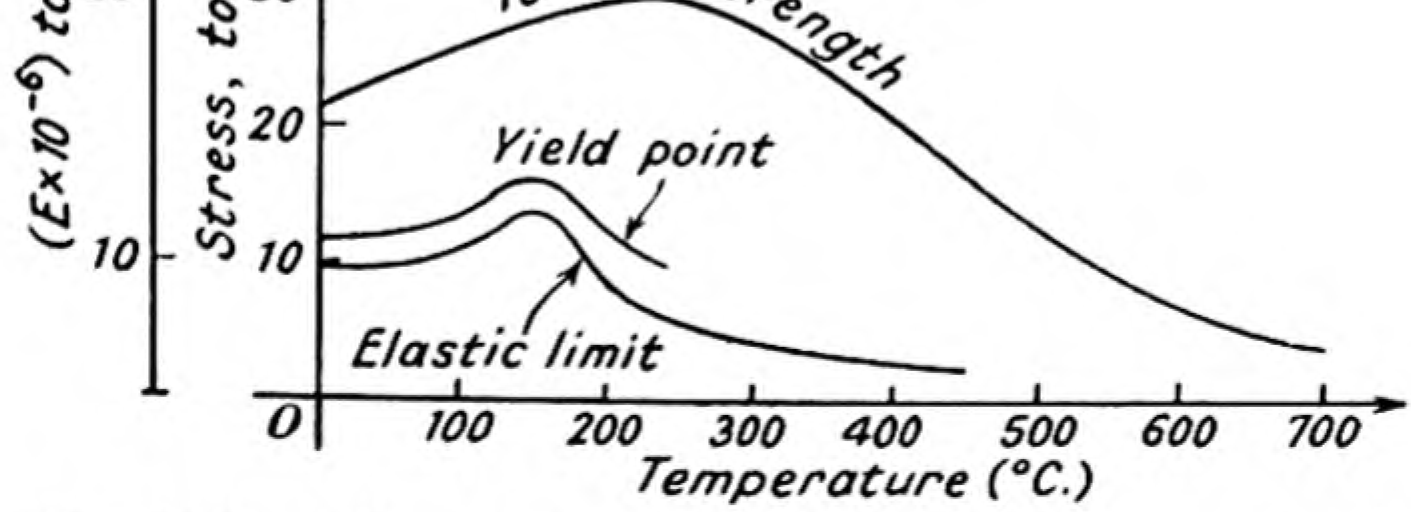


Fig. 7·39.—Influence of temperature on elastic properties of iron (almost pure).

decreases almost uniformly with rise in temperature but the tensile strength has a maximum value of about 33 ton.-wt.in.$^{-2}$ at 250° C. The curves for the elastic limit and the yield-point each show a maximum at about 150° C. but the curves are not symmetrical about the maxima.

The impact of smooth spheres: Newton's experimental law.—Newton discovered by experiment that if two spheres impinge directly, i.e. the direction of motion is along the common normal at the point of impact, the relative velocity of the spheres along the line of centres immediately after impact is $-e$ times the relative velocity before impact, where e is a constant depending on the nature of the materials in the spheres. For hard substances such as steel and ivory e is nearly unity, but for soft substances such as lead it is small.

If the spheres impinge obliquely their relative velocity after impact resolved along their common normal at the instant of impact is $-e$ times their relative velocity before impact in the same direction.

The quantity e is essentially positive and is known as the ***coefficient of restitution*** or ***resilience.***

If u_1 and u_2 are the component velocities of two spheres before impact along their common normal at the instant of impact and v_1 and v_2 the component velocities in the same direction after impact, the law stated above may be expressed as

$$v_1 - v_2 = -e(u_1 - u_2),$$

or the velocity of separation is e times the velocity of approach.

Newton arrived at the above conclusion by carrying out an experiment on the following lines. Two spheres are suspended by parallel threads O_1A_1 and O_2A_2, Fig. 7·40, whose lengths are adjusted so that when hanging freely the spheres are just in contact with their centres in a horizontal line in the plane of the diagram.

The sphere A_1 is then raised to P, the string being tight, the point P being at a height z_1 above A_1 and released from rest. Its speed on hitting the sphere A_2 is $+\sqrt{2gz_1}$. If the spheres rise to respective heights z_2 and z_3 before coming momentarily to rest, their speeds immediately after the impact are $+\sqrt{2gz_2}$ and $+\sqrt{2gz_3}$ respectively.

If the spheres rebound, their velocity of separation is

$$\sqrt{2g}(\sqrt{z_2} + \sqrt{z_3}),$$

while the velocity of approach was $\sqrt{2gz_1}$. Newton found that $\dfrac{\sqrt{z_2} + \sqrt{z_3}}{\sqrt{z_1}}$ was independent of z_1 and the masses of the colliding spheres. It did depend on the nature of the spheres.

Instead of measuring the heights to which the spheres are displaced it is more accurate to measure their horizontal displacements, x. Then if the supporting threads are long compared with z, the

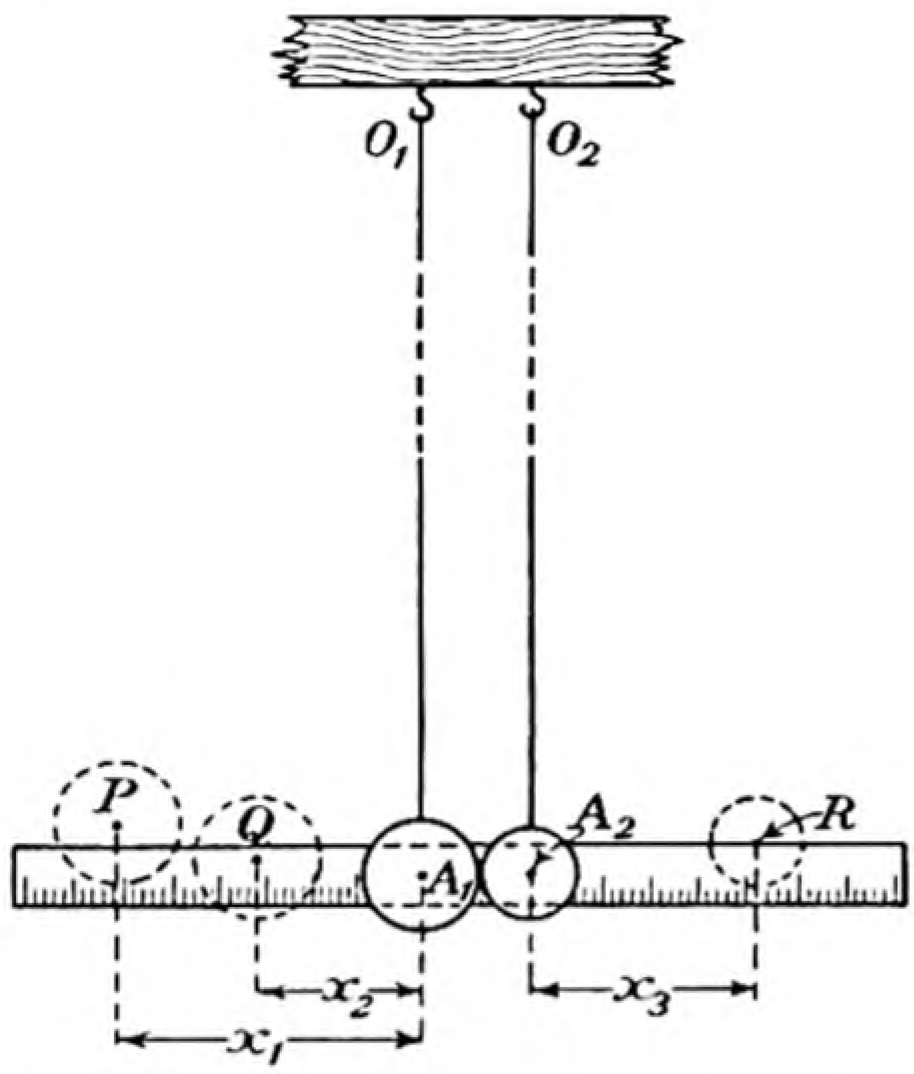


FIG. 7·40.—The impact of two smooth spheres.

velocity is directly proportional to x for $x^2 = z(2l - z) \doteqdot 2zl$, where l is the length of the supporting thread. But $v^2 = 2gz$, so that

$$v^2 = \frac{g}{l} x^2,$$

or v is directly proportional to x. Hence, with the notation indicated on the diagram,

$$e(x_1 - 0) = -(-x_2 - x_3),$$

or

$$e = \frac{x_2 + x_3}{x_1}.$$

Having shown that $e = \dfrac{x_2 + x_3}{x_1}$, if M is the mass of the sphere A_1 and m that of the sphere A_2, the momentum equation may be written

$$Mu_1 + m(0) = Mv_1 + mv_2.$$

$$\therefore\ Mx_1 = -Mx_2 + mx_3$$

$$= -M(ex_1 - x_3) + mx_3.$$

$$\therefore\ x_1 = \frac{M + m}{M}\left(\frac{1}{1 + e}\right)x_3.$$

By plotting $x = x_3$, $y = x_1$, we should therefore obtain a straight line of the slope $\left(\frac{1}{1+e}\right)\frac{M+m}{M}$. Thus a value for the coefficient of restitution may be obtained.

Kinetic energy lost by impact.—In general, when bodies collide, their total kinetic energy is reduced. Only the case of direct impact need be considered because when the impact is oblique it is only the velocity components in the line of centres that can effect a change in kinetic energy; the component velocities normal to the line of centres remain unaltered since there is no impulsive action in this direction.

If m_1 and m_2 are the masses of the colliding spheres, then, with the notation already defined,

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2,$$

and

$$v_1 - v_2 = -e(u_1 - u_2).$$

To eliminate v_2 from these two equations we have

$$v_2 = v_1 + e(u_1 - u_2),$$

so that

$$m_1(u_1 - v_1) = m_2[v_1 + eu_1 - (1 + e)u_2].$$

But

$$m_2(u_1 - v_1) = m_2u_1 - m_2v_1,$$

so that by adding together these two last equations we get

$$u_1 - v_1 = \frac{m_2}{m_1 + m_2}(1 + e)(u_1 - u_2),$$

which expresses the change in velocity of the first body in terms of the difference between the initial velocities of the two bodies.

The energy lost is

$$\tfrac{1}{2}m_1(u_1^2 - v_1^2) + \tfrac{1}{2}m_2(u_2^2 - v_2^2) = W \text{ (say)}$$

$$\begin{aligned}\therefore\ 2W &= m_1(u_1 - v_1)(u_1 + v_1) + m_2(u_2 - v_2)(u_2 + v_2)\\ &= m_1(u_1 - v_1)(u_1 + v_1) - m_1(u_1 - v_1)(u_2 + v_2)\\ &= m_1(u_1 - v_1)(u_1 - u_2 + v_1 - v_2)\\ &= m_1(u_1 - v_1)(u_1 - u_2)(1 - e)\\ &= \frac{m_1m_2}{m_1 + m_2}(u_1 - u_2)^2(1 - e^2),\end{aligned}$$

since

$$(u_1 - v_1) = \frac{m_2}{m_1 + m_2}(1 + e)(u_1 - u_2).$$

The loss in kinetic energy (which eventually appears as heat) is thus expressed in terms of three factors which are all essentially positive. The loss is zero either if e is unity, or if $u_1 = u_2$, in which

instance the bodies would not collide. Hence, in general, it may be said that there is always a loss of energy since $0 < e < 1$.

Experimental study of the impact of a steel sphere on a plane horizontal surface.—A steel ball-bearing is supported by a small electromagnet above the surface. On switching off the current [cf. however p. 597] the ball falls and the height reached by the rebounding ball may be determined by adjusting the retort-stand ring R, Fig. 7·41, so that the rebounding sphere just reaches the lowest plane of the ring. The distances Z and z are measured and the coefficient of restitution is found from the expression

$$(v_1 - v_2) = -e(u_1 - u_2).$$

Now $u_2 = v_2 = 0$ and $u_1 = +\sqrt{2gZ}$, while v_1 is $-\sqrt{2gz}$. Thus

$$e = +\sqrt{\frac{z}{Z}}.$$

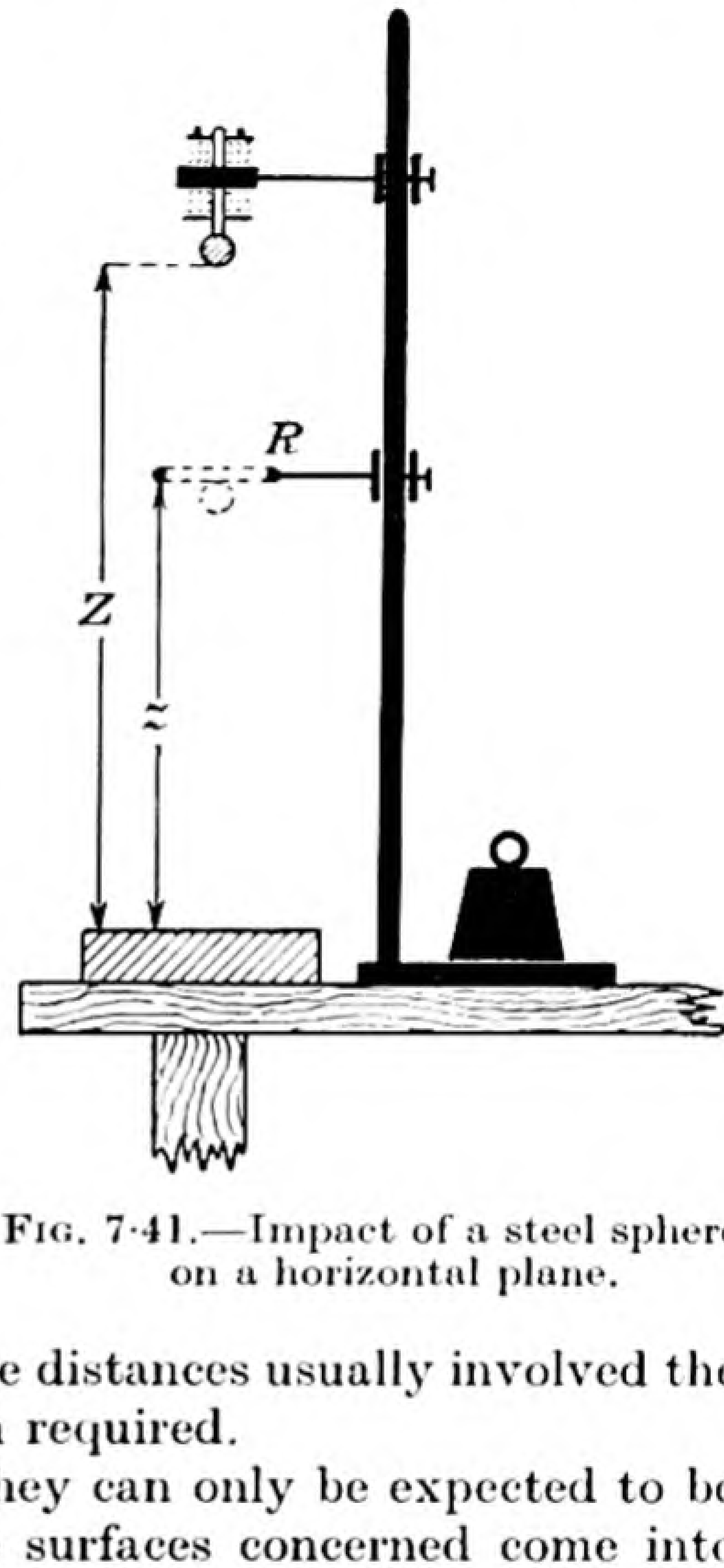


Fig. 7·41.—Impact of a steel sphere on a horizontal plane.

Strictly speaking the distances Z and z are those through which the centre of gravity of the sphere falls and then rises but with the large distances usually involved the correction on this account is seldom required.

If the observations are repeated they can only be expected to be concordant if fresh portions of the surfaces concerned come into contact since the deformation round the contact area may not disappear entirely.

Example.—A small sphere falls from a height z upon an immovable fixed plane. If e is the coefficient of resilience, prove that when the sphere ceases to rebound it will have described a total distance $\frac{(1 + e^2)}{(1 - e^2)} z$ and that the particle is permanently at rest after a time $\frac{1+e}{1-e}\sqrt{\frac{2z}{g}}$ after its release.

The downward speed of the particle when it first hits the plane is $v = \sqrt{2gz}$; the speed of rebound is ev and this is the speed with which it is assumed that the sphere strikes the plane on the second occasion. Thus the speeds of rebound are

$$ev,\ e^2v,\ e^3v,\ \ldots$$

and the distances to which the sphere ascends after the first, second, third rebounds . . . are

$$\frac{(ev)^2}{2g}, \frac{(e^2v)^2}{2g}, \frac{(e^3v)^2}{2g}, \ldots$$

or

$$e^2z, e^4z, e^6z, \ldots$$

Thus the total distance traversed is

$$z + 2[e^2z + e^4z + \ldots] = z\,\frac{1+e^2}{1-e^2}.$$

Now the initial time of fall is $\sqrt{\frac{2z}{g}}$, and the times in which the speeds ev, e^2v, . . . each become zero are

$$\frac{ev}{g}, \frac{e^2v}{g}, \ldots \text{ or } e\sqrt{\frac{2z}{g}}, e^2\sqrt{\frac{2z}{g}}, \ldots$$

Hence the time spent on its complete journey is

$$\sqrt{\frac{2z}{g}} + 2\left[\sqrt{\frac{2z}{g}}(e + e^2 + e^3 + \ldots)\right]$$

$$= \left(\frac{1+e}{1-e}\right)\sqrt{\frac{2z}{g}}.$$

EXAMPLES VII

7·01. Give a short account of what occurs when a nickel wire is stretched under a gradually increasing load.

Define ***elastic limit, perfect elasticity, Poisson's ratio.***

When a rubber cord is stretched the change in volume is negligible compared with the change in shape. Show that Poisson's ratio for rubber is 0·5.

7·02. A particle of mass m hangs at the end of a light elastic string of unstretched length l and modulus of stretch λ. Prove that in the position in which the string is stretched by an amount z, the combined gravitational and elastic potential energy is $\frac{\lambda z^2}{2l} - mgz + \text{const.}$ By considering the stationary value of this expression, determine the position of equilibrium of the particle. $\left[z = mg\,\frac{l}{\lambda}\right]$

7·03. Explain how to find Young's modulus for copper provided in the form (a) of a wire about 20 cm. long and of about 1 mm. radius, and (b) of a rod about 1 metre long and of rectangular cross-section about 2 cm. × 0·5 cm. Give experimental details in both cases and an account of the theory in one of them. (G)

7·04. A bar of high tensile steel 1·783 in. in diameter is placed in a testing machine. The extension is $6{\cdot}65 \times 10^{-3}$ in. for a load of mass 36 tons. If the extension is measured on a length 15·0 in. find a value for Young's modulus for the steel. [$3{\cdot}25 \times 10^4$ ton-wt. in.$^{-2}$]

7·05. Derive an expression for the thrust inwards due to a rope under tension passing round a smooth curve.

Calculate the limiting pressure inside a cylindrical boiler of 3 ft. radius, the sides being $\frac{1}{8}$ in. thick and made of a material which can stand a limiting tension of 40 ton-wt. in.$^{-2}$. [42 atmos.]

7·06. A fly-wheel consists of a uniform lead rim fastened to the edge of a circular steel plate. If lead has a density of 11·4 gm.cm.$^{-3}$ and a tensile strength of $1{\cdot}60 \times 10^8$ dyne.cm.$^{-2}$, calculate the maximum velocity of the rim before breakage will occur; and find also the maximum number of revolutions per second that can be made when the diameter of the rim is 20 cm. [188]

7·07. A sheet of indiarubber, 10 cm. square and 2 cm. thick, has one face fastened to a vertical wall, and to the other face a piece of wood is cemented. When a load of mass 30 kg. is hung from the wood, the wood is found to be finally lowered by 0·030 cm. Deduce the coefficient of rigidity of the rubber and the energy stored per cubic centimetre of the rubber. [$1{\cdot}96 \times 10^7$ dyne.cm.$^{-2}$, $2{\cdot}21 \times 10^3$ erg.cm.$^{-3}$]

7·08. Describe and discuss two methods of producing a shearing strain. Show by consideration of two different types of strain that the strain energy per unit volume of an isotropic solid is

$$\tfrac{1}{2}(\text{final stress} \times \text{final strain}).$$

A vertical rubber cord, fixed at its upper end, extends 10 cm. when a load is applied gradually at its lower end. Find the maximum extension of the cord when the load is applied suddenly, explaining the method of calculation. [20 cm., 0·63 sec.]

Find also the period of the subsequent oscillations. (G)

7·09. Derive an expression for the rigidity of a metal in the form of a wire, in terms of the period of the torsional oscillations about the axis of the wire, and other quantities.

Discuss the relative importance of errors in the various measurements you would make to obtain the value of the rigidity of the metal by this method.

7·10. Describe and explain how the values of Young's modulus, E, and of Poisson's ratio, σ, could be determined for glass using a uniform bar.

If $E = 7{\cdot}2 \times 10^{11}$ dyne.cm.$^{-2}$ and $\sigma = 0{\cdot}25$ for glass, calculate the modulus of rigidity of this substance assuming it to be isotropic, deriving the formula required. (S) [$2{\cdot}88 \times 10^{11}$ dyne.cm.$^{-2}$]

7·11. Define the coefficient of rigidity of an isotropic solid. From the definition derive an expression for the couple required to twist a wire about its axis. A wire of length 50·2 cm. and diameter 0·092 cm. was suspended vertically from a fixed support. To the lower end of the wire a short vertical cylinder of 4·36 cm. diameter was fixed. Two fine threads were wrapped round the cylinder, from which they passed horizontally over pulleys and supported scale-pans at their free ends. When a load of mass 100 gm. was placed in each scale-pan the lower end of the wire was found to be twisted through 162°. Calculate a value for the coefficient of rigidity for the material of which the wire was made. [$2{\cdot}70 \times 10^{11}$ dyne.cm.$^{-2}$]

7·12. The upper end of a vertical uniform wire of length l cm. and of radius a cm. is fixed and a horizontal couple of moment G dyne.cm. is applied at its lower end. Deduce, from first principles, an expression for the angle of twist of the lower end, the modulus of rigidity of the material of the wire being n dyne.cm.$^{-2}$.

Describe a method for determining the modulus of rigidity of the wire and discuss how the accuracy of the result is affected by errors of one per cent. in the measurements of the length and radius of the wire. (G)

7·13. Show that, in order to twist a wire of radius a through an angle ϕ per unit length, couples of moment $\frac{1}{2}\pi n a^4 \phi$ must be applied to its ends, n being the rigidity modulus of the material of the wire.

When a certain mass is attached to the end of a vertical wire 0·6 mm. in diameter, the extension produced is 1·12 mm. and the period of torsional oscillation is 15 sec. If the radius of gyration of the mass about the axis of the wire is 3 cm., calculate the ratio of Young's modulus to the rigidity modulus. (G) [2·49]

7·14. It is desired to replace a solid shaft of circular cross section by a hollow shaft that shall be equally twisted by the application of equal and opposite couples at its ends. If the external diameter of the hollow shaft is to be equal to twice its internal diameter, what fraction of material will be saved by the change? [0·13]

7·15. A metal bar, soldered to the lower end of a wire, vibrates torsionally. How will the time of oscillation be changed by halving (*a*) the linear dimensions of the wire, (*b*) the linear dimensions of the metal bar, (*c*) the linear dimensions of both bar and wire?

[(*a*) Increased $2\sqrt{2}$ times; (*b*) reduced $4\sqrt{2}$ times; (*c*) halved.]

7·16. A steel rod, of circular cross-section, is twisted about its axis by the application of equal and opposite couples at its ends. Assuming that the coefficient of rigidity for steel is $8{\cdot}5 \times 10^{11}$ dyne.cm.$^{-2}$, and that the maximum stress that the steel can stand is $5{\cdot}5 \times 10^{9}$ dyne.cm.$^{-2}$, calculate the maximum average energy per cubic centimetre that can be stored in the steel. [Express your answer in mechanical and in thermal units.] [$1{\cdot}78 \times 10^{9}$ erg.cm.$^{-3}$, 0·42 cal.cm.$^{-3}$]

7·17. Derive a formula which would enable you to calculate the bulk modulus of a solid in terms of other elastic moduli. Outline a method of finding the bulk modulus of brass by direct experiment.

7·18. If the equation $\dfrac{1}{E} = \dfrac{1}{3n} + \dfrac{1}{9\beta}$ is used to find β for a metal for which $E = 2 \times 10^{12}$ dyne.cm.$^{-2}$ and $n = 1 \times 10^{12}$ dyne.cm.$^{-2}$, find the error in the calculated value of β, if E and n are each correct to within 1 per cent. [$0{\cdot}67 \times 10^{12}$ dyne.cm$'^{-2}$, 0·050]

7·19. Obtain an expression for Young's modulus for an isotropic substance in terms of other elastic moduli. Define Poisson's ratio and discuss its limiting values for isotropic materials.

7·20. A cylindrical rod of radius a cm. and length l cm. is fixed at one end, and its other end is twisted through an angle ϕ radians by means of a couple applied there. Derive an expression for the modulus of rigidity of the material of the rod. If Young's modulus for the material were known, how would you deduce a value for the bulk modulus?

7·21. Define the three principal moduli of elasticity, and Poisson's ratio. Show that if E, β, and n be respectively Young's modulus, the bulk modulus and the modulus of rigidity, then $E = \dfrac{9n\beta}{3\beta + n}$.

A hollow cylindrical cast-iron column is 10·0 in. in external diameter and 8·0 in. in internal diameter, and 10·0 ft. long. How much will it contract under a load of 60 tons?

[Young's modulus for cast-iron may be taken as 8000 ton.-wt. in.$^{-2}$.] [0·032 in.]

7·22. A uniform glass tube is suspended from a rigid support so that a length of 2 metres lies below the support. This part of the tube is filled with water. When the tube is loaded the portion below the support stretches by 0·13 cm. while the length of tube occupied by the water column lengthens by only 0·080 cm. Obtain a value for Poisson's ratio for glass. [0·19]

7·23. A vertical wire is loaded by weights which produce total extensions of 0·4 cm. and 0·6 cm. respectively. Assuming that the elastic limit for the material of the wire has not been exceeded, compare the amount of work done in producing these extensions. [1 : 2·25]

7·24. A mass m attached to the lower end of a light elastic vertical string of natural length l, causes an increase of amount a in the length of the string. When m is at rest and then displaced vertically show that the period of oscillation of the mass when released is given by $T = 2\pi\sqrt{\frac{a}{g}}$.

When a mass μ is added to m the time of oscillation is increased by 20 per cent. Compare the masses μ and m. [$\mu = 0{\cdot}44\ m$.]

7·25. Two cylindrical steel shafts have the same mass per unit length. One is solid, while the other which is hollow has an external radius twice the internal radius. Compare their ***torsional rigidities*** and the maximum strains produced by equal twisting couples. [5 : 3, 5 : 3]

7·26. A suspension thread consists of a wire of length l and radius a carrying below it a wire of length $l/10$ and radius $10a$. In each instance the material is the same. The top end is clamped while the lower end is twisted about a vertical axis. Find (a) the ***torsional rigidity*** of the whole suspension and (b) the ratio of the angles through which the lower ends of each part of the suspension are twisted.

$$\left[(a)\ \frac{1}{2(1 + 10^{-5})}\cdot\frac{na^4}{l};\qquad (b)\ 10^5.\right]$$

7·27. Define ***coefficient of restitution*** and describe how you would determine its value for (a) two steel spheres, and (b) a steel sphere striking a horizontal metal surface.

A metal ball of mass 200 gm. falls from a height of 121 cm. on to a horizontal marble floor and rebounds for the first time to a height of 100 cm. Calculate (a) the total distance traversed by the ball until it comes to rest on the floor, and (b) the total time (from the instant of release) which elapses until this occurs. How much energy does the ball lose as a result of the first two impacts with the floor?

[(a) 1273 cm., (b) 10·4 sec., $7{\cdot}52 \times 10^6$ erg.]

7·28. A helium atom moving in a straight line with a velocity u makes direct impact with a hydrogen atom at rest. If the two atoms behave like perfectly elastic spheres i.e. $e = 1$, and the mass of a helium atom is four times that of a hydrogen atom, obtain a value for (a) the velocity of the hydrogen atom after the collision, and (b) the fraction of its energy lost by the helium atom. [(a) $\frac{8}{5}$u, (b) 0·64]

7·29. From a point in a smooth horizontal plane a small sphere is projected with a velocity u at an angle θ to the horizon. If e is the coefficient of restitution, prove that the sphere will move a distance $\frac{u^2 \sin 2\alpha}{g\ 1 - e}$ along the plane before it ceases to rebound. Discuss what happens afterwards.

7·30. A stream of particles each of mass m and moving with constant velocity u impinges normally on a plate which is being moved towards the particles with a steady velocity u_0. If e is the coefficient of restitution show that the change in kinetic energy of each particle is

$$\tfrac{1}{2}m(1 + e)[u^2 - u_0{}^2 - e(u + u_0)^2],$$

and that a force $Nm(1 + e)(u + u_0)$ is required to maintain the plate in steady motion if N particles strike it per unit time.

7·31. A hard steel ball is dropped from a height of 30 cm. on to a smooth block of glass; the rebound is 28·5 cm. Obtain values for (*a*) the coefficient of restitution of steel against glass, (*b*) the time the ball continued to bounce after it was first released, (*c*) the total distance traversed by the ball. [(*a*) 0·975, (*b*) 19·3 sec., (*c*) 1170 cm.]

7·32. An atom of mass $1{\cdot}66 \times 10^{-24}$ gm. and velocity 200 km.sec.$^{-1}$ collides head on with an atom of mass $6{\cdot}64 \times 10^{-24}$ gm. which is at rest. Assuming the collision to be perfectly elastic find the velocity of each atom after impact and the change in energy of each atom. Express the changes (*a*) in ergs, (*b*) in electron volts. [The electron charge may be taken as $-4{\cdot}80 \times 10^{-10}$ e.s.u. and 300 volts as 1 e.s.u. of potential.]

[(He) 80 km.sec.$^{-1}$, (H) -120 km.sec.$^{-1}$; $1{\cdot}62 \times 10^{-12}$ erg. or 132 eV.]

7·33. A thin uniform horizontal rod of length $2a$ is supported from its ends by vertical fibres, each of length l. Determine the frequency, f, of small oscillations of the rod about a vertical axis through its C.G. in terms of l and g, the acceleration due to gravity, and justify any approximation made.

Show (*a*) that if the rod supports a load concentrated at its centre, f^2 becomes a linear function of the load, and (*b*) how the expression for f is modified if the suspending fibres have torsional rigidity. [It may be assumed that when one end of a fibre is twisted through an angle ϕ with respect to the other, the restoring couple is $b\phi$, where b is a constant.]

G(alt'd).

CHAPTER VIII

THE THEORY OF THE BENDING OF BEAMS AND OF HELICAL SPRINGS

Loaded beams. Bending moment.—Let us consider an elastic body, in equilibrium under the action of a system of external forces, to be divided arbitrarily into two sections. Then the system of external forces acting on either part must be in equilibrium with the internal forces acting on the part considered due to the other part. The discussion of the equilibrium of beams under the action of external forces which follows will be limited to uniform beams, the longitudinal axes of which are horizontal when strains are absent in them. It will further be supposed that the external forces on the beam are all vertical, and that the stresses in the beam, when loaded, do not exceed the elastic limit.

Suppose that Fig. 8·01(a) represents a portion of a horizontal beam divided by a vertical plane at A into two portions, called (i) and (ii) respectively. Then that portion of the beam lying to the right of A exerts on the left-hand portion a system of forces which must be in equilibrium with the external forces acting on that portion; this system must reduce to a vertical force and a couple. Similarly, the second portion of the beam must experience a system of forces due to the first portion; this system will also reduce to a vertical force and a couple, each respectively equal in magnitude but opposite in direction to those acting on the first portion of the beam. If portion (i) experiences a system of forces reducing to a force S and a couple M due to the second portion, portion (ii) will experience a system of forces reducing to a force $-$S and a couple $-$M, since the whole beam is in equilibrium. The forces S are the ***shearing forces***, and the couples M are the ***bending moments*** at the transverse section A in the beam. In the earlier part of this chapter we shall develop means whereby S and M may be found. As a convention we shall always calculate the values of S and M appropriate to that portion of the beam on the left-hand side of the imaginary dividing line. The values of S and M will be considered positive when they act in the directions shown in Fig. 8·01(a).

Consider a beam OX, of negligible weight per unit length, acted upon by the system of external forces indicated in Fig. 8·01(b). Suppose it is required to find the shearing force and the bending moment at the section A in the beam. Consider the portion of the

beam to the left of A as indicated in Fig. 8·01(c). Let x be the distance of A from the end O of the beam, where $x_2 \leqslant x \leqslant x_3$. If S_A and M_A are the shearing force and the bending moment

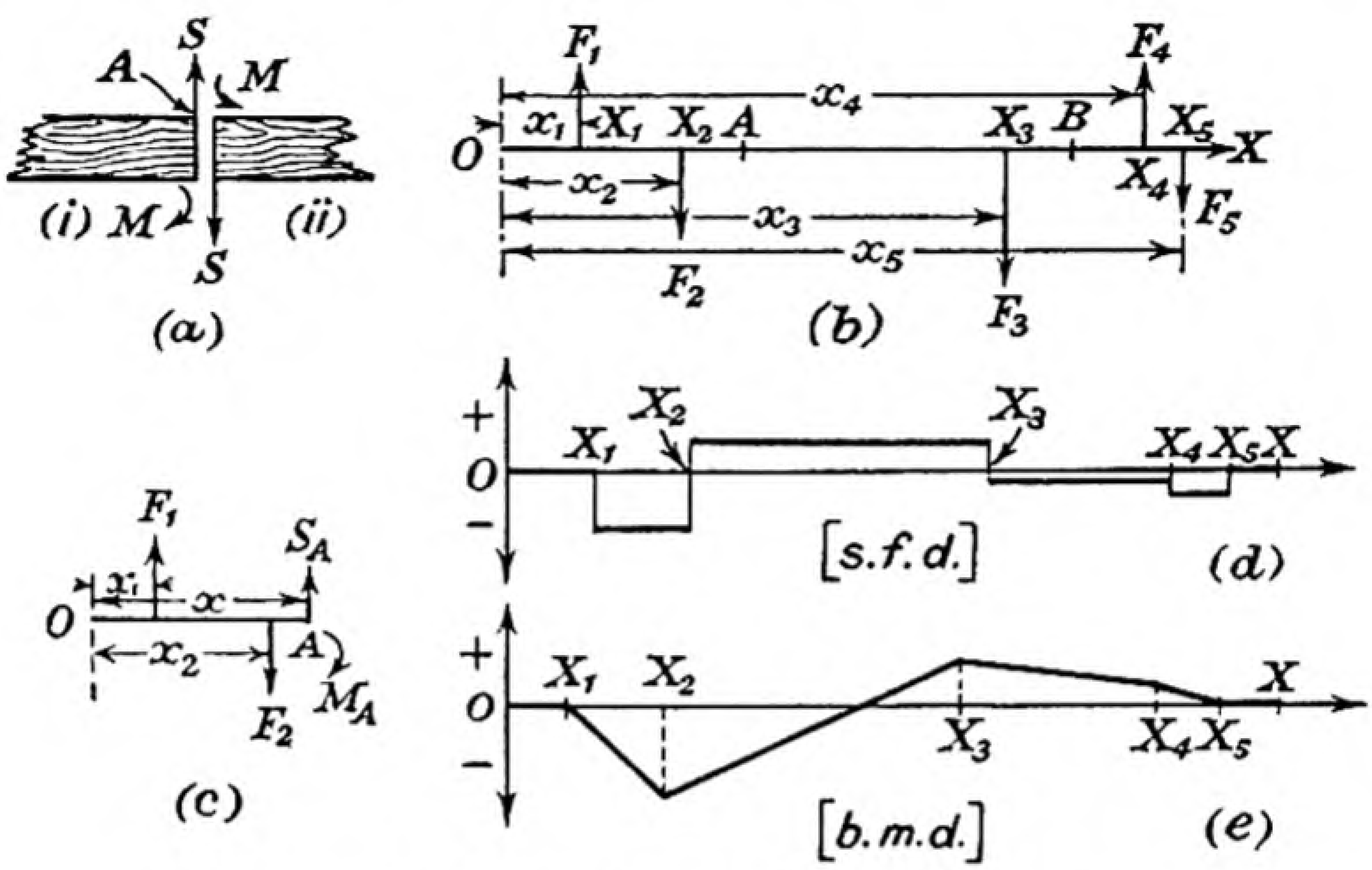


FIG. 8·01.—Shearing forces and bending moments.

respectively, the conditions for the equilibrium of this portion of the beam are

$$F_1 + S_A = F_2,$$

and $\quad M_A + F_2x_2 - F_1x_1 - S_Ax = 0.$

Hence $$S_A = F_2 - F_1,$$

and $$M_A = F_1(x_1 - x) - F_2(x_2 - x).$$

Similarly, if S_B and M_B are the shearing force and the bending moment at B, a section at a distance x from O, where $x_3 \leqslant x \leqslant x_4$, we have

$$F_1 + S_B = F_2 + F_3,$$

$$\therefore\ S_B = F_2 - F_1 + F_3,$$

$$= S_A + F_3.$$

Thus for a part of the beam considered, which is free from external forces, the shearing force is constant; but it is continuous at the point of application of an external force [e.g. at a point of support or at a point where a load is carried]. The change in the value of the shearing force, viz. $S_B - S_A$, in passing from the part in which A may lie to that in which B may lie, is equal to the external force acting on the portion of the beam between A and B.

Also at B,

$$M_B + F_2x_2 + F_3x_3 - S_Bx - F_1x_1 = 0.$$

Thus M is continuous, for $[M_A]_{x=x_3} = [M_B]_{x=x_3}$.

Suppose that the values of the shearing force of the bending moment have been calculated at a large number of transverse sections in the above beam. Let the values thus obtained be plotted to scale at right angles to a base line representing the length of the beam. The graphs thus obtainable are known as the ***shearing force diagram***, and the ***bending moment diagram***, respectively. For the beam considered, the former will consist of a set of straight lines parallel to the base line, with discontinuities at the points where the external forces act, while the latter will be straight lines with differing slopes—cf. Fig. 8·01(d), and (e).

Before proceeding to illustrate the above remarks by examples, let us consider a beam resting with its axis in a horizontal position. Let w be its weight per unit length, i.e. $w = \mu g$, where μ is the mass per unit length (linear density) of the beam, and g is the intensity of gravity. Let the forces and couples on a small element of this beam, of length δx, be as indicated in Fig. 8·02. They consist of its weight, $w\ \delta x$, acting vertically downwards through its mid-point, the shearing forces S and (S + δS), and the couples M and (M + δM). For the equilibrium of this element we must have

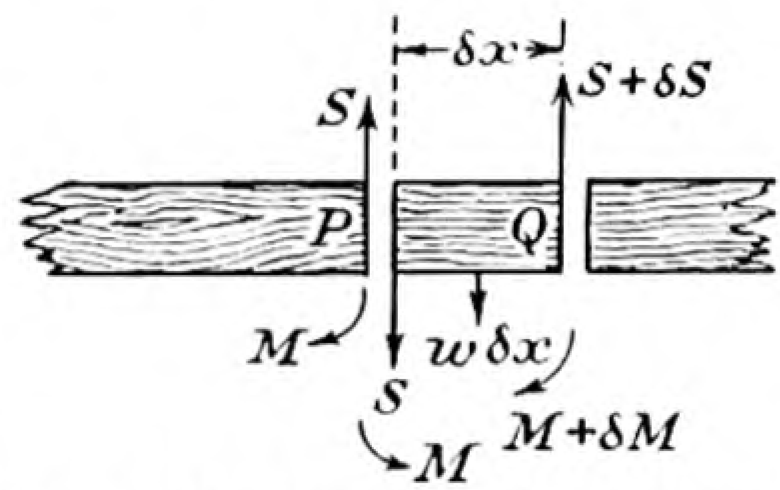


FIG. 8·02.—Forces and couples on an element of a beam.

$$\mathrm{S} + w\,\delta x = \mathrm{S} + \delta\mathrm{S},$$

i.e.

$$w = \frac{d\mathrm{S}}{dx},$$

and

$$\mathrm{M} + \delta\mathrm{M} - \mathrm{M} + w\,\delta x\left(\frac{\delta x}{2}\right) - (\mathrm{S} + \delta\mathrm{S})\,\delta x = 0,$$

i.e.

$$\mathrm{S} = \frac{d\mathrm{M}}{dx},$$

neglecting small quantities of the second order. The two equations just established give

$$\frac{d^2\mathrm{M}}{dx^2} = w.$$

The following examples will illustrate these facts, and also show how the fact that the beam itself possesses weight affects the shearing force diagram [s.f.d.] and the bending moment diagram [b.m.d.].

Example (i) —Consider a light horizontal beam carrying a load of weight W at its mid-point, and supported at its ends.

The reactions on the beam due to the supports are each $\frac{W}{2}$, and act vertically upwards as shown in Fig. 8·03(a). First consider the shearing

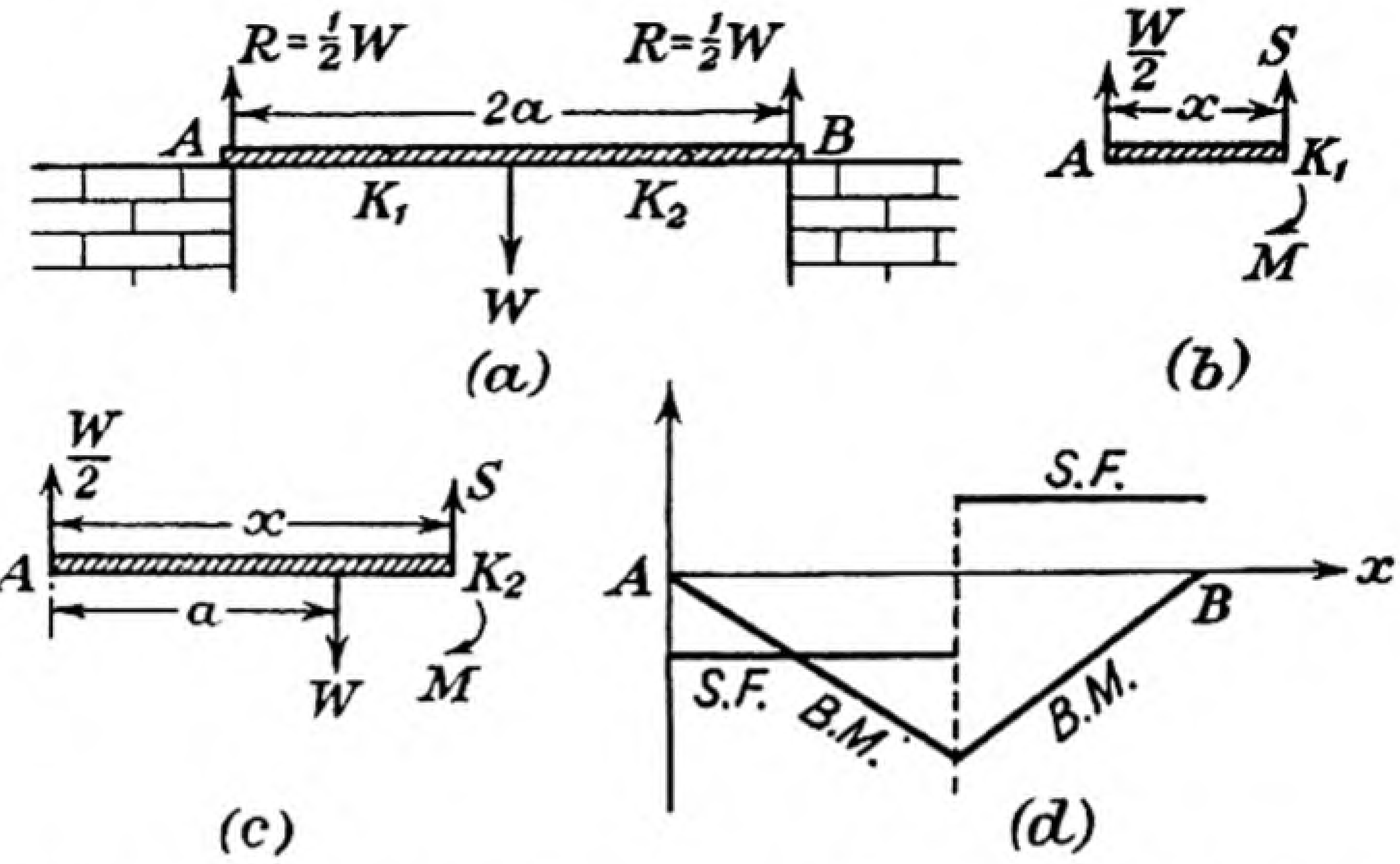


FIG. 8·03.—Freely supported light beam with central load.

force and the bending moment at a section K_1 in the beam at a distance x from the end A, where $0 < x < a$. Let the shearing force and the couple acting on the portion AK_1 of the beam be as indicated in Fig. 8·03(b). Then the conditions for equilibrium are

$$\frac{W}{2} + S = 0, \qquad \text{and} \qquad M - Sx = 0.$$

$$\therefore\ S = -\frac{W}{2}, \qquad \text{and} \qquad M = Sx = -\tfrac{1}{2}Wx.$$

Thus the shearing force on the portion to the left of K_1 at this point is directed downwards, while the bending moment is anticlockwise.

Similarly, if K_2 is a section at a distance x from A, where $a < x < 2a$ —cf. Fig. 8·03(c)—the conditions for equilibrium are

$$\frac{W}{2} + S = W, \qquad \text{i.e. } S = \frac{W}{2},$$

and

$$M + Wa - Sx = 0, \qquad \text{i.e. } M = -Wa + \tfrac{1}{2}Wx.$$

In each instance we see that

$$\frac{dM}{dx} = S, \qquad \text{and} \qquad \frac{d^2M}{dx^2} = 0,$$

as required by the formula established above.

The shearing force diagram and the bending moment diagram appropriate to this problem are indicated in Fig. 8·03(d).

Example (ii) —A uniform beam of weight w per unit length and length $2(a + b)$ is supported in a horizontal position by supports each

at a distance b from the nearer end. Discuss the values of the shearing force and of the bending moment at different transverse sections in the beam.

Let the reaction at each support be R, as shown in Fig. 8·04(a). Then $R = (a + b)w$. Consider the equilibrium of the portion AK_1, where x is

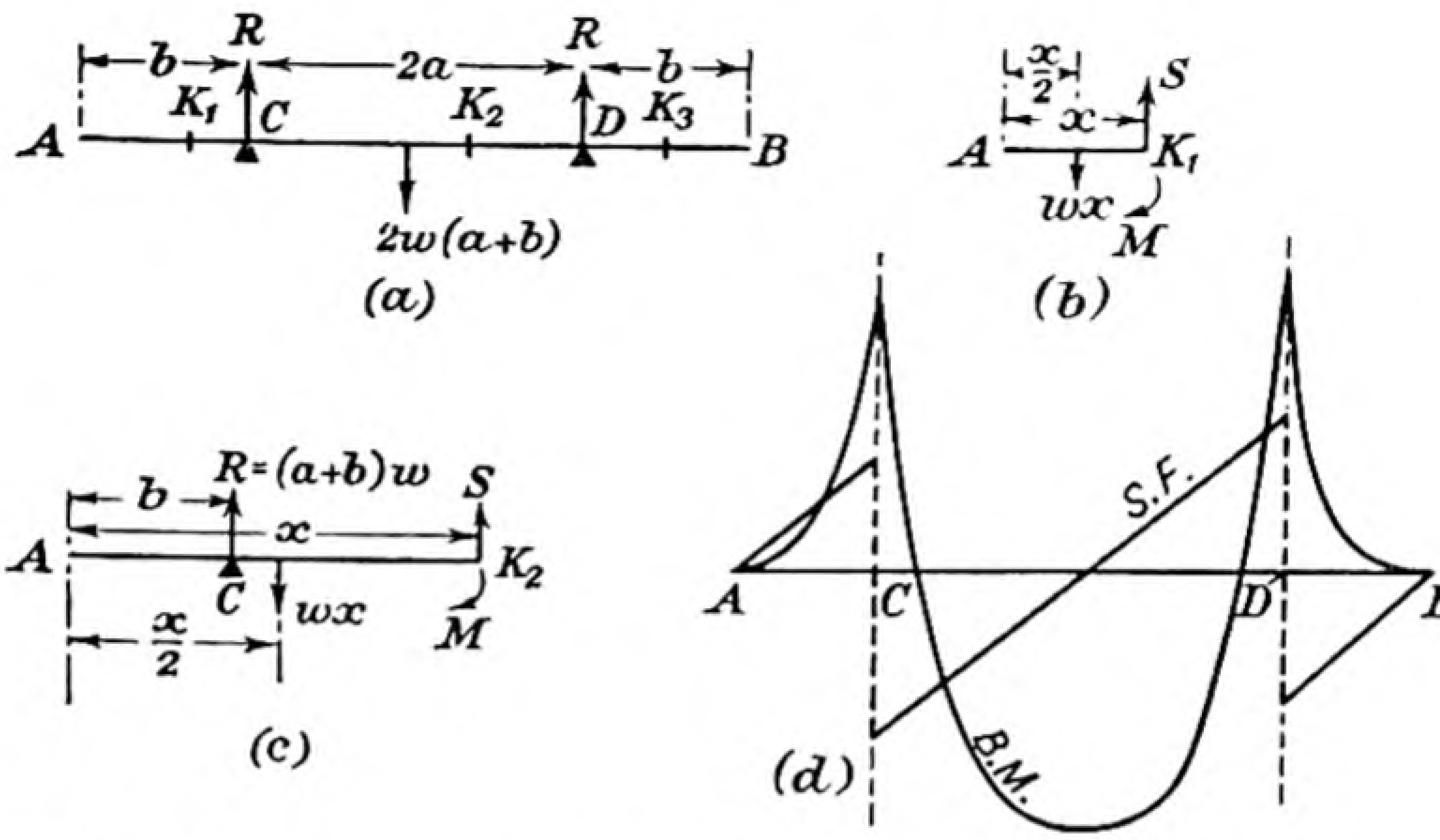


Fig. 8·04.—Heavy uniform beam, without external load, on symmetrically placed supports.

the distance of K_1 from the end A of the beam, and $0 < x < b$. Let S and M be as shown in Fig. 8·04(b). Then the conditions for equilibrium require

$$S = wx,$$

and

$$M = Sx - wx(\tfrac{1}{2}x) = \tfrac{1}{2}wx^2.$$

Now consider the portion AK_2 of the beam, where x, the distance of K_2 from A, is such that $b < x < (b + 2a)$. For equilibrium, cf. Fig. 8·04(c), we must have

$$S + (a + b)w = wx,$$

and

$$M + R(x - b) - \tfrac{1}{2}wx^2 = 0.$$

$$\therefore\ S = (x - a - b)w,$$

and

$$M = \tfrac{1}{2}wx^2 - w(a + b)(x - b).$$

It is not necessary to extend the formal argument to the portion of the beam beyond the second support, for the shearing force and the bending moment at K_3 at a distance z from B, where $b > z > 0$, will be equal in magnitude but opposite in sign to the corresponding quantities at a distance $x = |z|$ from A. The shearing force and the bending moment diagrams are as shown in Fig. 8·04(d).

The cantilever.—The cantilever consists of a uniform horizontal rod, AB, Fig. 8·05(a), of length l, fixed at one end into a wall or other rigid support. We shall first suppose that its own weight is negligible, and that it carries a weight W at its free end. Let R be the vertical reaction at the fixed end, and Z the couple which the

wall exerts on the rod. Considering the equilibrium of the rod as a whole, we have

$$R - W = 0,$$

and

$$Z + Wl = 0,$$

or

$$R = W, \quad \text{and} \quad Z = -Wl.$$

Thus the couple is anticlockwise, according to the convention here adopted.

Now consider the equilibrium of the portion AK of the lever,

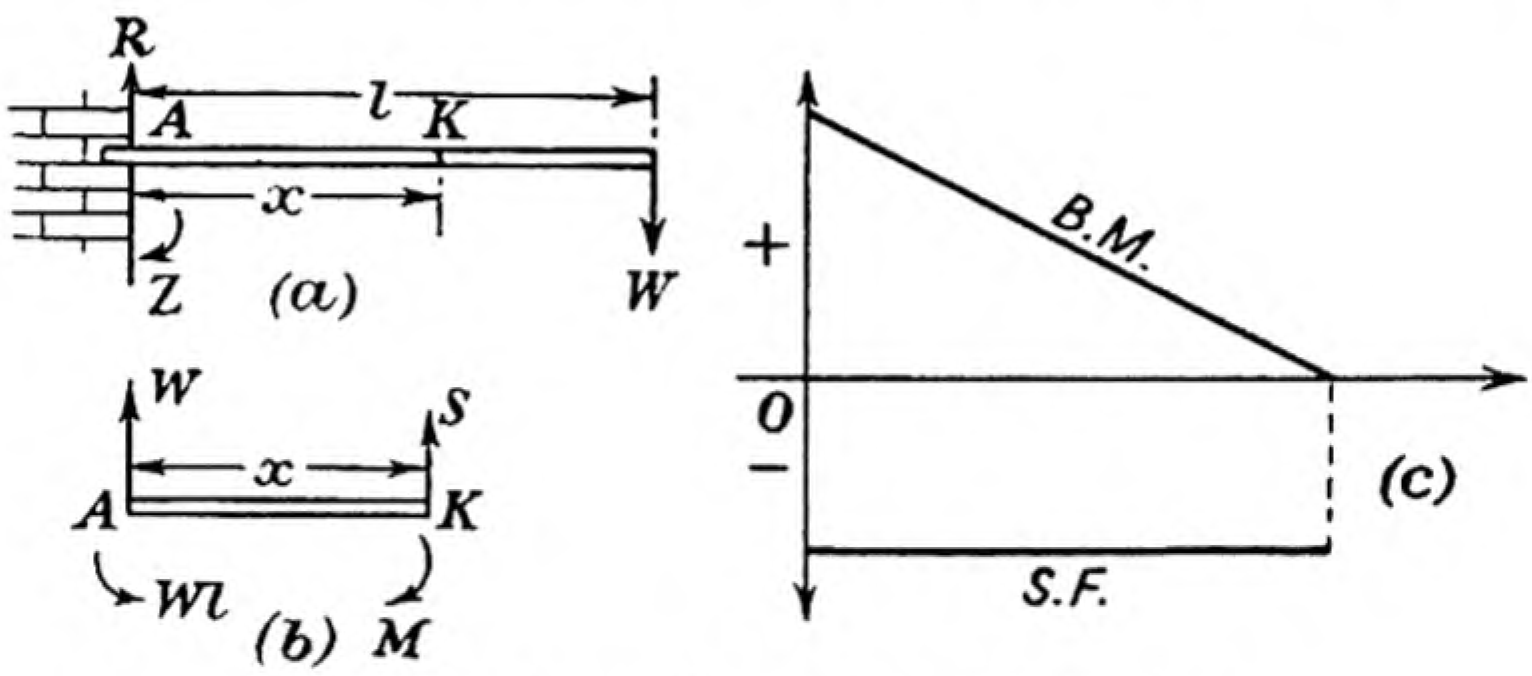


FIG. 8·05.—A cantilever of negligible weight.

where K is a section at a distance x from A, such that $0 < x < l$. The forces and couples on this portion of the lever are as shown in Fig. 8·05(b). The conditions for equilibrium require

$$W + S = 0, \quad \text{or} \quad S = -W,$$

and

$$M - Wl - Sx = 0,$$

i.e.

$$M = W(l - x).$$

The shearing force diagram and the bending moment diagram are therefore as indicated in Fig. 8·05(c).

Suppose now that the above lever has a weight w per unit length. Let R, Fig. 8·06(a), be the vertical reaction at the fixed end, Z being

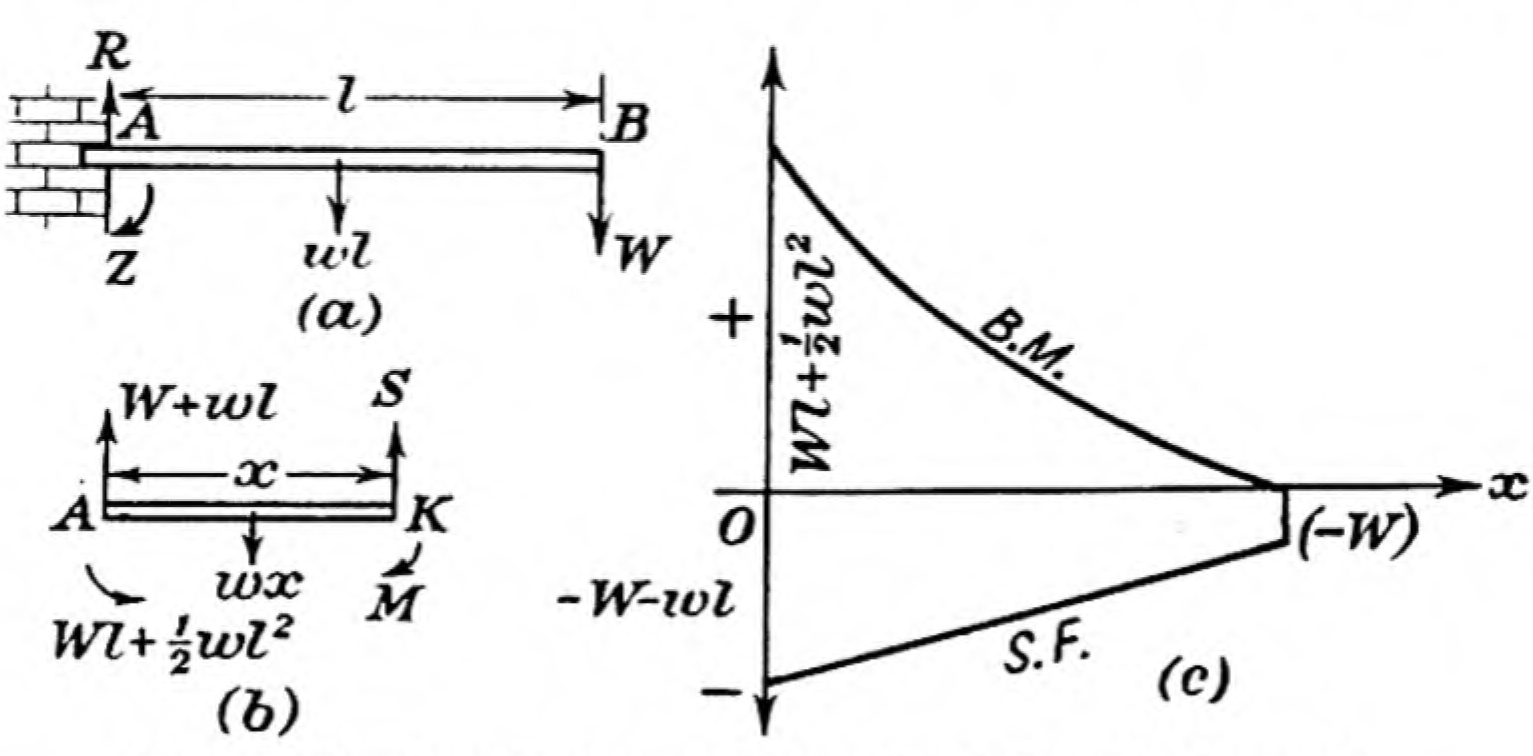


FIG. 8·06.—A uniform cantilever loaded at its free end.

the couple which the wall exerts on the rod. Considering the equilibrium of the lever as a whole, the conditions are

$$R - wl - W = 0,$$

and $$Z + Wl + \tfrac{1}{2}wl^2 = 0.$$

Hence $$R = W + wl,$$

and $$Z = -Wl - \tfrac{1}{2}wl^2,$$

i.e. this couple has a magnitude $(Wl + \frac{1}{2}wl^2)$, and acts in an anti-clockwise direction.

Now let S be the shearing force and M be the bending moment at a transverse section of the lever at a distance x from its fixed end, where $0 < x < l$. Then the forces and couples acting on this portion of the lever are as shown in Fig. 8·06(b). For equilibrium, we have

$$W + wl + S = wx,$$

i.e. $$S = -W - w(l - x),$$

and taking moments of forces about A

$$M + \tfrac{1}{2}wx^2 - Wl - \tfrac{1}{2}wl^2 - Sx = 0,$$

$$\therefore\ M = [-W - w(l - x)]x - \tfrac{1}{2}wx^2 + Wl + \tfrac{1}{2}wl^2$$
$$= W(l - x) + \tfrac{1}{2}w(l - x)^2.$$

Example.—A light but rigid rod OA, Fig. 8·07(a), of length $2a$ is

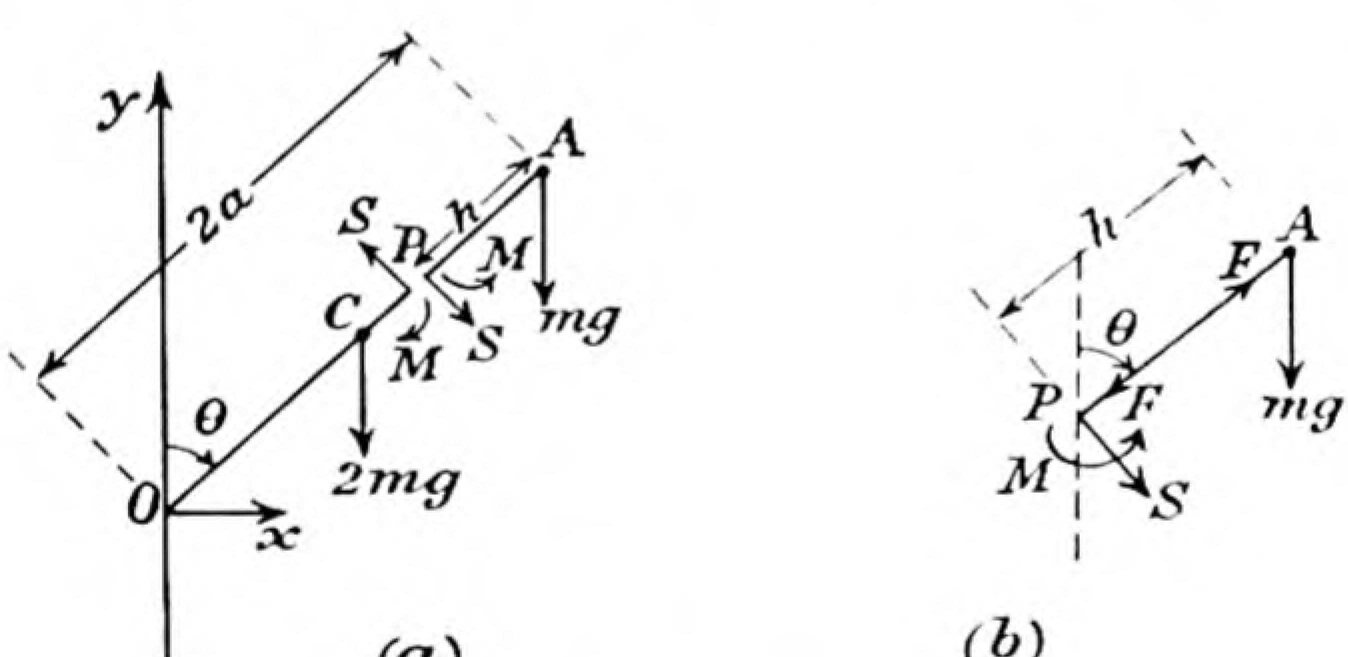


FIG. 8·07.

smoothly hinged to a fixed point O so that it can rotate in a vertical plane. A particle of mass m is rigidly attached to the rod at A, while another of mass $2m$ is fixed at its centre C. The rod is released from rest at an angle α with the upward vertical. Show that when it makes an angle θ with the upward vertical, its angular velocity $\dot{\theta}$ is given by

$$3a\dot{\theta}^2 = 4g(\cos\alpha - \cos\theta).$$

Also determine the angular acceleration of the rod, and the shearing force and bending moment at a section in the rod at a distance $h (h < a)$ from A. For what values of $\cos\theta$ is the upper half of the rod in compression?

When the rod makes an angle θ with Oy the external forces on the rod are mg at A, $2mg$ at C and the reaction at O. Since the rod is released from rest, the energy principle gives us

$$\tfrac{1}{2}.2m.a^2\dot{\theta}^2 + \tfrac{1}{2}.m.4a^2.\dot{\theta}^2 + 2mg.a\cos\theta + mg.2a\cos\theta$$
$$= 2mg.a\cos\alpha + mg.2a\cos\alpha,$$

i.e. $$3a\dot{\theta}^2 = 4g(\cos\alpha - \cos\theta).$$

Hence $$\ddot{\theta} = \frac{2}{3}\cdot\frac{g}{a}\cdot\sin\theta.$$

Now consider the shearing stresses, SS, and the bending moments, MM, at the section P where PA $= h$; the rod is rigid so that all considerations of elasticity may be omitted, cf. below. Now the acceleration of A is $2a\ddot{\theta}$ at right angles to OA and $2a\dot{\theta}^2$ along AO. Hence, resolving forces on PA perpendicular to PA, we have

$$m(2a\ddot{\theta}) = mg\sin\theta + \text{S},$$

and taking moments of forces about P,

$$m(2a\ddot{\theta})h = mgh\sin\theta - \text{M}.$$

$$\therefore\ \text{S} = 2am\cdot\frac{2}{3}\cdot\frac{g}{a}\sin\theta - mg\sin\theta$$
$$= m[\tfrac{4}{3}g\sin\theta - g\sin\theta] = \tfrac{1}{3}mg\sin\theta,$$

and $$\text{M} = -2am\cdot\frac{2}{3}\cdot\frac{g}{a}\cdot\sin\theta.h + mgh\sin\theta$$
$$= -\tfrac{1}{3}mgh\sin\theta.$$

Now suppose that F is the thrust in the portion PA of the rod. The forces acting on the mass m at A are as shown in Fig. 8·07(b). Resolving forces along AP we obtain

$$m(2a.\dot{\theta}^2) = mg\cos\theta - \text{F}.$$
$$\therefore\ \text{F} = mg\cos\theta - 2a\dot{\theta}^2.m$$
$$= mg\cos\theta - 2am.\frac{4g}{3a}(\cos\alpha - \cos\theta)$$
$$= \tfrac{1}{3}mg[11\cos\theta - 8\cos\alpha].$$

$\therefore$ F is positive when $\theta = \alpha$, decreases as θ increases, and changes sign at $\cos\theta = \frac{8}{11}\cos\alpha$.

$\therefore$ F is originally a compression and becomes a tension when $\cos\theta < \frac{8}{11}\cos\alpha$.

Stresses induced by bending.—Let ABCD, Fig. 8·08(a), be the longitudinal section of an element of a beam which is straight in the unstrained position; the beam, however, is no longer rigid, so that, as the sequel shows, its modulus of elasticity has to be considered. The material of the beam is assumed to be isotropic and

homogeneous. The shape of the section normal to the plane of the diagram is immaterial. Fig. 8·07(*b*) shows the longitudinal section of the same element of the beam after it has been bent by the application of a couple; when the bending is produced in this manner it is said to be ***pure***. It will be assumed that the limit of perfect elasticity has not been exceeded at any point in the beam. Let the position of any point in this longitudinal section be referred to

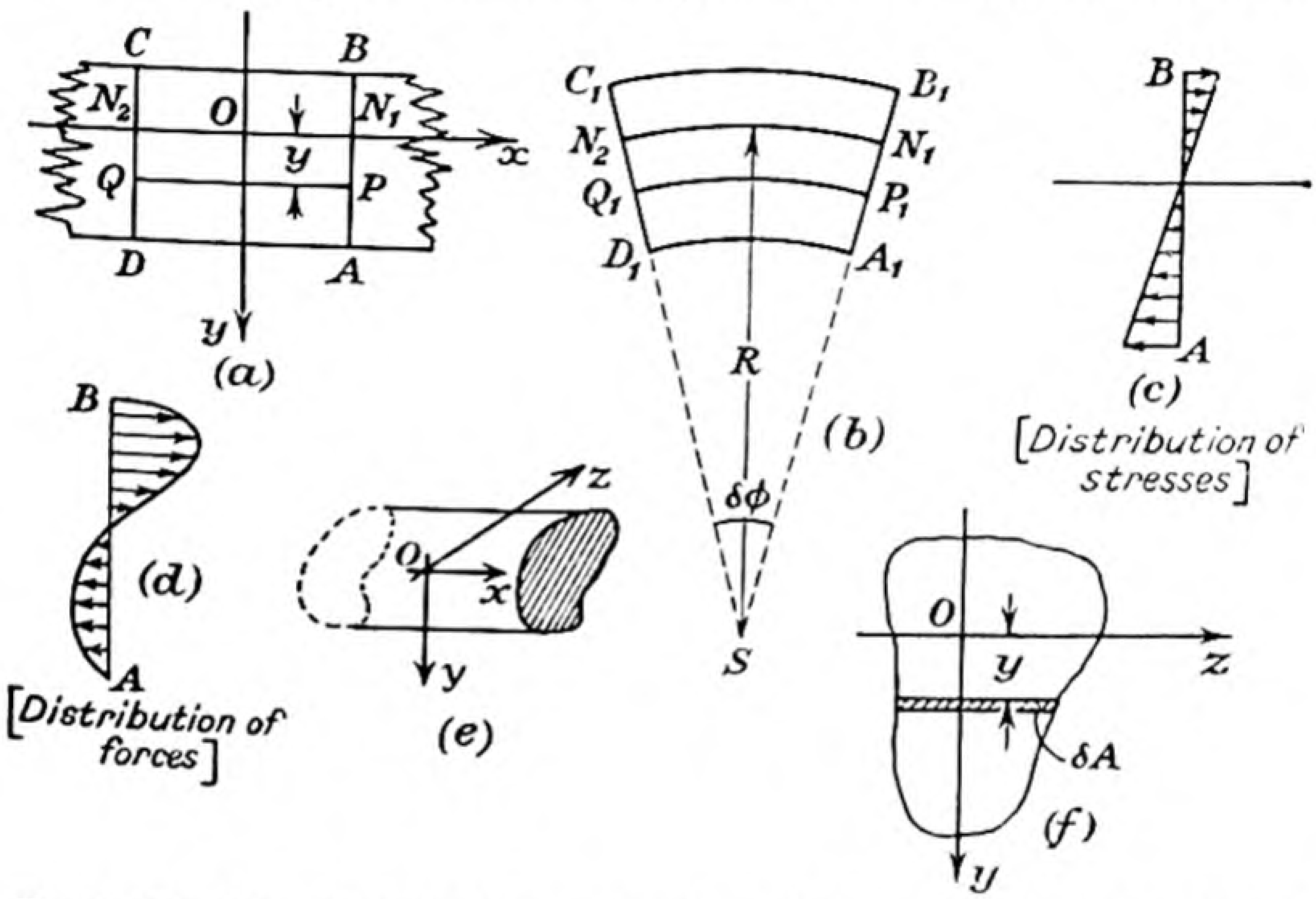


FIG. 8·08.—Beam bent by the application of a couple. [Pure bending.]

rectangular axes Ox, Oy: it will be noted that the positive direction of y is considered to be downwards.

If the bending moment at the section AB is M, then Fig. 8·07(*b*) shows that the filaments in the upper part of the beam will be stretched while those in the lower part will be compressed. The filaments on a certain surface will not be strained; this is the ***neutral surface*** and its trace N_1N_2 is taken as the x-axis of coordinates; this trace is often referred to as the ***neutral axis*** of the beam. The strain along P_1Q_1 [considered positive in the direction of x increasing] is

$$\frac{P_1Q_1 - PQ}{PQ} = \frac{(R - y)\delta\phi - R\,\delta\phi}{R\,\delta\phi} = -\frac{y}{R},$$

where R is the distance of the point of intersection of B_1A_1 and C_1D_1 from N_1N_2, and $\delta\phi$ is the angle indicated. The stress is $-\dfrac{E}{R}y$, where E is Young's modulus for the material of the beam. In stating this it has been assumed that every filament is free to expand longitudinally, and contract laterally, i.e. its behaviour under stress

is the same as if it were separate from all other layers. If this were not so E would not be Young's modulus for the material of the beam, but some other elastic constant.

The stress is therefore a linear function of the distance of the point at which it is considered from the neutral surface, cf. Fig. 8·08(*c*). Now according to the convention we have adopted, the stress at a point in AB is due to the portion of the beam to the right of AB and therefore the moment of the system of forces acting on AB, cf. Fig. 8·08(*d*), must be the bending moment M.

Having selected our x- and y-axes the third or z-axis will be as shown in Fig. 8·08(*e*). Let $\delta\mathrm{A}$, Fig. 8·08(*f*), be an element of area in a transverse section of the beam; the length of this element is parallel to the z-axis and its breadth is δy. Now, for this element of area, the force on it, in the direction of x increasing, is $\left(-\dfrac{\mathrm{E}}{\mathrm{R}}y\right)\delta\mathrm{A}$, so that the total force on the transverse section AB is

$$-\frac{\mathrm{E}}{\mathrm{R}}\int y\,d\mathrm{A},$$

and this is zero since the bending is assumed to be pure. [The integration extends over the whole cross-section considered.]

$$\therefore \int y\,d\mathrm{A} = 0.$$

But
$$\int y\,d\mathrm{A} = \bar{y}\mathrm{A},$$

where $\bar{y}$ is the distance of the centroid of the section from Oz, i.e. $\bar{y} = 0$ or the neutral axis passes through the centroid of the section.

The moment in a ***clockwise*** direction of the forces on the section about a ***line through*** the neutral surface is

$$-\left[-\frac{\mathrm{E}}{\mathrm{R}}\int y\,.\,y\,d\mathrm{A}\right] = \frac{\mathrm{E}}{\mathrm{R}}\int y^2\,d\mathrm{A} = \frac{\mathrm{EI}}{\mathrm{R}},$$

where I is the moment of inertia for the cross-section considered. [It must be emphasised that I is not a true moment of inertia for it does not involve mass; I is merely used to denote $\int y^2\,d\mathrm{A}$ which is the ***quadratic moment of the area*** with respect to the neutral line. It is sometimes called the ***second areal moment*** of the area. Its value is determined solely by the size and shape of the cross-section and the position of the neutral surface.]

$$\therefore\ \mathrm{M} = \frac{\mathrm{EI}}{\mathrm{R}} = \mathrm{EI}\frac{d^2y}{dx^2},$$

since the curvature of the beam is so small that R^{-1} may be replaced by $\dfrac{d^2y}{dx^2}$. [If $\dfrac{d^2y}{dx^2}$ is negative, the radius of curvature will be negative,

i.e. the nearer surface of the beam will be concave when viewed from a point on the negative side of y.]

The factor EI is termed the ***flexural rigidity*** of the beam.

Since $|p| = \frac{E}{R}|y|$, the longitudinal stress is a maximum at the boundary (of the section) which is most distant from the neutral surface. Since $M = \frac{EI}{R}$, it follows that $|p| = \frac{M}{I}|y|$.

Ordinary bending.—A discussion of simple bending has just been given: it refers to the particular instance of bending in which there is no shearing force. In most practical cases, however, bending is always accompanied by shearing forces tending to produce a vertical shear across transverse sections of the beam. When this occurs, the forces across any transverse section have, in general, to balance a couple and also the shearing forces at that section, i.e. the longitudinal stress will be accompanied by a tangential stress. The physicist and engineer find, however, that in actual practice it is sufficient to apply the theory of simple bending to the problems which occur in designing a structure. Accordingly, we shall assume in all subsequent work that the curvature of the beam under stress is connected with the bending moment by the simple formula we have established.

Transverse bending; anticlastic curvature.—When a beam is bent longitudinally, i.e. in a plane containing its length, a transverse bending with opposite curvature must necessarily be associated with it, if each filament is perfectly free to expand or to contract laterally when subjected to a compressive or a tensile stress respectively. In the beam considered in Fig. 8·08 the filaments above the neutral surface are extended; each must therefore suffer a lateral contraction, while the filaments which are below are compressed so that they suffer a lateral expansion.

The following simple experiment will help to clarify these remarks. If a piece of india-rubber about 5 cm. × 3 cm. × 0·5 cm. is bent lengthways into an arc of a circle, the shape assumed by the rubber is similar to that shown in Fig. 8·09(a). Not only the longitudinal fibres but also the fibres normal to them are bent into circular arcs. If the longitudinal fibres are bent so that they are concave with respect to a point below the india-rubber, then the transverse fibres are concave with respect to a point above the rubber. The bending which takes place in a plane normal to the longitudinal plane gives to the specimen an ***anticlastic curvature***.

Now consider a beam of rectangular cross-section subjected to pure bending, and in such a manner that filaments above the neutral surface are extended while those below are compressed. Thus let

ABCD, Fig. 8·09(b), be a side of an element of the beam when bent, this side being parallel to the plane of bending. Let LN be the line along which this side is cut by the neutral surface. Let $A_1B_1C_1D_1$ be the opposite side of the element, the neutral surface cutting it along L_1N_1. We have to find an expression for R_2, the radius of anticlastic curvature, i.e. for the radius of LL_1 (or of NN_1) in terms of

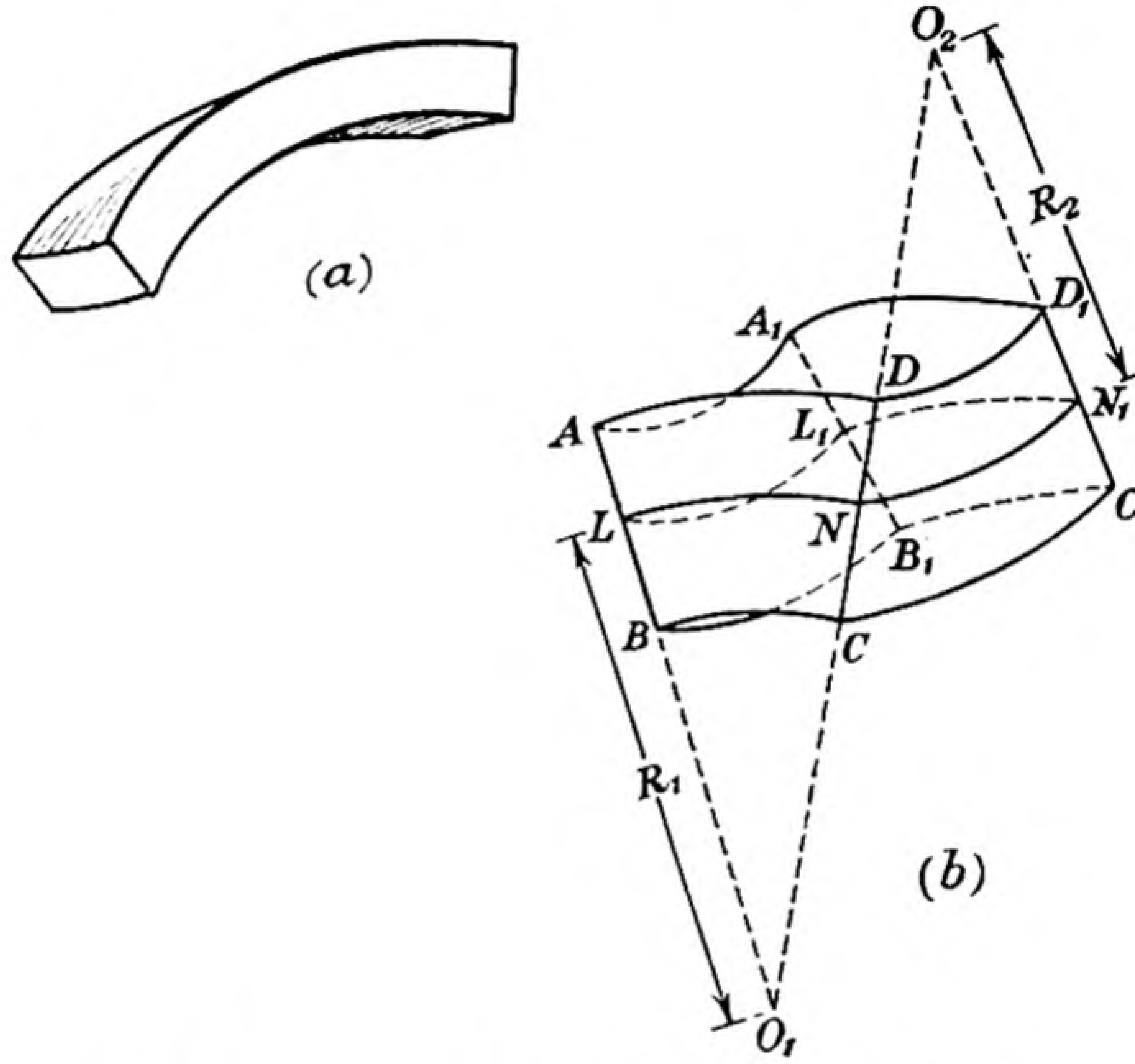


FIG. 8·09.—Transverse bending; anticlastic curvature.

R_1, the radius of curvature of LN, and of σ, Poisson's ratio for the material of the beam. Let AB and DC produced meet in O_1, while CD and C_1D_1 produced meet in O_2.

Let $LN = L_1N_1 = a$, and $NN_1 = LL_1 = b$. Then if AL, etc. $= h$, we have

$$\frac{AD}{LN} = \frac{R_1 + h}{R_1}, \qquad \text{i.e. } AD = a\left(1 + \frac{h}{R_1}\right).$$

Also
$$\frac{DD_1}{NN_1} = \frac{R_2 - h}{R_2}, \qquad \text{i.e. } DD_1 = b\left(1 - \frac{h}{R_2}\right).$$

But $\dfrac{AD - LN}{LN}$ = longitudinal strain along AD,

and

$$\frac{NN_1 - DD_1}{NN_1} = \text{lateral contraction per unit width along } DD_1.$$

$$\therefore\ \sigma\left[\frac{\mathrm{AD}}{a} - 1\right] = \left[1 - \frac{\mathrm{DD_1}}{b}\right],$$

i.e. $$\mathrm{R_1} = \sigma \mathrm{R_2}.$$

On the bending of a beam which is not quite straight initially.—Let ABCD, Fig. 8·10(a), be an element of a beam slightly curved initially. Let PQ be the neutral surface (or rather

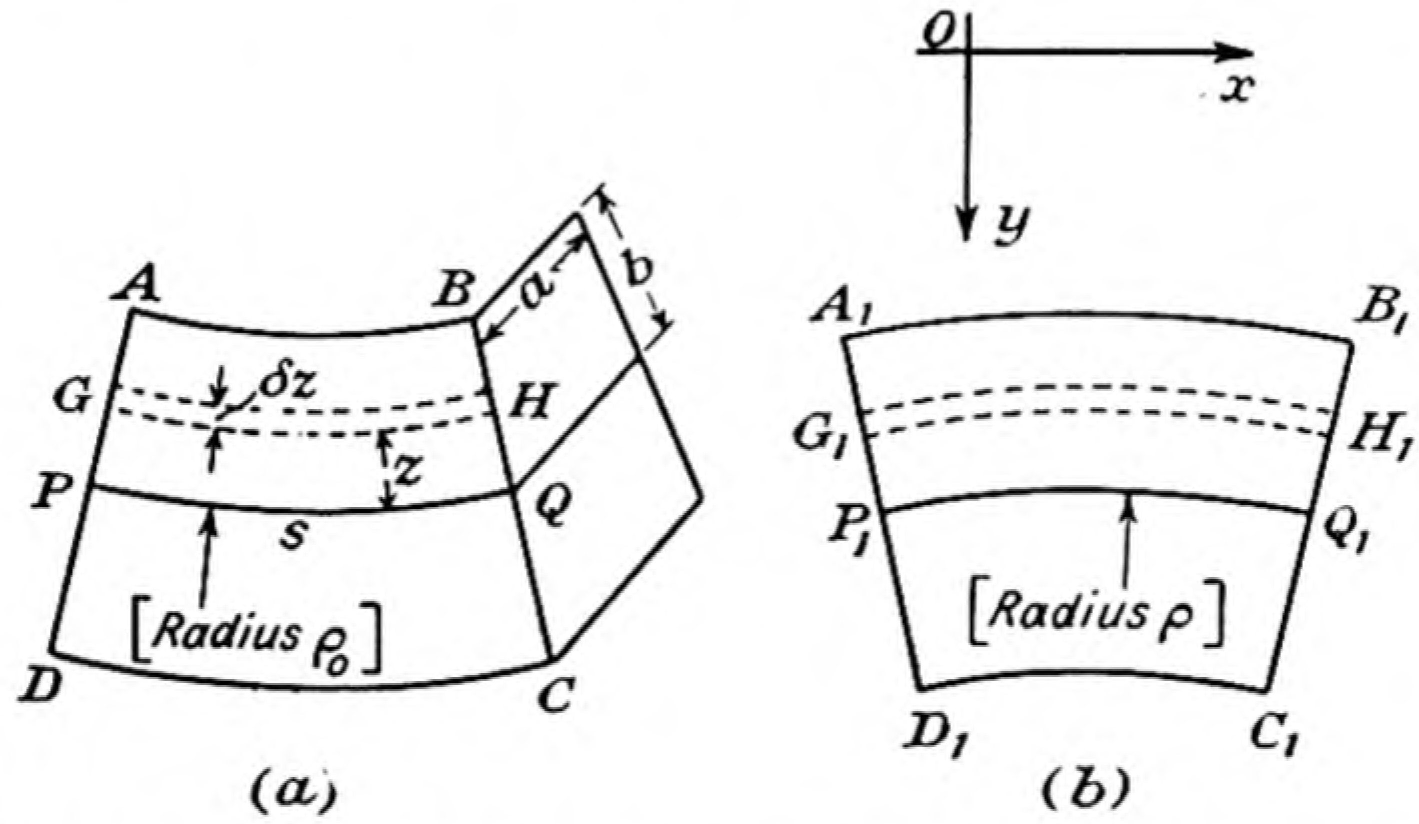


FIG. 8·10.—Bending of a beam with a slight initial curvature.

its trace in the plane of the diagram). If PQ $= s$, a small finite length, then AB $= s - \Delta s_0$, say. Hence, with the notation indicated in the diagram,

$$\frac{s - \Delta s_0}{s} = \frac{\rho_0 - b}{\rho_0}, \quad \text{or} \quad \frac{\Delta s_0}{s} = \frac{b}{\rho_0}.$$

If AB becomes $\mathrm{A_1B_1}$, cf. Fig. 8·10(b), after a given system of forces has been applied to the bar to produce bending, call $\mathrm{A_1B_1} = s + \Delta s$. Then

$$\frac{\Delta s}{s} = \frac{b}{\rho},$$

where ρ is the new radius of curvature of the neutral surface and all quantities are considered numerically. Thus AB has increased in length by $\Delta s + \Delta s_0 = sb\left[\frac{1}{\rho} + \frac{1}{\rho_0}\right]$.

Consider now a filament GH (shown dotted), at height z above PQ. The extension η produced when this becomes $\mathrm{G_1H_1}$ is given by

$$\frac{\eta}{\Delta s + \Delta s_0} = \frac{z}{b}, \qquad \text{or} \qquad \eta = zs\left[\frac{1}{\rho} + \frac{1}{\rho_0}\right].$$

If E is Young's modulus for the material of the bar [and, as usual, we assume E to be the same when the bar is in compression as when it is in tension], then

$$\mathrm{E} = \frac{\dfrac{\text{Stretching force exerted on GH by adjacent filaments}}{a\,\delta z}}{zs\left[\dfrac{1}{\rho} + \dfrac{1}{\rho_0}\right] \div s\left[1 - \dfrac{z}{\rho_0}\right]}$$

$$= \frac{\text{Stretching force}}{a\,\delta z} \div z\left[\frac{1}{\rho} + \frac{1}{\rho_0}\right], \qquad \text{if } \frac{z}{\rho_0} \to 0.$$

$$\therefore \text{ Stretching force} = a\mathrm{E}\left[\frac{1}{\rho} + \frac{1}{\rho_0}\right] z\,\delta z.$$

The moment of this force about the straight line through Q in which the neutral surface intersects the end of the element shown is

$$\mathrm{E}\left[\frac{1}{\rho} + \frac{1}{\rho_0}\right] z^2\,\delta z.$$

Since the total stretching force is zero, PQ must lie in a plane passing through the centroid of the cross-section of the element and the total moment, M, is

$$\mathrm{E}\left[\frac{1}{\rho} + \frac{1}{\rho_0}\right] \int a z^2\,\delta z,$$

where the integration extends over the whole depth of the bar.

$$\therefore\ \mathrm{M} = \mathrm{EI}\left[\frac{1}{\rho} + \frac{1}{\rho_0}\right], \qquad \text{if } \mathrm{I} = \int a z^2\,dz.$$

Strictly speaking ρ_0 is negative if ρ is positive and hence the above formula is better written

$$\mathrm{M} = \mathrm{EI}\left[\frac{1}{\rho} - \frac{1}{\rho_0}\right],$$

i.e. M = EI $\times$ change in curvature.

Alternative treatment: To straighten the element of the beam considered a bending moment M_0, given by $\mathrm{M}_0 = \dfrac{\mathrm{EI}}{\rho_0}$ would have to be applied. To produce the additional curvature an additional bending moment given by $\dfrac{\mathrm{EI}}{\rho}$ is necessary. Hence the total bending

moment required to change the curvature from ρ_0 to ρ is, apart from signs, given by

$$M = EI\left[\frac{1}{\rho} + \frac{1}{\rho_0}\right]$$

$$= EI \times \text{change in curvature.}$$

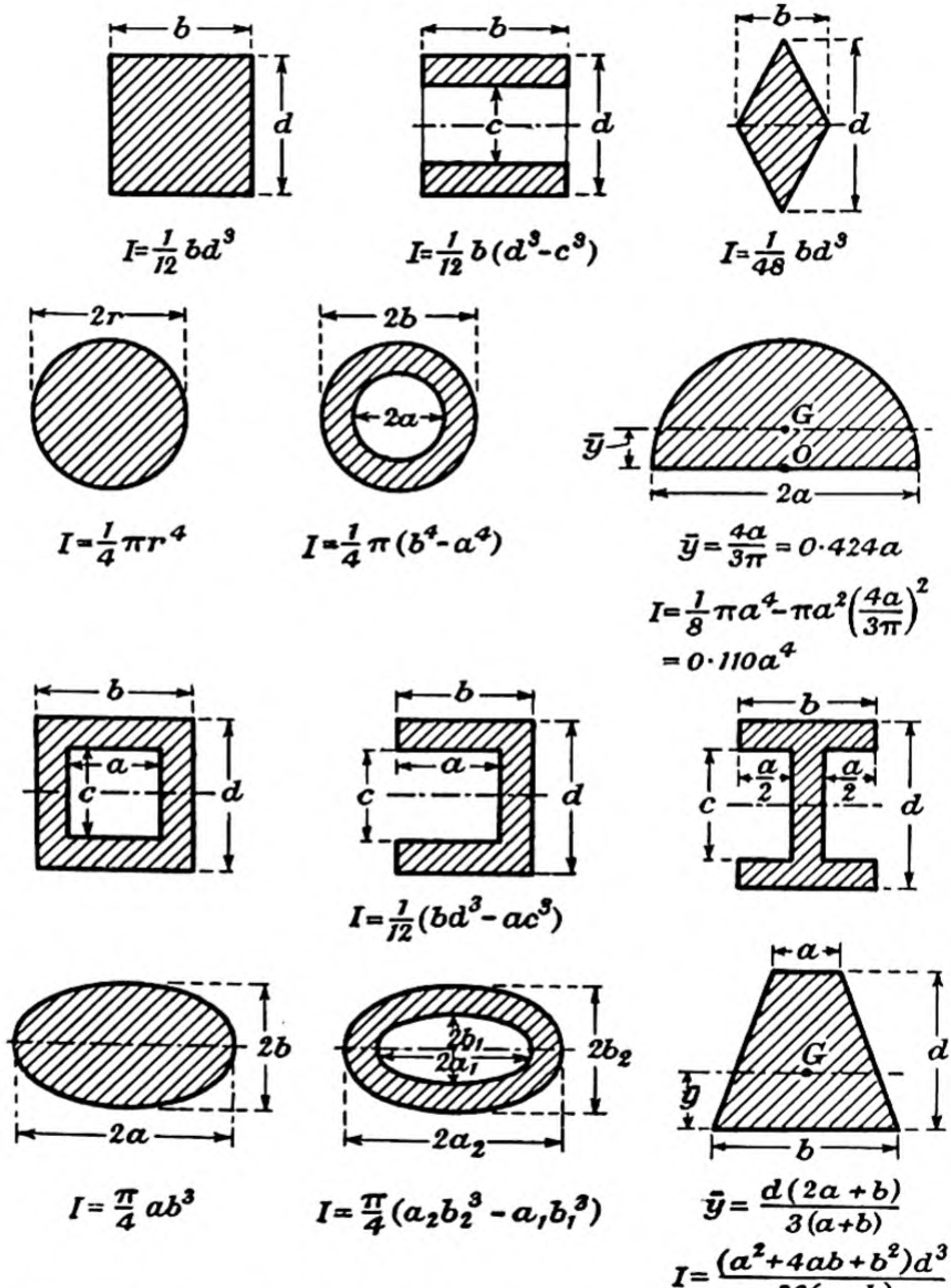


TABLE OF 'MOMENTS OF INERTIA', OR SECOND AREAL MOMENTS, FOR DIFFERENT CROSS-SECTIONS; EACH VALUE REFERS TO A HORIZONTAL AXIS THROUGH THE 'CENTRE OF GRAVITY' OF THE AREA.

A light beam supported at its ends and carrying a central load.—Let AB, Fig. 8·11(a), be the beam resting in a horizontal position on supports at its ends. Let $2a$ be the length of the beam and suppose it carries a load of weight W at its mid-point; the beam is 'light' so that its own weight is negligible compared with W. The reactions at the supports are each equal to $\frac{1}{2}$W and act

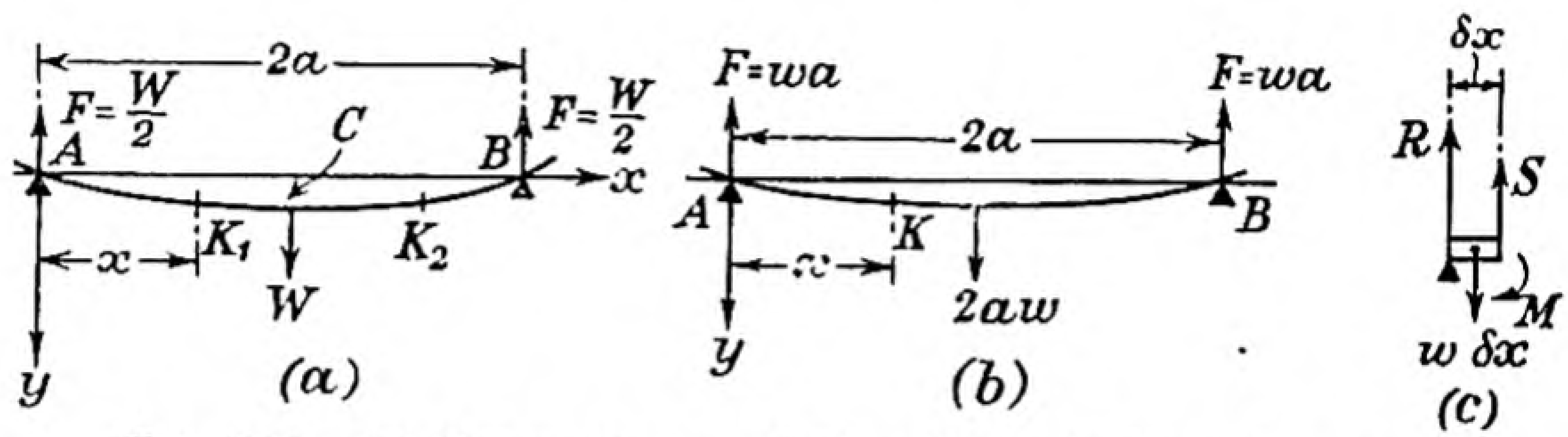


FIG. 8·11.—(a) Non-uniform and (b) uniform loading of a beam.

vertically upwards. Then, in the usual way, we find that the shearing force S, at K_1, at a distance x from A, where $0 \leqslant x \leqslant a$, is $-\frac{1}{2}$W, while M, the bending moment, is given by $M = -\frac{1}{2}Wx$.

Hence

$$EI.\frac{d^2y}{dx^2} = M = -\tfrac{1}{2}Wx, \qquad \text{or} \qquad \frac{d^2y}{dx^2} = -\frac{1}{2}.\frac{W}{EI}.x.$$

Integrating this equation, we obtain

$$\frac{dy}{dx} = -\frac{1}{4}.\frac{W}{EI}.x^2 + A,$$

where A is an integration constant. Since $\frac{dy}{dx}$ is zero at the point $x = a$, we have

$$\left(\frac{dy}{dx}\right)_{x=a} = 0 = -\frac{1}{4}.\frac{W}{EI}.a^2 + A.$$

$$\therefore\ A = \frac{1}{4}.\frac{W}{EI}.a^2.$$

Integrating again, we get

$$y = -\frac{1}{12}.\frac{W}{EI}.x^3 + \frac{1}{4}.\frac{W}{EI}.a^2x,$$

no added constant being necessary since the deflexion is zero at $x = 0$.

The deflexion at the centre of the beam is therefore given by

$$[y]_{x=a} = \frac{1}{6}.\frac{W}{EI}.a^3.$$

Also, the slope of the beam at the supports is the value of $\frac{dy}{dx}$ when x is zero. If the beam makes an angle θ with the horizontal at A,

$$\tan\theta = \left(\frac{dy}{dx}\right)_{x=0} = \frac{1}{4}.\frac{W}{EI}.a^2.$$

The deflexion at K_2, a point in the portion CB of the beam, may in this particular instance be written down by considering the symmetry of the arrangement. In order, however, to illustrate the method of dealing with more complicated problems, a complete solution follows. Let the distance of K_2 from A be x, where $a \leqslant x \leqslant 2a$. Then M, the bending moment at K_2, is given by

$$M = -Wa + \frac{W}{2}.x.$$

$$\therefore \frac{d^2y}{dx^2} = -\frac{W}{EI}.a + \frac{1}{2}.\frac{W}{EI}.x.$$

$$\therefore \frac{dy}{dx} = -\frac{W}{EI}.ax + \frac{1}{4}.\frac{W}{EI}.x^2 + A_1,$$

where A_1 is a constant. It is determined by the fact that at $x = a$, $\frac{dy}{dx}$ is zero. Hence

$$0 = -\frac{W}{EI}.a^2 + \frac{1}{4}.\frac{W}{EI}.a^2 + A_1.$$

$$\therefore A_1 = \frac{3}{4}.\frac{W}{EI}.a^2.$$

Integrating again, we obtain,

$$y = -\frac{W}{2EI}.ax^2 + \frac{1}{12}.\frac{W}{EI}.x^3 + \frac{3}{4}.\frac{W}{EI}.a^2x + B_1,$$

B_1 being a constant of integration. Since, at $x = 2a$, $y = 0$, we have

$$0 = \frac{W}{EI}[-\tfrac{1}{2}a.4a^2 + \tfrac{1}{12}.8a^3 + \tfrac{3}{4}a^2.2a + B_1].$$

$$\therefore B_1 = -\frac{Wa^3}{6EI}.$$

$$\therefore \Big[y\Big]_{x=a} = \frac{1}{6}.\frac{W}{EI}.a^3,$$

as before.

A uniformly loaded beam.—Now consider a horizontal beam supported at its ends and carrying a uniformly distributed load of weight w per unit length—this will include the weight per unit length of the beam, or it may be only this weight. Let AB, Fig. 8·11(b), be the beam of length $2a$. The directions of the coordinate axes are taken in accordance with the usual plan. Let x be the distance of a transverse section at K at which the bending moment is required. The thrust at A is wa. Now consider the equilibrium of the portion AK of the beam, the shearing force and bending moment at K being S and M, respectively. Then

$$wa + \mathrm{S} = wx, \qquad \text{i.e. } \mathrm{S} = w(x - a),$$

and

$$\mathrm{M} + \tfrac{1}{2}wx^2 - \mathrm{S}x = 0.$$

Hence

$$\mathrm{M} = w\left(\frac{x^2}{2} - ax\right).$$

[N.B.—Since we do not require the value of S, we could avoid determining it by taking moments of forces about K, when

$$wax + \mathrm{M} - \tfrac{1}{2}wx^2 = 0,$$

so that

$$\mathrm{M} = w\left(\frac{x^2}{2} - ax\right), \qquad \text{as before.]}$$

Hence

$$\frac{d^2y}{dx^2} = \frac{1}{\mathrm{EI}}[\tfrac{1}{2}wx^2 - wax].$$

$$\therefore \frac{dy}{dx} = \frac{1}{\mathrm{EI}}[\tfrac{1}{6}wx^3 - \tfrac{1}{2}wax^2] + \mathrm{A},$$

where A is an integration constant. But since the beam is horizonta at its mid-point,

$$\left[\frac{dy}{dx}\right]_{x=a} = 0 = \frac{1}{\mathrm{EI}}[\tfrac{1}{6}wa^3 - \tfrac{1}{2}wa^3] + \mathrm{A},$$

i.e.

$$\mathrm{A} = \frac{1}{3}\cdot\frac{1}{\mathrm{EI}}\cdot wa^3.$$

Integrating again, we have,

$$y = \frac{1}{\mathrm{EI}}[\tfrac{1}{24}wx^4 - \tfrac{1}{6}wax^3 + \tfrac{1}{3}wa^3x],$$

the constant of integration vanishing since, when $x = 0$, $y = 0$.

$$\therefore \Big[y\Big]_{x=a} = \frac{5}{24}\cdot\frac{wa^4}{\mathrm{EI}}.$$

If we call W_0 the weight of the complete load carried by the beam, i.e. $W_0 = 2aw$, the deflexion at the middle is

$$\frac{5}{48} \cdot \frac{W_0 a^3}{EI},$$

i.e. the effect of distributing a load uniformly over a beam, instead of concentrating it at the centre, is to reduce the maximum deflexion to five-eighths its former value.

The stiffness of a beam.—***The stiffness of a beam is defined as the ratio of the maximum deflexion of the beam to its span.*** If the stiffness is denoted by η^{-1}, then for steel girders of large span $1000 < \eta < 2000$. For timber beams η should not be less than about 400.

General method of determining the deflexion of a beam.—The solutions of any of the problems relating to the bending of beams hitherto discussed, have all depended on the fact that it has been possible to use the equation

$$M = EI \frac{d^2y}{dx^2},$$

the general value for M being calculated by the methods already given. Now it is only possible to calculate the value for the bending moment when the number of supports does not exceed two, for before calculating M in a more complicated case it is necessary to determine the thrusts due to the supports on the beam, and the methods of statics do not permit us to determine these when the number of supports exceeds two. But it has already been shown [cf. p. 335] that

$$\frac{dS}{dx} = \frac{d^2M}{dx^2} = w,$$

the weight of the beam per unit length. Hence the differential equation becomes

$$EI \frac{d^4y}{dx^4} = w,$$

which will be written

$$EIy^{IV} = w.$$

The integration of this equation introduces four arbitrary constants which the remaining conditions of the problem suffice to determine. We shall illustrate this method of dealing with a problem concerning the bending of a beam, by considering again the deflexion of a uniformly loaded beam supported in a horizontal position by props at its ends. Before doing this, however, we have to establish an important theorem.

Lemma.—To show that at a supported end of a beam, the bending moment is zero.

Suppose that R is the thrust due to the support on the beam at the end which is taken to be the origin of coordinates. Let S and M be the shearing force and the bending moment, respectively, at a transverse section of the beam at a distance δx from the support—cf. Fig. 8·11(c). For this element of the beam to be in equilibrium the following conditions must be satisfied.

(a) Resolving forces vertically,

$$\mathrm{S} + \mathrm{R} - w\,\delta x = 0,$$

so that in the limit when $\delta x \to 0$, $\mathrm{S} = -\mathrm{R}$.

(b) Taking moments of forces about the origin, we have,

$$\mathrm{M} + w\,\delta x(\tfrac{1}{2}\delta x) - \mathrm{S}\,\delta x = 0.$$

Neglecting quantities of the second order,

$$\begin{aligned}\mathrm{M} &= \mathrm{S}\,\delta x\\ &= 0,\end{aligned}$$

when we proceed to the limit $\delta x \to 0$. Thus the theorem is established.

Uniformly loaded beam supported at its ends.—[Alternative Treatment.] Integrating the equation

$$\mathrm{EI}y^{\mathrm{IV}} = w,$$

we obtain

$$\mathrm{EI}y''' = wx + \mathrm{A},$$

where A is a constant to be determined. Integrating again, we get

$$\mathrm{EI}y'' = \tfrac{1}{2}wx^2 + \mathrm{A}x + \mathrm{B},$$

the constant B being zero, since the bending moment, and therefore y'', is zero at the fixed end, $x = 0$.

Another integration gives

$$\mathrm{EI}y' = \tfrac{1}{6}wx^3 + \tfrac{1}{2}\mathrm{A}x^2 + \mathrm{C},$$

the constant C being determined by the fact that at $x = a$, $y' = 0$. This gives

$$\mathrm{C} = -\tfrac{1}{6}wa^3 - \tfrac{1}{2}\mathrm{A}a^2.$$

Integrating again, we have,

$$\mathrm{EI}y = \tfrac{1}{24}wx^4 + \tfrac{1}{6}\mathrm{A}x^3 - \tfrac{1}{6}wa^3x - \tfrac{1}{2}\mathrm{A}a^2x,$$

no constant of integration being added since the deflexion is zero

at the point $x = 0$. To determine A we use the fact that y is also zero when $x = 2a$. This gives

$$\mathrm{EI}[y]_{x=2a} = 0 = \tfrac{2}{3}wa^4 + \tfrac{4}{3}\mathrm{A}a^3 - \tfrac{1}{3}wa^4 - \mathrm{A}a^3.$$

$$\therefore\ \mathrm{A} = -wa.$$

Hence
$$\mathrm{EI}y = \tfrac{1}{24}wx^4 - \tfrac{1}{6}wax^3 - \tfrac{1}{6}wa^3x + \tfrac{1}{2}wa^3x$$
$$= \tfrac{1}{24}wx^4 - \tfrac{1}{6}wax^3 + \tfrac{1}{3}wa^3x,$$

as before.

Built-in (or encastré) beams.—A beam which is so firmly fixed at each end that the supports prevent any change in the

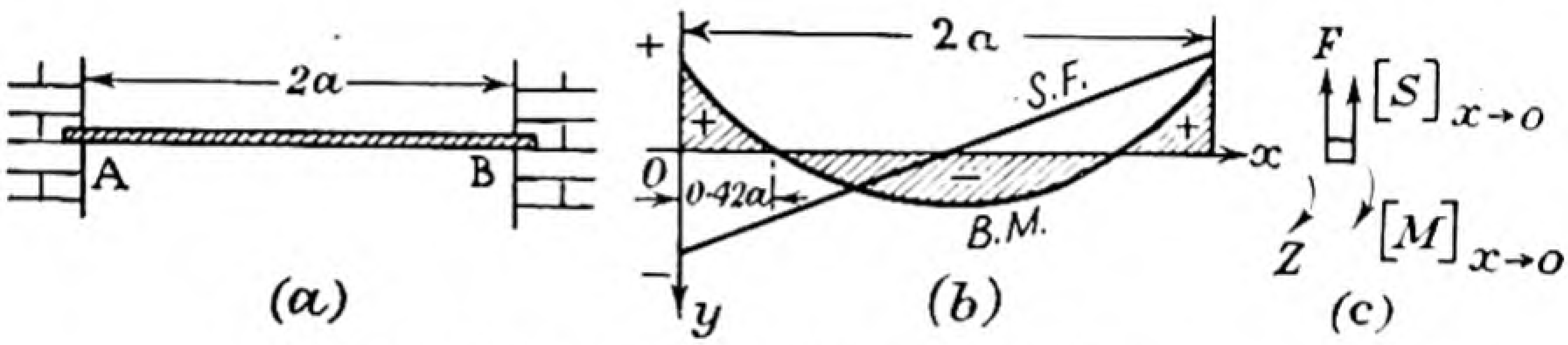


FIG. 8·12.—A built-in (or encastré) beam.

inclination of the beam at its ends, is termed a ***built-in beam.*** We shall only consider such a beam when its ends are in the same horizontal plane. It will be found that the effect of this method of securing a uniform beam is to reduce the deflexion at all points, i.e. there will be less tendency for the beam to break.

Let AB, Fig. 8·12(a), be a uniform built-in beam of length $2a$, and weight w per unit length, with its ends fixed in a horizontal plane. Then the differential equation, whose solution will give the shape of the neutral surface in the beam, is

$$\mathrm{EI}y^{\mathrm{IV}} = w,$$

the coordinate axes being as indicated.

$$\therefore\ \mathrm{EI}y'' = \tfrac{1}{2}wx^2 + \mathrm{A}x + \mathrm{B},$$

and
$$\mathrm{EI}y' = \tfrac{1}{6}wx^3 + \tfrac{1}{2}\mathrm{A}x^2 + \mathrm{B}x + \mathrm{C},$$

where A, B and C are integration constants.

Now
$$y' = 0 \text{ when } x = 0, \text{ and when } x = 2a.$$
$$\therefore\ \mathrm{C} = 0,$$

and
$$\mathrm{EI}[y']_{x=2a} = 0 = \tfrac{4}{3}wa^3 + 2\mathrm{A}a^2 + 2\mathrm{B}a.$$
$$\therefore\ \mathrm{EI}y = \tfrac{1}{24}wx^4 + \tfrac{1}{6}\mathrm{A}x^3 + \tfrac{1}{2}\mathrm{B}x^2,$$

the integration constant being zero, since $y = 0$ when $x = 0$.

Using the fact that the deflexion is also zero at the point $x = 2a$, we have,

$$EI[y]_{x=2a} = \tfrac{2}{3}wa^4 + \tfrac{4}{3}Aa^3 + 2Ba^2 = 0.$$

Solving the two equations in A and B, we have

$$A = -wa, \quad \text{and} \quad B = \tfrac{1}{3}wa^2.$$

$$\therefore\ EIy = \tfrac{1}{24}wx^4 - \tfrac{1}{6}wax^3 + \tfrac{1}{6}wa^2x^2.$$

The deflexion of the beam is a maximum when $x = a$. It is given by

$$EI[y]_{x=a} = \tfrac{1}{24}wa^4,$$

i.e. the deflexion at the centre is only one-fifth of what it would be if the same beam rested freely on end supports.

The bending moment at any point in a built-in horizontal beam is given by

$$M = EIy'' = \tfrac{1}{2}wx^2 - wax + \tfrac{1}{3}wa^2.$$

At $x = 0$, $\qquad [M]_{x=0} = \tfrac{1}{3}wa^2;$

at $x = a$, $\qquad [M]_{x=a} = -\tfrac{1}{6}wa^2;$

and at $x = 2a$, $\qquad [M]_{x=2a} = \tfrac{1}{3}wa^2.$

Also M is zero, when $x = a \pm \dfrac{a}{\sqrt{3}} = 0{\cdot}42a$ or $1{\cdot}58a$.

Since S, the shearing force, is given by $S = \dfrac{dM}{dx}$, we have $S = w(x - a)$. The bending-moment and shearing-force diagrams for a built-in beam are therefore as shown in Fig. 8·12(*b*).

To determine the vertical force, F, and the couple, Z, which the walls exert on the end of the beam we may proceed as follows. Consider the equilibrium of an element of the beam near to the end $x = 0$, cf. Fig. 8·12(*c*). We have

$$F = -[S]_{x\to 0} = wa, \qquad \text{and} \qquad Z = -[M]_{x\to 0} = -\tfrac{1}{3}wa^2,$$

i.e. Z is a couple of magnitude $\tfrac{1}{3}wa^2$ acting in an anticlockwise direction.

Example.—A beam of span l and of negligible weight is fixed horizontally at each end. Two equal loads, W, are placed at equal distances a from the ends of the beam. Investigate the deflexion of the beam at its mid-point.

It is at once apparent that the reaction of the support on the beam at each end is a force W, directed upwards, and there may possibly be a torque. At the end $x = 0$ let this torque be Z and suppose its sense is counter-clockwise.

Consider the equilibrium of a portion of the beam of length x, where

$0 < x < a$. The shearing forces are everywhere zero. Hence, with the usual notation,

$$\mathrm{EI}y'' = \mathrm{Z} - \mathrm{W}x,$$

which on integration gives

$$\mathrm{EI}y' = \mathrm{Z}x - \tfrac{1}{2}\mathrm{W}x^2,$$

the constant of integration being zero since at $x = 0$, $y' = 0$.

Integrating again we get

$$\mathrm{EI}y = \tfrac{1}{2}\mathrm{Z}x^2 - \tfrac{1}{6}\mathrm{W}x^3,$$

and the integration constant is again zero since at $x = 0$, $y = 0$.

Now consider the equilibrium of a portion of the beam defined by $a < x < \dfrac{l}{2}$. We have

$$\mathrm{EI}y'' = \mathrm{Z} - \mathrm{W}a,$$

which on integration yields

$$\mathrm{EI}y' = (\mathrm{Z} - \mathrm{W}a)x + \mathrm{A},$$

where A is an integration constant. Since $y' = 0$ at $x = \dfrac{l}{2}$, we have $\mathrm{A} = -(\mathrm{Z} - \mathrm{W}a)\dfrac{l}{2}$, so that

$$\mathrm{EI}y' = (\mathrm{Z} - \mathrm{W}a)\left(x - \frac{l}{2}\right),$$

and on integrating again we have

$$\mathrm{EI}y = \tfrac{1}{2}(\mathrm{Z} - \mathrm{W}a)\left(x - \frac{l}{2}\right)^2 + \mathrm{B},$$

where B is an integration constant.

Now in the equations so far obtained we have left the unknowns Z and B; their values may be found as follows. To determine Z we use the fact that y' is continuous at $x = a$, so that

$$\mathrm{Z}a - \tfrac{1}{2}\mathrm{W}a^2 = (\mathrm{Z} - \mathrm{W}a)\left(a - \frac{l}{2}\right),$$

or

$$\mathrm{Z} = \frac{\mathrm{W}a}{l}(l - a) = \mathrm{W}a\left[1 - \frac{a}{l}\right].$$

Also, at $x = a$ the deflexion is the same whichever part of the beam is considered. Hence

$$\tfrac{1}{2}(\mathrm{Z} - \mathrm{W}a)\left(a - \frac{l}{2}\right)^2 + \mathrm{B} = \tfrac{1}{2}\mathrm{Z}a^2 - \tfrac{1}{6}\mathrm{W}a^3,$$

which gives

$$\mathrm{B} = \tfrac{1}{2}\mathrm{W}a^2\left[\frac{l}{4} - \frac{a}{3}\right].$$

Thus the deflexion at any point in the beam is known and we get at once that

$$\Big[y\Big]_{x=\frac{1}{2}l} = \frac{\mathrm{W}a^2}{\mathrm{EI}}\left[\frac{l}{8} - \frac{a}{6}\right].$$

Experimental determination of Young's modulus by the non-uniform bending of a beam.—Let AB, Fig. 8·13(a), be a uniform beam for whose material Young's modulus is to be found. Let it rest on knife-edges C and D lying in a horizontal plane on two very rigid supports. A scale pan P is attached to the beam at the point H midway between C and D; the load, of weight W, in

the pan may be varied. To measure the deflexion of the beam at its mid-point a short length of a steel needle [a gramophone needle does well] is attached to the mid-point of the beam with wax and its point well illuminated, as shown in Fig. 8·13(*b*). A horizontal microscope, its eye-piece provided with a horizontal cross-wire, is

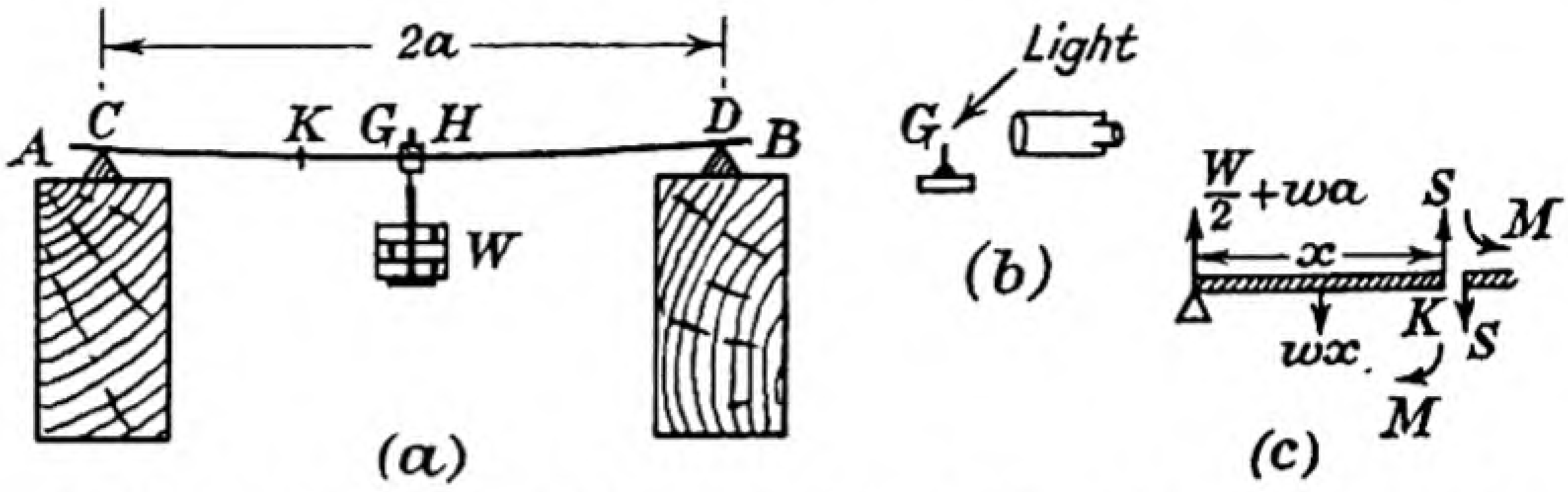


FIG. 8·13.—Experimental determination of Young's modulus by the non-uniform bending of a beam.

focused on the tip of the needle, the microscope itself standing on a rigid stand.

If $2a$ is the distance between C and D, w the weight per unit length of the beam, the bending moment at the transverse section K, Fig. 8·13(*c*), is given by

$$\mathrm{M} + \left(\frac{\mathrm{W}}{2} + wa\right)x - wx\left(\frac{x}{2}\right) = 0.$$

$$\therefore\ \mathrm{M} = -\frac{\mathrm{W}}{2}x - wax + \tfrac{1}{2}wx^2.$$

$$\therefore\ \mathrm{EI}y'' = -\frac{\mathrm{W}}{2}x - wax + \tfrac{1}{2}wx^2.$$

$$\therefore\ \mathrm{EI}y' = -\frac{\mathrm{W}}{4}x^2 - \tfrac{1}{2}wax^2 + \tfrac{1}{6}wx^3 + \mathrm{A},$$

where A is a constant which may be determined by the fact that $y' = 0$ when $x = a$. This gives

$$0 = -\frac{\mathrm{W}}{4}a^2 - \tfrac{1}{2}wa^3 + \tfrac{1}{6}wa^3 + \mathrm{A}.$$

$$\therefore\ \mathrm{A} = \frac{\mathrm{W}}{4}a^2 + \tfrac{1}{3}wa^3.$$

$$\therefore\ \mathrm{EI}y' = -\frac{\mathrm{W}}{4}x^2 - \tfrac{1}{2}wax^2 + \tfrac{1}{6}wx^3 + \frac{\mathrm{W}}{4}a^2 + \tfrac{1}{3}wa^3.$$

$$\therefore\ \mathrm{EI}y = -\frac{\mathrm{W}}{12}x^3 - \tfrac{1}{6}wax^3 + \tfrac{1}{24}wx^4 + \frac{\mathrm{W}}{4}a^2x + \tfrac{1}{3}wa^3x,$$

the constant of integration being zero, since $y = 0$ when $x = 0$.

The maximum deflexion of the beam occurs when $x = a$, and, if it is denoted by ξ, is given by

$$\mathrm{EI}\xi = -\frac{\mathrm{W}}{12}a^3 - \tfrac{1}{6}wa^4 + \tfrac{1}{24}wa^4 + \frac{\mathrm{W}}{4}a^3 + \tfrac{1}{3}wa^4,$$

$$= \tfrac{1}{6}\mathrm{W}a^3 + \frac{5}{24}wa^4.$$

$$\therefore\ \xi = \frac{a^3}{6\mathrm{EI}}\mathrm{W} + \text{constant}$$

$$= \frac{a^3 g}{6\mathrm{EI}}(m + p) + \text{constant},$$

if m is the mass carried in the pan of mass p, and g is the intensity of gravity,

$$\therefore\ \xi = \frac{a^3 g}{6\mathrm{EI}}m + \text{constant}.$$

If therefore, having obtained a series of corresponding values of m and ξ, we plot them as abscissae and ordinates respectively, we should obtain a straight line whose slope is

$$\frac{a^3 g}{6\mathrm{EI}}.$$

For a beam of circular cross-section, of radius r, $\mathrm{I} = \frac{1}{4}\pi r^4$; for a beam of rectangular cross-section of breadth a and depth b, $\mathrm{I} = \frac{1}{12}ab^3$; while for a tube with a cross-section of internal radius r_1 and external radius r_2, $\mathrm{I} = \frac{1}{4}\pi(r_2^4 - r_1^4)$. The value for E may therefore be calculated for the material of any beam likely to be under investigation.

Determination of Young's modulus when a beam is uniformly bent.—Let us examine the conditions necessary for a uniform beam of isotropic material to be uniformly bent, i.e. the radius of curvature is constant. Since $\mathrm{M} = \frac{\mathrm{EI}}{\mathrm{R}}$, it follows that M must be constant over any portion of a beam which is bent so that its neutral surface is part of the circumference of a circle. This is obtained in practice in the following way.

Suppose that AB, Fig. 8·14(a), is a uniform beam of length $(l + 2a)$ resting on piers at C and D each at a distance a from the end of the beam nearer to it. Let the points of support lie in a horizontal plane, and suppose that the beam carries a load of weight W at each end. Then, by symmetry, the thrusts on the beam

at C and D are equal, the actual thrust being W, if the beam is so light that its weight per unit length is negligible. Let A be the origin of rectangular coordinates. Then at a transverse section at

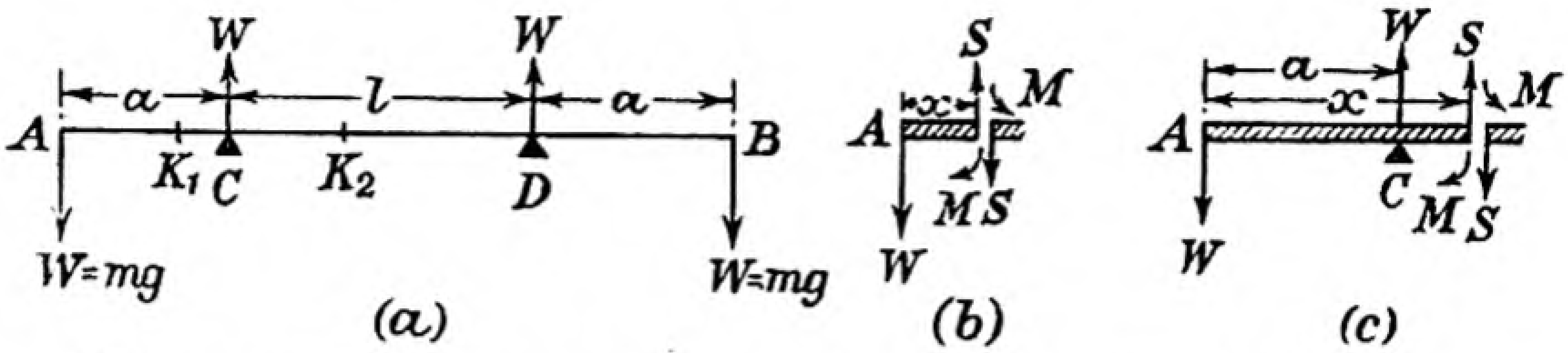


FIG. 8·14.—Experimental determination of Young's modulus by the uniform bending of a light beam.

K_1, at a distance x from A, where $0 \leqslant x \leqslant a$, we have, cf. Fig. 8·14(b), in the usual way,

$$S = W,$$

and
$$M - Sx = 0.$$

$$\therefore\ M = Wx.$$

At K_2, at distance x from A, where $a \leqslant x \leqslant (a + l)$, we have, cf. Fig. 8·14(c),

$$S = 0,$$

and hence taking moments of forces about C,

$$M = Wa = \text{constant},$$

since W and a are constants in any particular instance. It therefore follows that the central span of the beam is uniformly bent, its radius of curvature being given by

$$\frac{EI}{R} = Wa = mga,$$

if m is the mass of each end load.

Let h be the elevation at the mid-point, which may be measured as in the previous experiment. Then, by the well-known property of a circle,

$$(2R - h)h = \left(\frac{l}{2}\right)^2.$$

$$\therefore\ R = \frac{l^2}{8h},$$

since h is small in all practical cases where the elastic limit must not be exceeded.

$$\therefore\ E = \frac{l^2}{8h}\cdot\frac{mga}{I}.$$

[N.B. If the beam possesses a weight per unit length which is not negligible, the shearing force at sections in the central span of the beam cannot vanish, and therefore M cannot be constant since, in general,

$$\frac{dM}{dx} = S,$$

and a constant value for M requires S to be zero.]

Alternative treatment: Since $M = Wa$, we have

$$EIy'' = Wa$$

$$\therefore \; EIy' = Wax + A,$$

where A is an integration constant. Now at the centre of the beam, i.e. at $x = b$, say, $y' = 0$.

$$\therefore \; EIy' = Wa(x - b).$$

$$\therefore \; EIy = Wa(\tfrac{1}{2}x^2 - bx) + B,$$

where B is an integration constant determined by the fact that $y = 0$ when $x = a$.

$$\therefore \; 0 = Wa(\tfrac{1}{2}a^2 - ab) + B.$$

$$\therefore \; EIy = Wa[\tfrac{1}{2}x^2 - bx - \tfrac{1}{2}a^2 + ab].$$

To get $y_{\text{max.}}$, i.e. the deflexion at the centre of the beam, put $x = b$.

$$\therefore \; EIy_{\text{max.}} = -\tfrac{1}{2}Wa[b - a]^2 = -\tfrac{1}{2}mag[b - a]^2 = -\tfrac{1}{8}magl^2.$$

so that $y_{\text{max.}}$ is negative since E and I are each essentially positive.

Experimental determination of Poisson's ratio for the material of a beam.—It has been shown [cf. p. 345] that when a beam of isotropic material is bent the ratio of the radius of curvature in the plane of bending to the radius of anticlastic curvature is equal to σ, Poisson's ratio for the material of the beam. To determine these radii of curvature for an iron bar (say), and hence a value for σ, let the bar be loaded as indicated in Fig. 8·14(a), i.e. the central span will be uniformly bent since the bending moment is constant at all transverse sections in this part of the bar, provided, of course, that the weight per unit length of the beam is small. The radius of curvature for the neutral surface is easily determined by measuring the distance between the knife-edges C and D and, with the aid of a travelling microscope, the elevation of the mid-point of the front edge of the bar when the load is applied. To determine the radius of anticlastic curvature is more difficult. It is sometimes done by soldering to the edges of the bar at their mid-points two

brass connectors, provided with set screws; such connectors are often used in electrical experiments but here we shall call them clamps. Their axes must be normal to the upper face of the bar. Two needles, not necessarily straight, each about 40 cm. long, are securely fastened to the clamps. One of these needles carries at its free end a mm. scale and the free end of the second needle just rests against the scale. When the bar is bent the two needles rotate about horizontal axes towards each other, the relative displacement of the end of one needle with respect to that of the other being determined directly with the aid of the mm. scale. If the length of

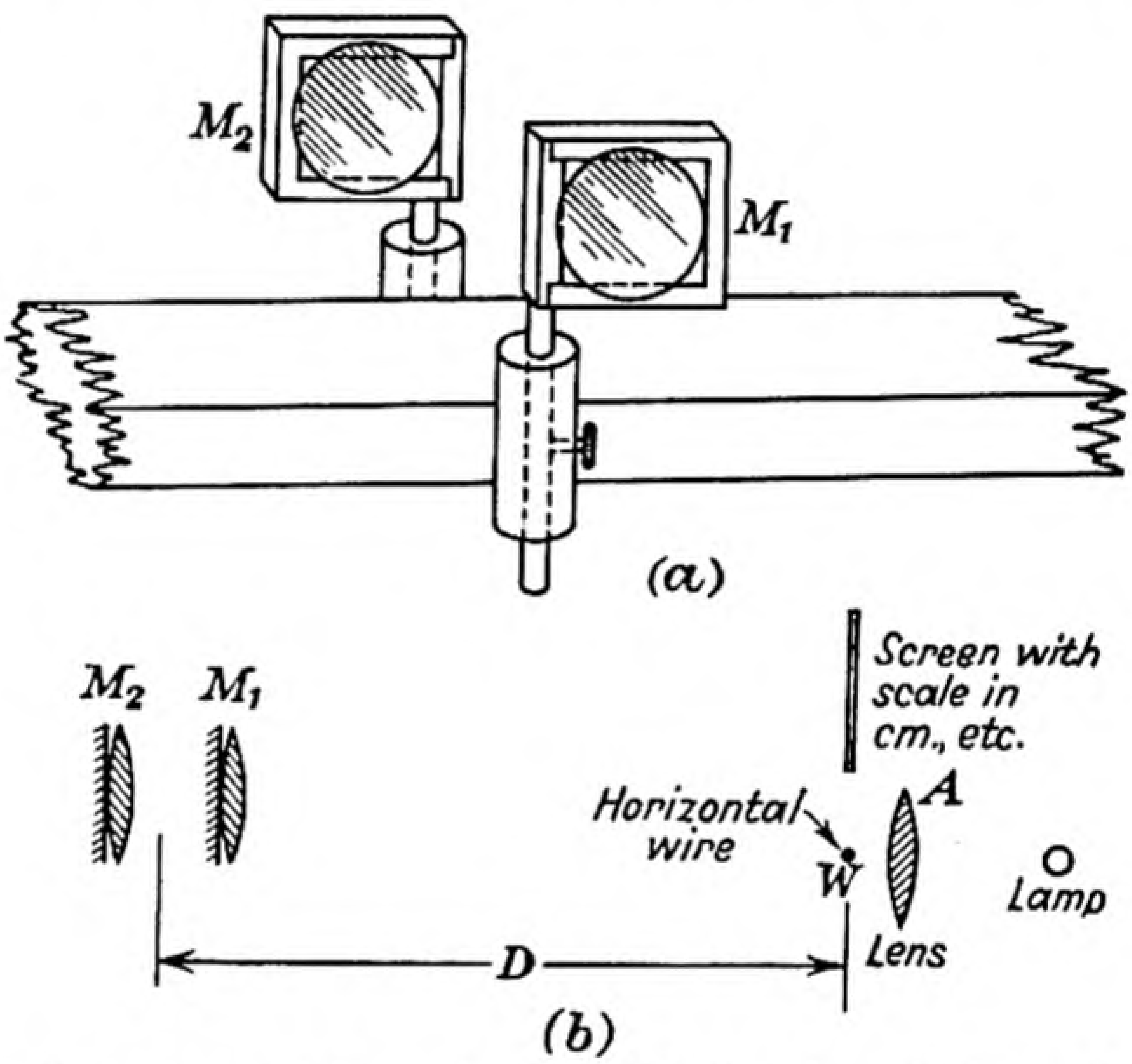


FIG. 8·15.—Experimental determination of Poisson's ratio from observations on a uniformly bent beam.

the needle above the neutral axis of the bar is known, as well as the width of the bar, the radius of anticlastic curvature is readily derived. For iron bars of a convenient size the relative movement does not amount to more than 3 or 4 mm. if the elastic limit for the material of the bar is not to be exceeded; a microscope should therefore be used to measure this displacement if accurate results are required.

The author has found the following arrangement more convenient and accurate. The needles are replaced by two small plane mirrors M_1 and M_2, Fig. 8·15(a), supported in copper frames soldered to short lengths of thick copper wire which fit the clamps. These copper wires may be slightly bent so that the mirrors are parallel or almost parallel to each other. To the front of each mirror there is fixed a converging lens of two metres focal length. A horizontal

beam of light from a filament lamp, cf. Fig. 8·15(*b*), is then directed on to the mirrors. This is done with the aid of a converging lens, A, across the front of which a fine straight wire, W, is stretched. Two images of this wire, one from each plane mirror and its attached lens, are obtained on a vertical translucent screen placed near to the wire W. This is possible on account of the large depth of focus possessed by a converging lens of long focal length. When the beam is loaded the images of the wire are displaced; from a knowledge of these displacements and the mean distance of the mirrors from the scale, the changes in inclination of the mirrors can be calculated. If the width of the beam is known, the radius of anticlastic curvature is at once determinable.

If d is the sum of the displacements of the images, D, the mean distance between the mirrors and the scale, the change in the inclination of the mirrors when the bar is loaded is given by

$$\phi = \frac{d}{2\mathrm{D}};$$

each mirror will move through half this angle. If b is the breadth of the bar,

$$\mathrm{R}_2 = \text{radius of anticlastic curvature} = \frac{b}{\phi} = \frac{2b\mathrm{D}}{d}.$$

If y_m is the elevation at the centre of the bar, then R_1, the radius of curvature in the plane of bending, is given by

$$2\mathrm{R}_1 y_m = \tfrac{1}{4}l^2,$$

where l is the distance between the knife-edges. But $\sigma\mathrm{R}_2 = \mathrm{R}_1$, i.e.

$$\frac{2\sigma b\mathrm{D}}{d} = \frac{l^2}{8y_m},$$

or

$$y_m = \frac{l^2}{16\sigma b\mathrm{D}}d.$$

If, therefore, a series of loads is applied to the bar and we plot $y_m = y$ and $d = x$, a straight line with slope $\dfrac{l^2}{16\sigma b\mathrm{D}}$ should be obtained. From this slope a value for Poisson's ratio for the material of the bar, may be deduced.

To calculate a value for Young's modulus for iron, use is made of the formula developed on p. 358.

Experimental determination of Young's modulus for the material of a cantilever.—A cantilever is a horizontal uniform

beam clamped at one end and it has been shown [cf. p. 339] that when a cantilever of weight w per unit length carries a load of weight W at its free end the bending moment at a distance x from the free end is given by

$$\mathrm{EI}y'' = \mathrm{M} = \mathrm{W}(l - x) + \tfrac{1}{2}w(l - x)^2.$$

$$\therefore\ \mathrm{EI}y' = -\tfrac{1}{2}\mathrm{W}(l - x)^2 - \tfrac{1}{6}w(l - x)^3 + \tfrac{1}{2}\mathrm{W}l^2 + \tfrac{1}{6}wl^3,$$

the constant terms being obtained from the condition that $y' = 0$ when $x = 0$.

Integrating again, we get

$$\mathrm{EI}y = \tfrac{1}{6}\mathrm{W}(l - x)^3 + \tfrac{1}{24}w(l - x)^4 + \tfrac{1}{2}\mathrm{W}l^2x + \tfrac{1}{6}wl^3x + \mathrm{C},$$

where C is a constant of integration; since $y = 0$ when $x = 0$,

$$\mathrm{C} = -\tfrac{1}{6}\mathrm{W}l^3 - \tfrac{1}{24}wl^4.$$

$$\therefore\ \mathrm{EI}y = \tfrac{1}{2}\mathrm{W}lx^2 - \tfrac{1}{6}\mathrm{W}x^3 + \tfrac{1}{24}w(l - x)^4 + \tfrac{1}{6}wl^3x - \tfrac{1}{24}wl^4.$$

$$\therefore\ \mathrm{EI}[y]_{x=l} = \tfrac{1}{3}\mathrm{W}l^3 + \tfrac{1}{8}wl^4.$$

$$\therefore\ [y]_{x=l} = \xi = \frac{l^3}{3\mathrm{EI}}\,[\mathrm{W} + \tfrac{3}{8}\mathrm{W}_0],$$

where $\mathrm{W}_0 = wl$, the total weight of the beam.

(*a*) ***Statical method:*** If therefore a series of known loads is applied to the free end of the cantilever and the deflexion, ξ, measured for each load, the graph obtained by plotting ξ against W will be linear with a slope $\dfrac{l^3}{3\mathrm{EI}}$, so that Young's modulus for the material of the cantilever may be obtained. Strictly speaking, we do not measure ξ, the displacement of the free end of the lever below a horizontal plane through its fixed end but a distance χ which is the change in position of the free end when the load is applied. But since $\chi + \alpha = \xi$, where α is a constant, the slope of the graph obtained by plotting χ against W is still $\dfrac{l^3}{3\mathrm{EI}}$.

For a long cantilever such as a wooden lath 150 cm. long, 0·5 cm. thick and 2·5 cm. wide, the initial deflexion of the beam is considerable so that the theory of bending developed in this chapter does not apply. This difficulty is overcome by clamping the lever OA, Fig. 8·16(*a*), so that it may only be deflected in a horizontal plane; the weight of the lever is then immaterial. To deflect the lever mounted in this way a light string, Fig. 8·15(*b*), is attached to the end of the lever and this string, after passing over a light pulley†

† It is better to use a light aluminium tube resting on a plane inclined at 45° to the horizontal; cf. Fig. 8·23(*b*), p. 379 and I.P. p. 769.

at least 5 cm. in diameter, carries a scale-pan. When the pan is loaded the beam is deflected and the deflexions can be measured first with the load in the pan being increased, and then decreased. Moreover, the direction of the applied force can be reversed and thus a more reliable mean value for the change in deflexion per unit change in the deflecting force may be obtained.

To correct for the presence of any small frictional forces at the

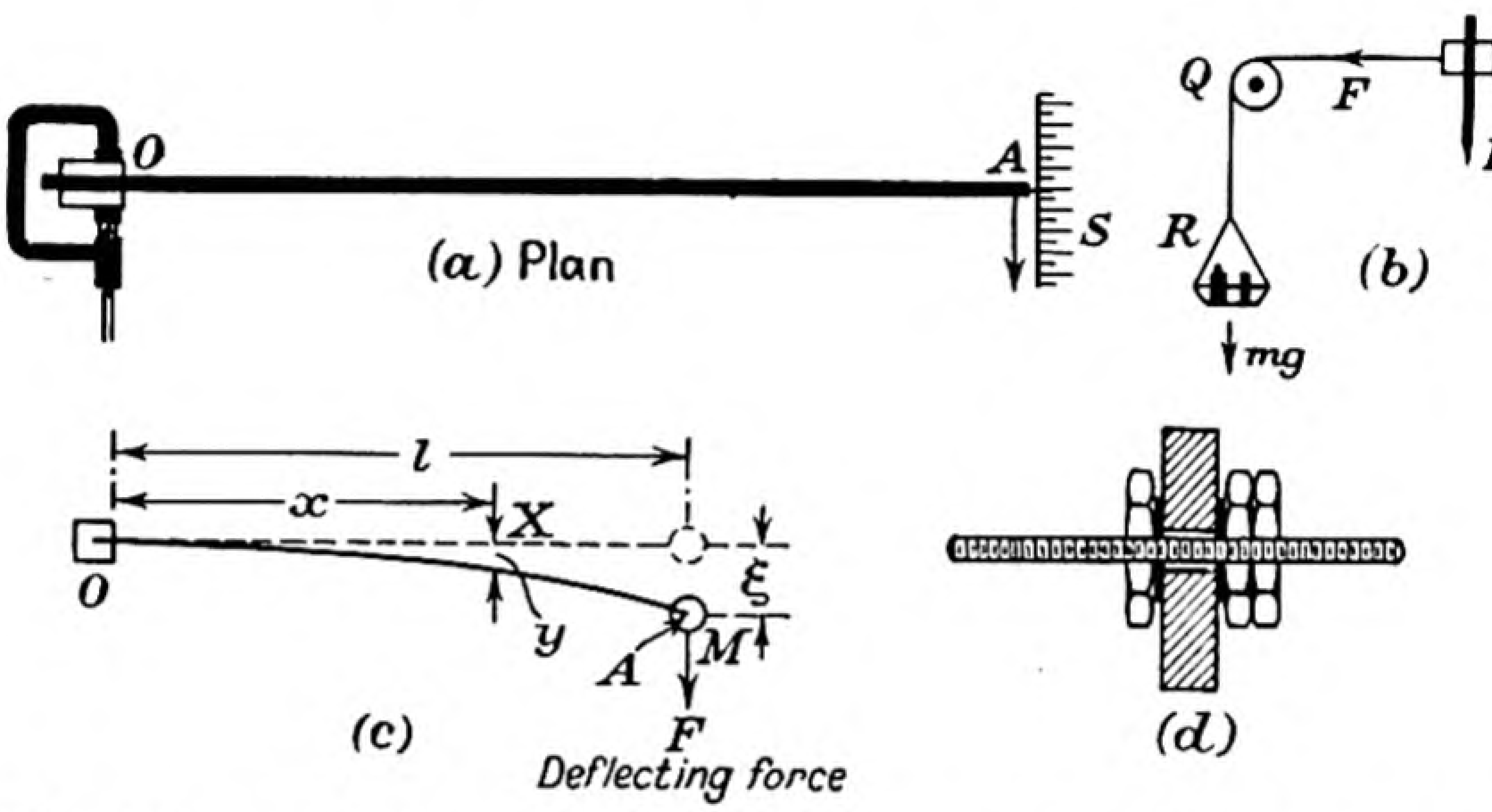


FIG. 8·16.—A cantilever: experimental determination of Young's modulus for its material.

pulley, which are not eliminated by taking observations on each side of the equilibrium position of the cantilever, we may write

$$\xi = \frac{mg + \beta}{\mathrm{EI}} \cdot \frac{l^3}{3},$$

where $mg = W$ and β is a small constant; the graphical method already suggested enables us to find E without a knowledge of β.

(*b*) ***Dynamical method:*** The above arrangement of a cantilever also enables us to find a value for Young's modulus for the material of the cantilever by determining the periods of vibration of the lever when different masses are attached to its end. For the particular cantilever specified the motion is relatively slow, so that the transits of the end of the lever past a fiducial mark can be seen and a stroboscopic method is not necessary to find the period T. The mass of the lever, however, while it does not affect the statical experiment cannot now be neglected; its effect may be examined in the following way.

Let F be the instantaneous value of the deflecting force [in a horizontal plane] at the end of the lever. Then the instantaneous

deflexion y at X, Fig. 8·15(c), at a distance x from the fixed end, is given by

$$\mathrm{EI}y = \tfrac{1}{2}\mathrm{F}lx^2 - \tfrac{1}{6}\mathrm{F}x^3,$$

a fact easily verified since, cf. p. 338,

$$\mathrm{EI}y'' = \mathrm{M} = \mathrm{F}(l - x),$$

where F is now written for W.

At $x = l$, the instantaneous deflexion ξ is given by

$$\mathrm{EI}\xi = \tfrac{1}{3}\mathrm{F}l^3,$$

so that

$$y = \left[\frac{3}{2}\frac{x^2}{l^2} - \frac{1}{2}\frac{x^3}{l^3}\right]\xi.$$

Hence the instantaneous velocity of a particle in the beam at a distance x from the fixed end is given by

$$\dot{y} = \left[\frac{3}{2}\frac{x^2}{l^2} - \frac{1}{2}\frac{x^3}{l^3}\right]\dot{\xi}.$$

If μ is the mass per unit length of the cantilever, its kinetic energy is

$$\int_0^l \tfrac{1}{2}\mu\left[\frac{3}{2}\frac{x^2}{l^2} - \frac{1}{2}\frac{x^3}{l^3}\right]^2 \dot{\xi}^2\, dx = \frac{33}{280}\mu l\dot{\xi}^2.$$

The kinetic energy of the mass M carried at the end of the lever is $\tfrac{1}{2}\mathrm{M}\dot{\xi}^2$.

The potential energy of the beam in any position is equal to the work done in causing it to take up that position. In the position considered it is therefore given by

$$\int_0^\xi \mathrm{F}\, d\xi = \int_0^\xi \frac{3\mathrm{EI}}{l^3}\,\xi\, d\xi = \frac{3}{2}\frac{\mathrm{EI}}{l^3}\,\xi^2.$$

Since the total kinetic energy + the potential energy is constant,

$$\left(\tfrac{1}{2}\mathrm{M} + \frac{33}{280}\mu l\right)\dot{\xi}^2 + \frac{3}{2}\frac{\mathrm{EI}}{l^3}\,\xi^2 = \text{constant},$$

so that by differentiating with respect to t, we get

$$\left(\tfrac{1}{2}\mathrm{M} + \frac{33}{280}\mu l\right)\ddot{\xi} + \frac{3}{2}\frac{\mathrm{EI}}{l^3}\,\xi = 0,$$

as the equation of motion. The motion is therefore simple harmonic, with a period given by

$$\mathrm{T} = 2\pi\sqrt{\frac{(\mathrm{M} + \frac{33}{140}\mu l)l^3}{3\mathrm{EI}}}.$$

Again, a graphical method, in which T^2 is plotted against M, serves to determine both E and μ. Now this theory of the vibrating beam is probably not quite complete and therefore it is not desirable to find μ by this method. The term $\frac{33}{140}\mu l$ may be regarded as a constant and this assumption may be tested by using the graphical method already indicated.

Alternative treatment: If the cantilever is loaded at its free end with a mass M and then this end is further displaced by an amount χ a force will be called into play tending to restore the cantilever to its equilibrium position. This force is directly proportional to χ and when the lever is released this restoring force will cause the mass M to execute simple harmonic motion about its equilibrium position.

Now for a load of mass m the displacement ξ is given by

$$\xi = \frac{mgl^3}{3EI},$$

so that the restoring force per unit displacement is

$$\frac{mg}{\xi} = \frac{3EI}{l^3}.$$

Hence for a displacement χ from the equilibrium position of the mass M, the restoring force is $\frac{3EI}{l^3}\chi$ and this operates on the mass M and on the cantilever.

The motion of the load M is therefore given by the equation

$$(M + m_0)\ddot{\chi} + \frac{3EI}{l^3}\chi = 0,$$

where m_0 is a correction term, assumed constant, on account of the mass of the lever. The period of oscillation is therefore

$$T = 2\pi\sqrt{\frac{(M + m_0)l^3}{3EI}}.$$

If therefore a series of corresponding values of M and T is obtained and M plotted against T^2, the slope of the graph is constant and equal to $\frac{4\pi^2 l^3}{3EI}$ if the assumptions made in our argument are correct. Thus a value for Young's modulus for the material of the cantilever may be found.

In the dynamical experiment it is not possible to load the cantilever with a scale-pan and so a brass screw is passed through the

lever at its free end and fixed in position with brass nuts—cf. Fig. 8·16(*d*). Brass discs of known mass are threaded on to the brass screw to increase the mass carried by the lever. If the observations are dealt with graphically it is not necessary to know the mass of the brass screw and the fixing nuts.

[It is interesting to note that if the mass of the cantilever is negligible then

$$T = 2\pi\sqrt{\frac{Ml^3}{3EI}}$$

and since the displacement due to a load of mass M is given by

$$\xi = \frac{1}{3}\frac{Mgl^3}{EI},$$

we have
$$T = 2\pi\sqrt{\frac{\xi}{g}},$$

Thus a value for the intensity of gravity may be obtained.]

Determination of the diameter of a fine wire by using the wire as a cantilever.—From the equations obtained on p. 362 we find that the deflexion of an unloaded cantilever is given by

$$\xi = \frac{1}{8}\frac{l^3}{EI}W_0 = \frac{1}{8}\frac{l^3}{EI}M_0 g = \frac{1}{8}\frac{l^3}{EI}(\pi r^2 \rho_0 l)g,$$

where W_0 is the weight and M_0 the mass of the lever, r its radius of cross-section and ρ_0 the density of its material. If the lever is completely immersed in a liquid of density ρ, the deflexion ξ_ρ is given by

$$\xi_\rho = \frac{1}{8}\frac{l^3}{EI}(\pi r^2 l g)(\rho_0 - \rho).$$

From these equations we have

$$\frac{\xi_\rho}{\xi} = \left(1 - \frac{\rho_0}{\rho}\right),$$

so that if ρ is known a value for ρ_0 can be found. If E is known, the radius of the wire may be determined.

An apparatus designed by the author for use in such an experiment is shown in Fig. 8·17(*a*)–(*b*). A brass disc A is clamped to a second brass disc B by means of a screw and nut C. A rod is soldered to B so that the apparatus may be supported in a clamp. FG is a brass rod soldered to A and this remains in a horizontal position when the screw D is in the position shown. When D is withdrawn the rod FG falls approximately into a vertical position and is held

exactly vertical by inserting the screw D into the aperture E in B, the centres of D and E having been made to subtend an angle $\frac{1}{2}\pi$ at the centre of the disc B.

One end of the wire under investigation is fixed in a small clamp P—cf. Fig. 8·17(*c*) for details concerning the structure of P. [P is movable along FG to accommodate wires of different lengths.] With FG and the wire pointing vertically downwards a pin R at the end of FG is adjusted until its tip touches the end of the wire. When

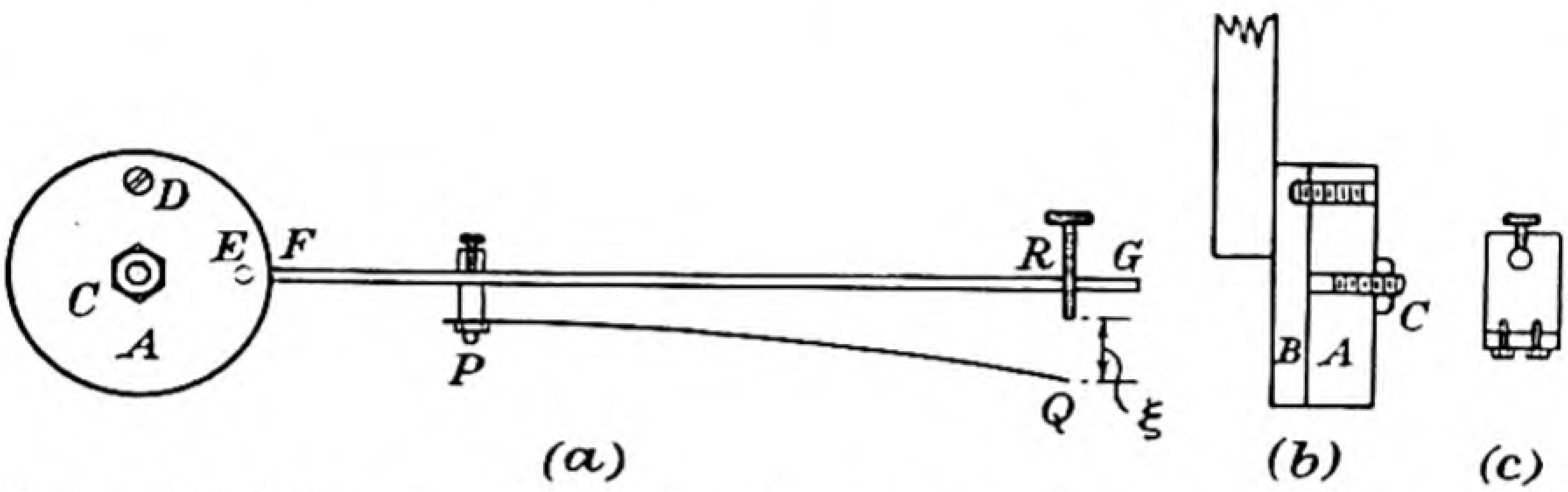


Fig. 8·17.—Determination of the density of the material of a fine wire.

FG is horizontal the wire is deflected from the horizontal position it would occupy in the absence of gravity and the displacement ξ may be measured with a travelling microscope.

An aqueous solution of hypo, of known density, is placed in a tank with vertical glass sides and the experiment repeated so that ξ_ρ is obtained. If E is known, a value for the radius of cross-section of the wire may be obtained in the manner already indicated.

An optical method for determining the deflexion of a beam.—Let AB, Fig. 8·18, be a beam of length $2a$ but of negligible weight per unit length, resting in a symmetrical position on two knife-edges lying in a horizontal plane. Let M_1 be a small plane mirror rigidly fixed to one end of the beam. S is a vertical scale

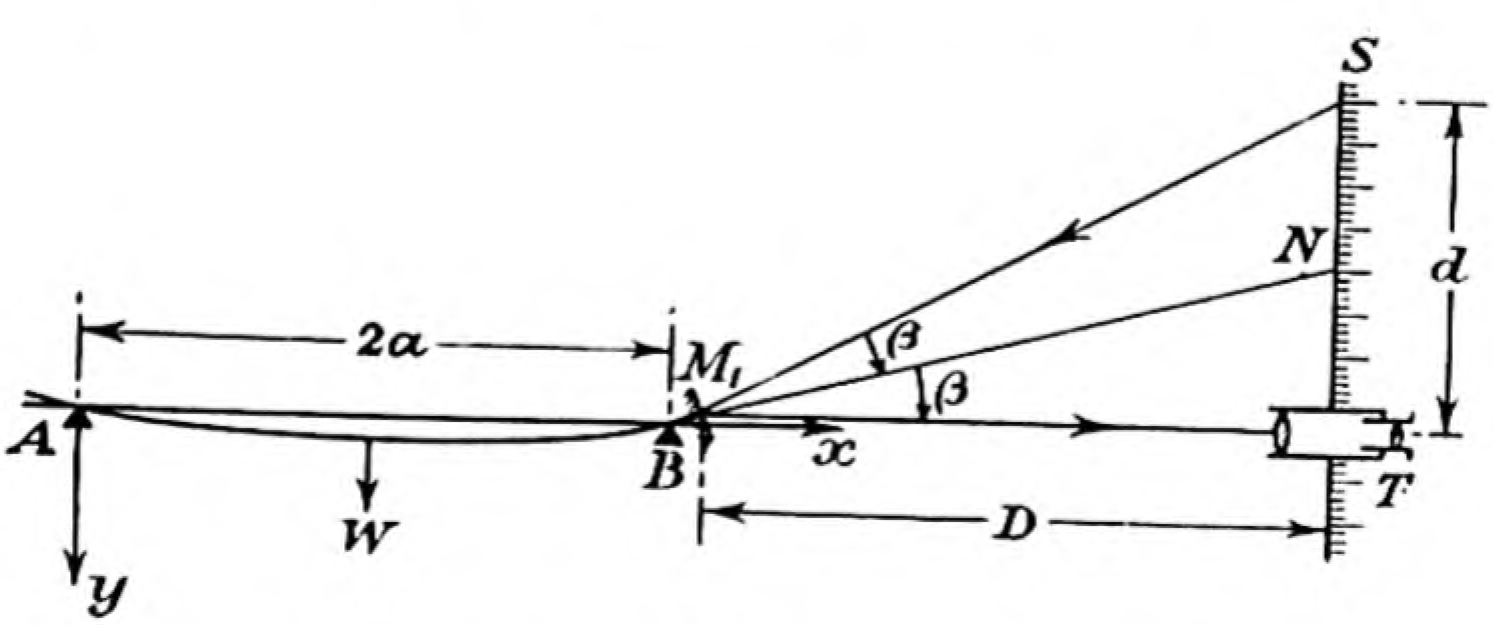


Fig. 8·18.—An optical method for determining the inclination at one end of a non-uniformly bent beam.

in centimetres, etc., situated in the vertical plane containing the beam at a distance D from M_1, its image formed by reflexion in the plane mirror being observed through a fixed telescope T. When the beam carries a load of weight W at its mid-point, let β be the angle of inclination of the beam to the horizontal as shown. If d is the difference between the scale readings whose images are observed on the cross-wires of the telescope when the beam is unloaded and then when it is loaded, since β is small,

$$\tan 2\beta = \frac{d}{\mathrm{D}}, \qquad \text{or} \qquad \tan \beta = \frac{d}{2\mathrm{D}}.$$

If M is the bending moment at a section K in the beam, at a distance x from A, where $a \leqslant x \leqslant 2a$, we have, in the usual way,

$$\frac{\mathrm{W}}{2}.x + \mathrm{M} - \mathrm{W}(x - a) = 0.$$

$$\therefore \ \mathrm{EI}y'' = \mathrm{W}\left(\frac{x}{2} - a\right).$$

$$\therefore \ \mathrm{EI}y' = \mathrm{W}\left(\frac{x^2}{4} - ax\right) + \mathrm{A},$$

where A is a constant whose value is determined by the fact that at $x = a$, $y' = 0$. This gives

$$\mathrm{A} = \tfrac{3}{4}a^2\mathrm{W}.$$

The slope at $x = 2a$ is $\tan(\pi - \beta)$ and is given by

$$\mathrm{EI}\tan(\pi - \beta) = \mathrm{W}[a^2 - 2a^2 + \tfrac{3}{4}a^2] = -\frac{\mathrm{W}a^2}{4}.$$

$$\therefore \ \tan\beta = \frac{\mathrm{W}a^2}{4\mathrm{EI}}.$$

Hence
$$d = \frac{ga^2\mathrm{D}}{2\mathrm{EI}}m,$$

where m is the mass of the load and g is the intensity of gravity.

In carrying out an experiment, several loads should be placed in the pan attached to the beam at its mid-point, and corresponding values of m and d plotted as abscissae and ordinates respectively. The slope of the best straight line drawn through the points thus obtained gives a mean value for $\dfrac{g\mathrm{D}a^2}{2\mathrm{EI}}$. This value is used in the above formula to find E.

The above proof assumes that the mirror M_1 is near to the end of the beam—otherwise it would suffer a translation as well as a rotation. To increase the accuracy of the method a second mirror may be placed at the other end of the beam and, with the aid of another telescope, the inclination found there also.

[The formula which we have derived could have been obtained by writing down the differential equation obtained by considering the equilibrium of the first half of the beam and calculating the slope at the end $x = 0$, which is $\tan \beta$. The proof given was adopted to illustrate the method to be applied in a more complicated instance.]

König's method for finding Young's modulus for the material of a uniform beam.—This method was devised by

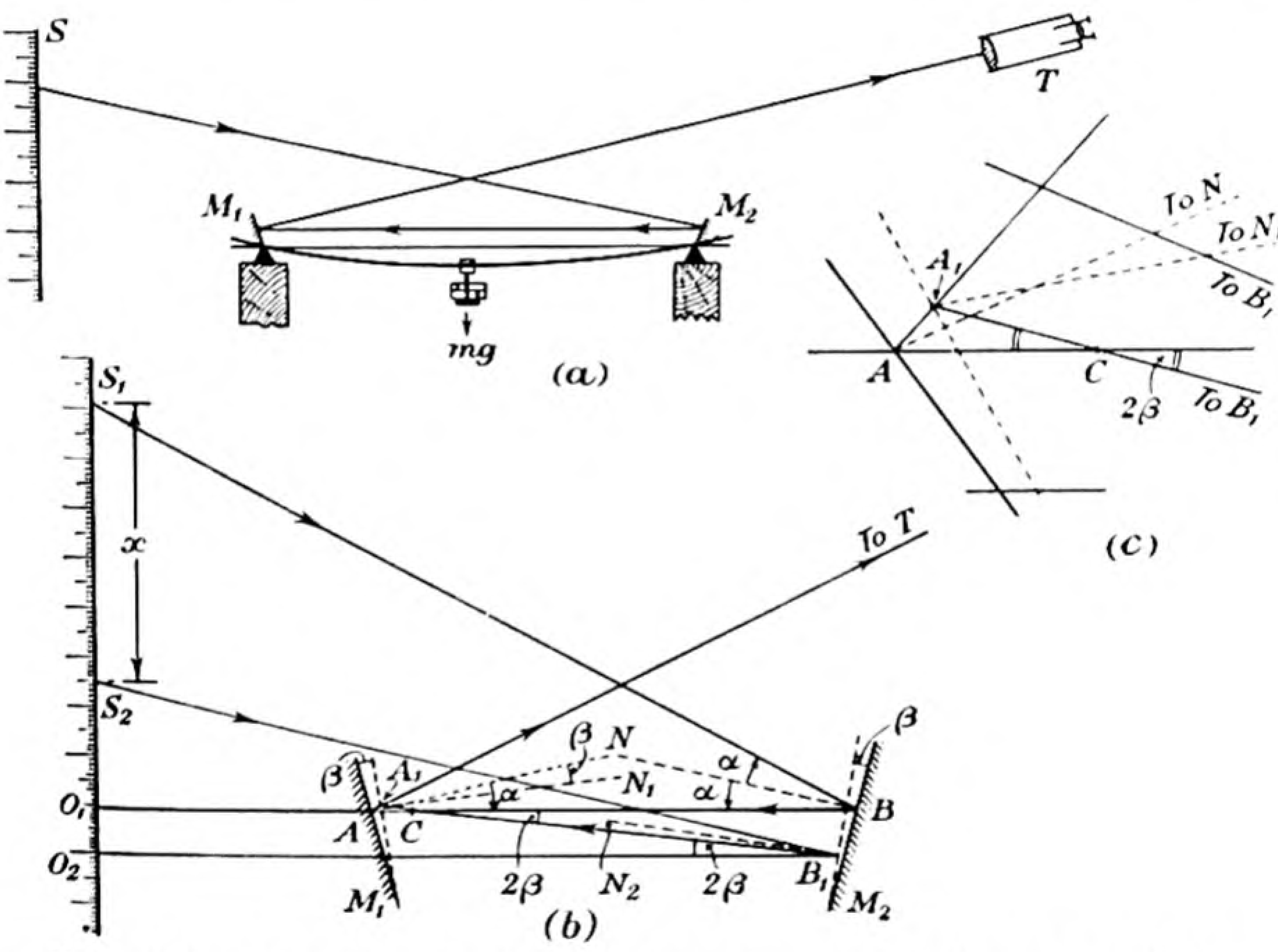


FIG. 8·19.—König's (optical) method for determining Young's modulus for the material of a beam.

KÖNIG† in 1885, the method just described being a simplified form of it. The advantage of König's method is that simultaneous use is made of the changes of inclination at each end of the beam. The apparatus is shown diagrammatically in Fig. 8·19(*a*). Two plane mirrors, M_1 and M_2, whose normals each make an angle α with the horizontal when the beam is unloaded, are rigidly fixed

† König, *Berlin Phys. Ges. Verh.*, 1885.

to the beam under investigation at points just beyond the knife-edges supporting the bar. The images of the divisions on a vertical scale S situated in the plane containing the apparatus are viewed through a telescope T. When the beam is loaded with a mass m at its centre, let β be the inclination of the beam at either end to the horizontal; then each mirror will rotate through an angle β. Suppose that x is the change in the scale reading observed through T, i.e. x is the distance between those divisions on S whose images are seen in T, firstly when the beam is unloaded, and secondly when it is loaded.

To proceed, let us imagine a ray of light to pass from the telescope along its axis to the scale after suffering reflexion at each mirror in turn. Let this ray strike M_1 at A, Fig. 8·19(b), and then proceed via AB to strike M_2 at B. It is then reflected along BS_1. Let BA produced cut the scale in O_1. Call $BO_1 = D$ and $AB = d$. It is now necessary to correlate s, d and D with the angle β. Let AN and BN be the normals to the mirrors at A and at B respectively. If α is the angle which the normal to each mirror at the point of incidence makes with the horizontal, then

$$S_1O_1 = D \tan 2\alpha.$$

When M_1 moves through an angle β, let the ray from T striking the first mirror at A_1 be reflected along A_1B_1, where B_1 is the point of incidence on the second mirror. If AB and A_1B_1 intersect at C, cf. Fig. 8·19(c) where the rays concerned are shown in greater detail,

$$A\widehat{C}A_1 = B\widehat{C}B_1 = 2\beta.$$

Now the normal, viz. B_1N_2, to M_2 at B_1 intersects the normal to that mirror at B at an angle β. Hence the normal at B_1 makes an angle $(\alpha - \beta)$ with the horizontal.

$$\therefore\ C\widehat{B_1}N_2 = (\alpha - \beta) - C\widehat{B_1}O_2 \quad [\because\ B_1O_2 \text{ is parallel to AB}]$$

$$= \alpha - 3\beta.$$

Now $$O_2\widehat{B_1}S_2 = C\widehat{B_1}O_2 + 2.N_2\widehat{B_1}C_1$$

$$= 2\beta + 2(\alpha - 3\beta) = 2\alpha - 4\beta.$$

Also $$\frac{S_1O_1}{D} = \tan 2\alpha = 2\alpha,$$

if α is small, and

$$\frac{S_2O_2}{D} = \tan (2\alpha - 4\beta) = 2\alpha - 4\beta,$$

β being small. In these equations it is assumed that B and B_1 are equidistant from the scale.

Now
$$\begin{aligned} x &= S_1S_2 = O_1S_1 - O_1S_2 \\ &= O_1S_1 - [O_2S_2 - O_2O_1] \\ &= 2\alpha(D) - (2\alpha - 4\beta)D + 2\beta(d) \\ &= 4\beta D + 2\beta d. \end{aligned}$$
$$\therefore\ \beta = \frac{x}{4D + 2d}.$$

But, cf. p. 368, $\tan\beta = \frac{1}{4}\frac{W}{EI}a^2$, where $d = 2a$, so that a value for E may be obtained.

To ensure that the approximations we have contemplated shall be realized in practice, the scale is placed at a considerable distance from the mirrors and the telescope arranged so that the rays of light just clear the tops of the mirrors.

When this method is used to determine Young's modulus for the material of a beam, greater accuracy is attained than by using a method in which the depression at the centre is measured, because the angle β can be measured more accurately than can the depression; also, if the knife-edges sink into the beam when the load is increased this does not affect the readings as when the depression is measured directly with a microscope. There is also less chance of damaging the beam by exceeding the elastic limit for its material, since smaller loads can be used.

Cornu's method of determining the elastic constants for glass.—In 1869 CORNU† published an account of a method of determining optically the elastic constants of glass by observations on the deformation of the surface of a rectangular slab of glass subjected to a uniform bending moment. He produced the curvature of the surface by supporting the plate on two knife-edges lying in a horizontal plane with their edges normal to the axis of the beam, and suspending weights from the ends. A plane cover-glass rested on the beam and interference fringes localized in the film of air between the cover plate and the beam were obtained when the system was suitably illuminated with sodium light. The interference pattern consisted of two conjugate systems of hyperbolae as shown diagrammatically in Fig. 8·20(a). By measuring the fringes along and across the beam, the longitudinal and transverse curvatures were obtained, so that Poisson's ratio could be calculated at once. The value of Young's modulus for glass was derived from

† Cornu, *Compt. Rend.*, **69**, 333, 1869.

the longitudinal curvature in the usual way. It is known, however, that some of the results he obtained were faulty as the knife-edges were too close together in comparison with the width of the beam.

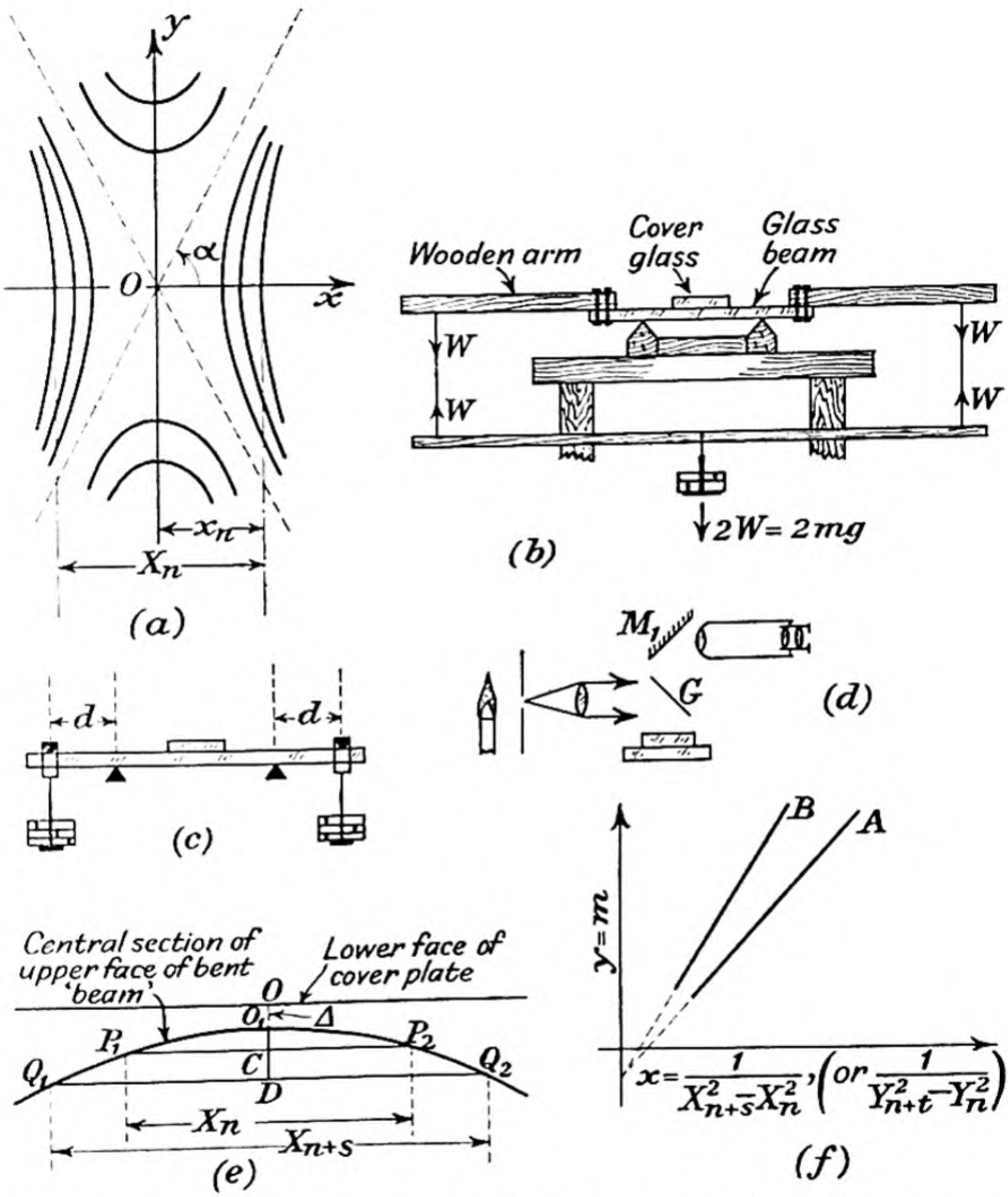


FIG. 8·20.—Cornu's interference method for finding E and σ for glass.

When this happens the distribution of the thrust on the knife-edges varies as the beam is deformed.

JESSOP† using Cornu's method, carried out in 1921 a careful series of measurements on the elastic constants of glass but took care to avoid the disturbing effects to which reference has just been made. The glass beam, Fig. 8·20(b), was supported on two knife-edges and

† Jessop, *Phil. Mag.*, **42**, 551, 1921.

the load applied by means of a mass suspended, in the manner indicated, from two wooden arms clamped to the ends of the beam. In this way a large moment could be applied without introducing excessive thrusts on the knife-edges, thus reducing the action of disturbing end-effects. To eliminate effects arising from the want of perfect flatness in the surface of the beam and that of the cover-glass the beam was turned over and the couple applied as before. But in this instance it was necessary to place the cover-glass below the beam and bring the surfaces together by means of three levelling screws supporting the cover-glass. By this means the mechanical difficulties of actually reversing the couple to obtain the same effect were avoided. The reason why this mode of procedure eliminates the effects of irregularities in the surface of the beam and cover-glass is that if the bending moment is reversed in direction while its magnitude is kept constant, the theoretical curvatures of the surface of the beam will be the same in magnitude but opposite in sign, while the surface irregularities will increase a particular curvature in one instance and decrease it when the couple is reversed. The mean value of the curvature will therefore be independent of these irregularities.

When this experiment is carried out in the laboratory as a practical exercise, the apparatus may take the more simple form shown in Fig. 8·20(c). If d is the distance between the point of support of the load of mass m and the knife-edge nearer to it, the bending moment at all transverse sections of the central span will be mgd, where g is the intensity of gravity. To obtain the fringes, light from a sodium flame ($\lambda = 5{\cdot}90 \times 10^{-5}$ cm.) is reflected on to the cover-glass and beam by means of a piece of glass G making an angle of about 45° with the horizontal. The fringes, formed between the upper surface of the glass beam and the cover-glass, may be viewed with a travelling microscope situated above the system. It is more convenient, however, to measure the fringes with a microscope whose axis is horizontal, using a plane mirror M_1, suitably arranged at about 45° to the horizontal, as shown in Fig. 8·20(d). With the horizontal traverse on the instrument the diameters of fringes in a direction parallel to the longitudinal axis of the beam may be measured, while with the vertical traverse the diameters of fringes in a direction at right-angles to the above, and passing through the centre of the fringe system, may be found. Measurements in these directions will be denoted by A and by B respectively. As the fringes will not be in focus even with a low power microscope, it is necessary to convert the latter into a telescope suitable for observing objects near to it. To do this, the objective is replaced by an achromatic converging lens of about 20 cm. focal length; a brass extension piece may be required as well.

The diameters of the 'horizontal' and the 'vertical' fringes are then measured for four different values of the load m, say from 0·5 to 2 kgm.

When the bending moment is M, i.e. mgd, the longitudinal radius of curvature, R_1, is given by

$$\frac{EI}{R_1} = M = mgd,$$

where E is Young's modulus for the material of the beam, and I is the 'moment of inertia' of a cross-section of the plate about an axis through its centroid and perpendicular to the plane of bending. It is given by $I = \frac{1}{12}ab^3$, where a is the breadth and b the thickness of the beam.

On account of the fact that the glass beam has an anticlastic curvature, and the cover plate is plane, the plate cannot be in contact with the beam at the centre O_1 of the fringe system, where the tangent plane is parallel to the plate. But the form and position of the fringe system is independent of the position of its centre. To prove this let us suppose that the diameters of the fringes have been measured along the directions we have called A and B. If P_1 and P_2, Fig. 8·20(e), represent the positions of the nth pair of fringes in the A direction, counted from the centre, the radius of curvature R_1 of the neutral surface is given by

$$2R_1 . O_1C = \left(\frac{X_n}{2}\right)^2,$$

where $X_n = P_1P_2$. [O_1C is small compared with $2R_1$ and so is neglected in the term $2R_1 - O_1C$.] If Q_1 and Q_2 are the $(n + s)$th pair of fringes,

$$2R_1 . O_1D = \left(\frac{X_{n+s}}{2}\right)^2.$$

$$\therefore\ 2R_1(O_1D - O_1C) = \tfrac{1}{4}(X_{n+s}^2 - X_n{}^2).$$

But $(O_1D - O_1C) = s\left(\dfrac{\lambda}{2}\right)$, where λ is the wavelength of the light used.

$$\therefore\ R_1 = \frac{1}{4\lambda s}(X_{n+s}^2 - X_n{}^2).$$

Similarly, if Y_{n+t} and Y_n refer to fringes in the direction across the beam, and R_2 is the radius of anticlastic curvature,

$$R_2 = \frac{1}{4\lambda t}(Y_{n+t}^2 - Y_n{}^2).$$

Suppose that the beam has an initial radius of curvature R_0 due to its own weight. Then [cf. p. 346],

$$EI\left[\frac{1}{R} - \frac{1}{R_0}\right] = mgd,$$

in general. This may be written

$$\frac{EI}{R_1} = mgd + \beta,$$

where β is a constant. Using the value for R_1 already obtained,

$$\frac{4EI\lambda s}{X^2_{n+s} - X_n{}^2} = mgd + \beta.$$

This equation suggests that if m is varied and corresponding values of $(X^2_{n+s} - X_n{}^2)$ are obtained in each instance, then calling $m = y$, and $\dfrac{1}{X^2_{n+s} - X_n{}^2} = x$, we have

$$\frac{4EI\lambda s}{gd}x = y + \frac{\beta}{gd}.$$

This equation represents the straight line A, Fig. 8·19(f), whose slope is $\dfrac{4EI\lambda s}{gd}$, so that E may be calculated when the slope has been determined.

In the same way

$$\frac{1}{4\lambda t}(Y^2_{n+t} - Y_n{}^2) = R_2 = \frac{1}{\sigma}.R_1 = \frac{1}{\sigma}.\frac{EI}{mgd + \beta}$$

where σ is Poisson's ratio for glass.

$$\therefore\ \frac{4\lambda t}{\sigma}\left[\frac{EI}{Y^2_{n+t} - Y_n{}^2}\right] = mgd + \beta.$$

Hence, if we plot $m = y$ and $\dfrac{1}{Y^2_{n+t} - Y_n{}^2} = x$, we should obtain a straight line, B, Fig. 8·19(f), whose slope is $\dfrac{1}{\sigma}$ times that of A.

$$\therefore\ \sigma = \frac{\text{slope of A}}{\text{slope of B}}.$$

Note on the shape of Cornu's fringes.—In the experimental determination of the elastic constants for glass by Cornu's method it is not necessary to consider the shapes of the interference fringes; it is nevertheless a matter of some interest to discuss these shapes.

Let ABCD, Fig. 8·21(*a*), be the lower surface of a flat glass plate. Let the horizontal tangent plane to the top surface of the bar, whose elastic constants are being determined, touch that surface at O_1, this tangent plane being parallel to ABCD. O is the point in ABCD such that OO_1 is perpendicular to ABCD. Let $OO_1 = \Delta$.

Take rectangular axes in ABCD, Ox parallel to the plane of the applied couples and Oy perpendicular to it. Let the radius of curvature in the plane of the couples be R_1 and in the plane normal to

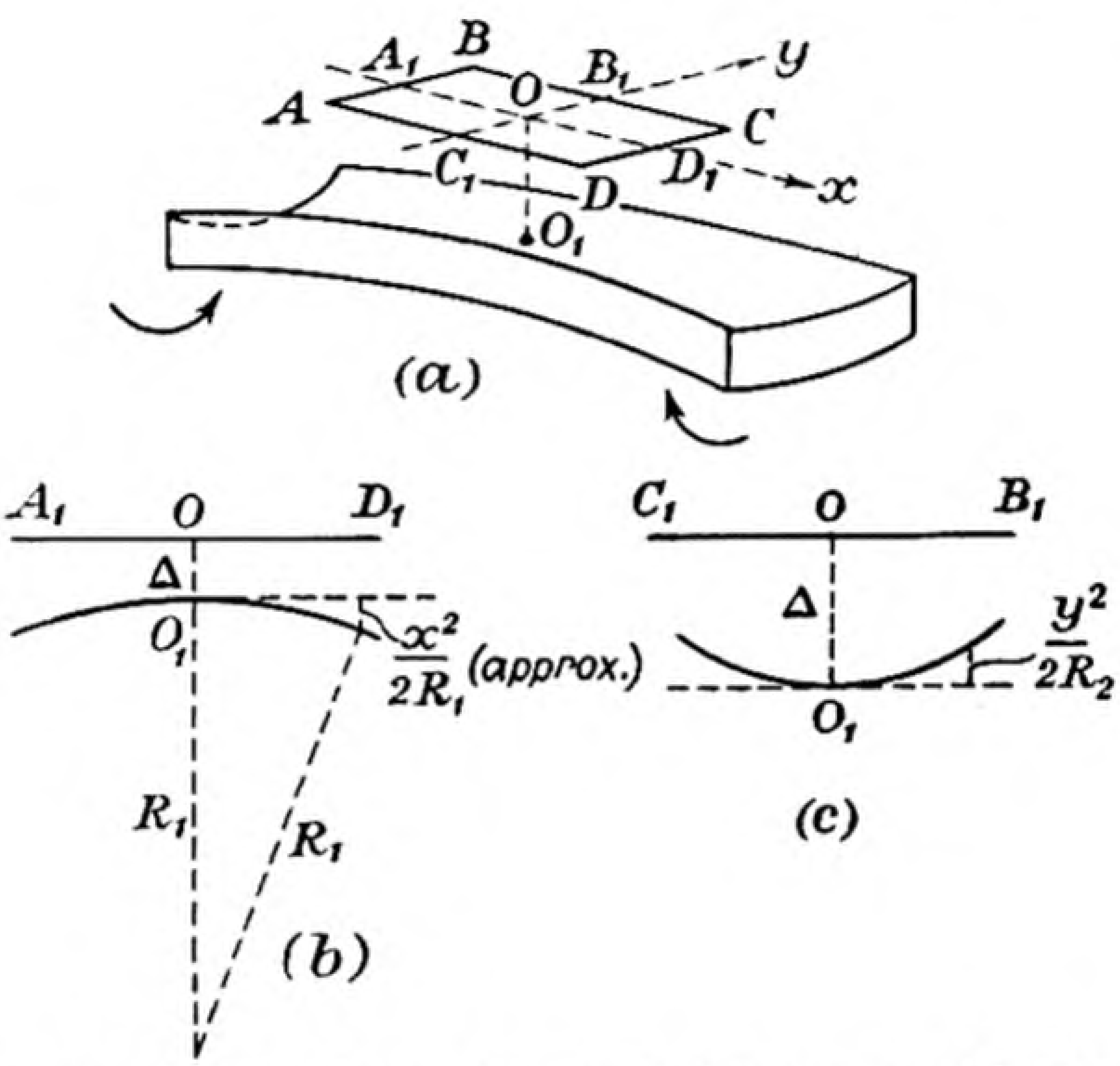


FIG. 8·21.—The shape of Cornu's interference fringes.

this R_2. Then the length of the straight line parallel to OO_1 and drawn from any point (x, y) in ABCD to the top of the bent beam as shown in Fig. 8·21(*b*) and (*c*), may be taken as $\Delta + \dfrac{x^2}{2R_1} - \dfrac{y^2}{2R_2}$. This distance is constant for any given fringe. Hence

$$\frac{x^2}{2R_1} - \frac{y^2}{2R_2} = \kappa \text{ (say)},$$

or

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

where $a^2 = 2R_1\kappa$ and $b^2 = 2R_2\kappa$, i.e. the fringes are hyperbolae.

Let α be the angle between the asymptotes and the axis Ox. Then Poisson's ratio, σ, is given by

$$\sigma = \frac{R_1}{R_2} = \frac{a^2}{b^2} = \cot^2 \alpha.$$

Again, the thickness of the air film at the point $(x_n, 0)$ is $\Delta + \dfrac{x_n^{\,2}}{2R_1}$ and at $(x_{n+m}, 0)$ it is $\Delta + \dfrac{x^2_{n+m}}{2R_1}$.

The path difference is twice this in each instance so that the change in path difference from the positions defined by x_n and x_{n+m} is

$$2\left[\frac{x^2_{n+m} - x_n^{\,2}}{2R_1}\right] = m\lambda,$$

where λ is the wavelength of the light used. If diameters are measured, as in practice,

$$X_n = 2x_n.$$

so that
$$R_1 = \frac{X^2_{n+m} - X_n^{\,2}}{4m\lambda}.$$

Similarly
$$R_2 = \frac{Y^2_{n+m} - Y_n^{\,2}}{4m\lambda}.$$

$$\therefore\ \sigma = \frac{R_1}{R_2} = \frac{X^2_{n+m} - X_n^{\,2}}{Y^2_{n+m} - Y_n^{\,2}}$$

Instead of a glass plate, a converging lens of long focal length may rest on the beam. Let r be the radius of curvature of the lens face which is nearest to the upper surface of the bent beam. Let O, Fig. 8·22(a), the lowest point in this surface of the lens, be the

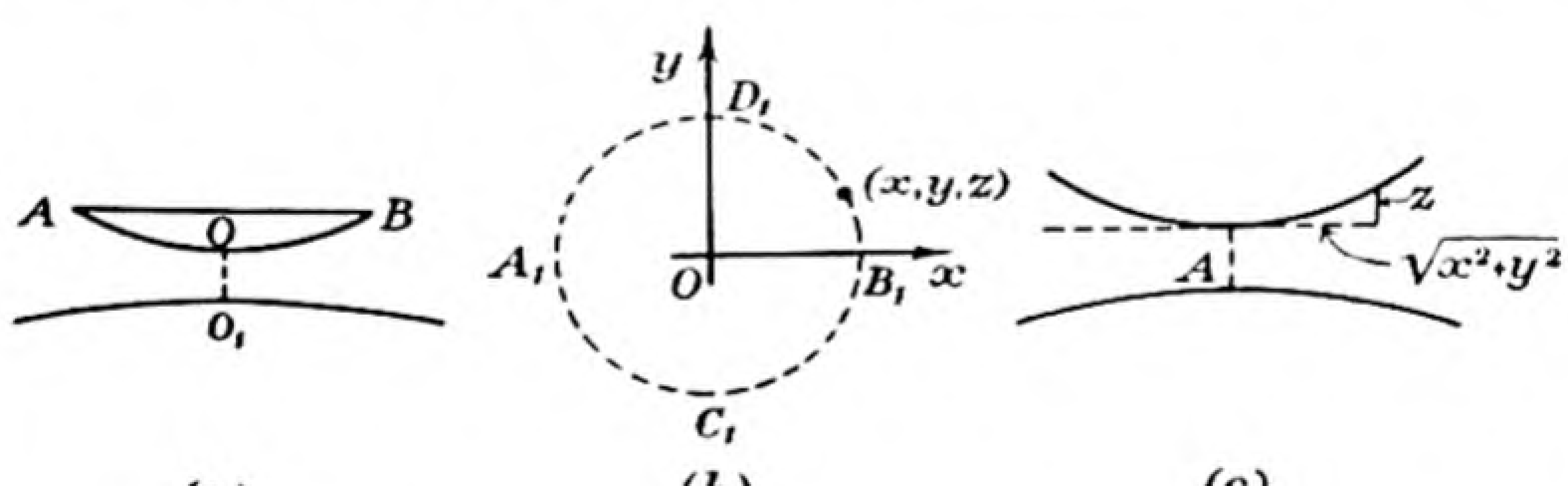


Fig. 8·22.—The shape of the fringes when a spherical surface rests on a bent beam.

origin of a system of rectangular coordinates; the plane x, y is horizontal through O and Oz is vertical. Then at any point (x, y, z) on the surface of the lens, cf. Fig. 8·22(b) and (c),

$$(x^2 + y^2) = 2rz,$$

since r is not small. Hence

$$z = \frac{x^2 + y^2}{2r}.$$

Hence, as in the preceding instance, for the fringe on which (x, y, z) lies,

$$\Delta + \frac{x^2}{2R_1} - \frac{y^2}{2R_2} + \frac{x^2 + y^2}{2r} = \kappa.$$

If $A = \kappa - \Delta$, this equation may be written

$$\therefore \frac{x^2}{A}\left[\frac{1}{2R_1} + \frac{1}{2r}\right] - \frac{y^2}{A}\left[\frac{1}{2R_2} - \frac{1}{2r}\right] = 1,$$

which is the equation to an ellipse or to an hyperbola according as $\frac{1}{2r} >$ or $< \frac{1}{2R_2}$. Now R_2 is usually large compared with r, i.e. the fringes are usually ellipses.

An optical method for determining the angular deflexion at the free end of a cantilever.—It has already been stated, cf. p. 362, that if a uniform bar of material is clamped at one end and in such a way that it may only be deflected in a horizontal plane, then the effect of its own weight on any bending to which it may be subjected is negligible. Consequently, its bending moment at a section at a distance x from the clamped end is given by

$$EIy'' = M = W(l - x),$$

where the symbols have their usual meanings. This gives at once

$$EIy' = -\tfrac{1}{2}W(l - x)^2 + \tfrac{1}{2}Wl^2,$$

since the angular deflexion is zero at $x = 0$. The angular deflexion, α, at the free end of the lever is given by

$$EI\alpha = EI[y']_{x=l} = \tfrac{1}{2}Wl^2.$$

To measure α the apparatus shown diagrammatically in Fig. 8·23(a) was designed by the author. The lever OA is a thin strip of glass clamped in a horizontal position (or it may hang vertically downwards) with its faces in vertical planes. The free end of the lever is silvered and immediately in front there is fixed a converging lens L of two metres focal length; the whole is supported on a suitable base board B. A ***horizontal*** force F is applied normally to the lever at a point A, i.e. at a distance l from the clamped end; then the silvered part may be used as a mirror to measure α. To apply F a light aluminium tube R, Fig. 8·23(b), is allowed to rest with its axis horizontal on a plane inclined at 45° to the horizontal and a light cord, attached to the lever at A, passes over the roller; the portion of the cord between the lever and the roller must be horizontal. At its other end this cord carries a known load W.

Since the tube is free to roll on the surface of the inclined plane, the tension in the string is very nearly equal to W, cf. I.P. p. 769. In any case we may write $F = W - f$, where f is a small correction term which includes the weight of any paper container holding W. Then

$$EI\alpha = \tfrac{1}{2}(W - f)l^2,$$

so that if a series of corresponding values of W and α is obtained a graphical method may be used to find a mean value for EI; hence E may be calculated.

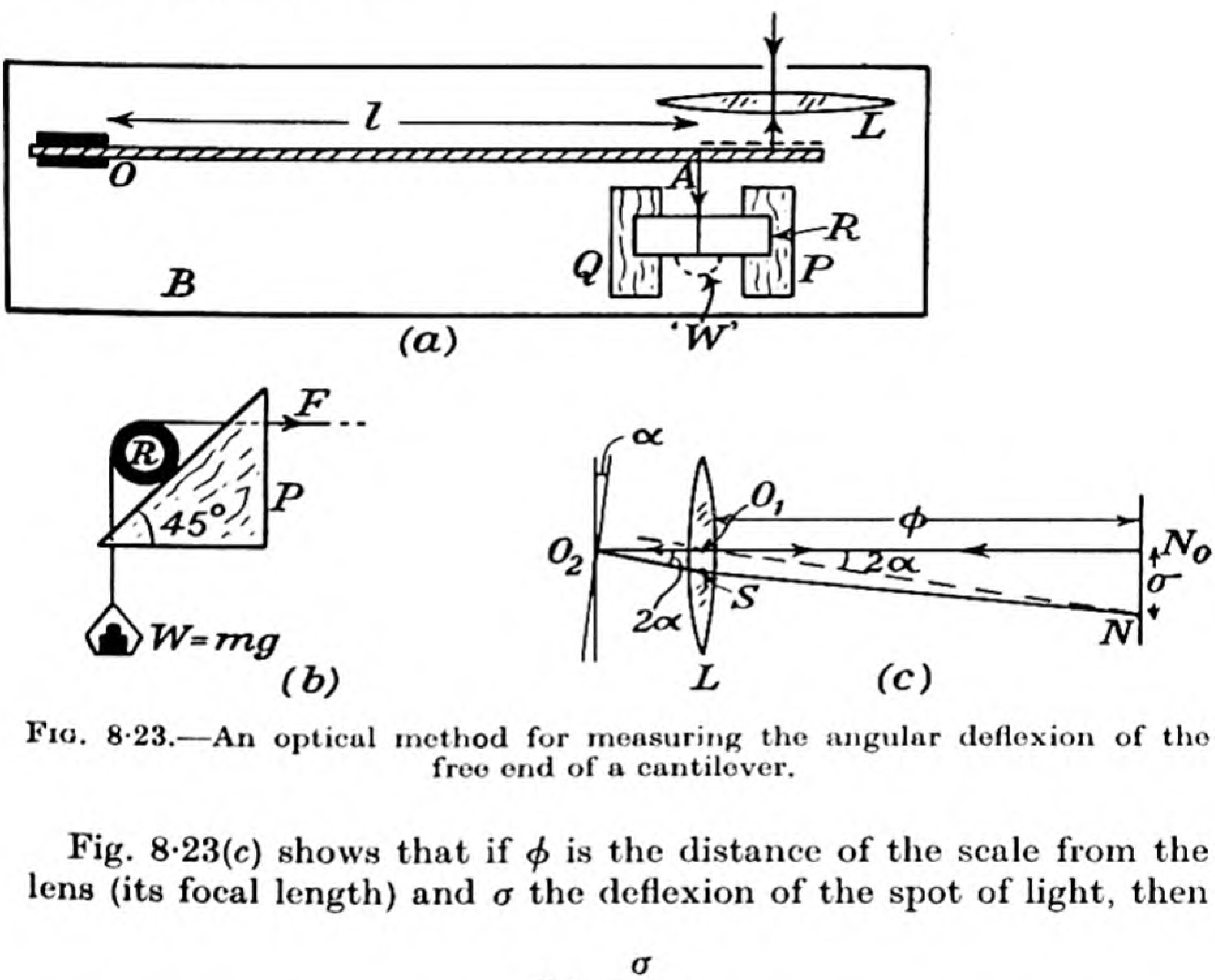


FIG. 8·23.—An optical method for measuring the angular deflexion of the free end of a cantilever.

Fig. 8·23(*c*) shows that if ϕ is the distance of the scale from the lens (its focal length) and σ the deflexion of the spot of light, then

$$\alpha = \frac{\sigma}{2\phi}.$$

[N.B. O_1N is that subsidiary axis parallel to the ray O_2S.]

Determination of the elastic constants of the material of a wire; Searle's apparatus.—The method to be described is due to SEARLE† and in his original paper it is pointed out that like most other methods of determining the elastic constants of a solid, it is limited to materials which are isotropic. AB and CD, Fig. 8·24(*a*), are two equal brass bars of square or circular cross-section. The wire under test, of which only a few centimetres are required, is

† Searle, *Phil. Mag.*, 99, 193, 1900.

securely clamped at its ends to the mid-points H and K of the bars—cf. Fig. 8·24(*b*) for details. Two small hooks are screwed into the bars at points directly above H and K—i.e. into those faces of the bars which are perpendicular to those pierced by the holes for the clamping screws. These hooks permit the system to be suspended by two parallel pieces of cotton, each about a metre long so

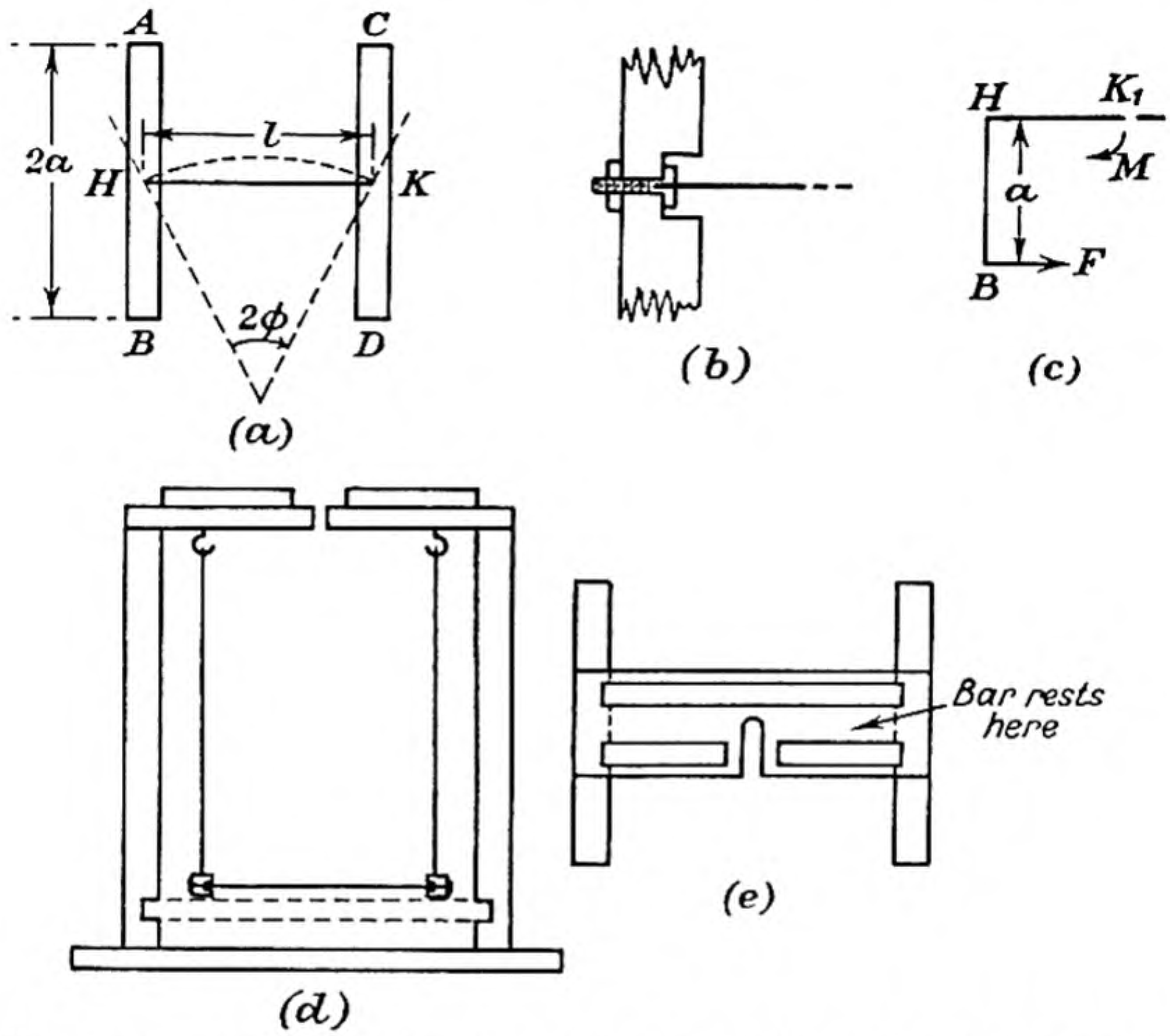


Fig. 8·24.—Searle's method for determining E, n, β and σ for the material of a wire.

that the couples due to torsion in these pieces of cotton are small compared with other couples applied to the system. Each bar is then able to oscillate in the same horizontal plane, and if the ends of the hooks where they engage the strings are about four centimetres above the faces of the bars, the horizontal position of the system is quite stable.

If the ends B and D of the bars are made to approach each other, the displacements of each end being identical in magnitude, and then released, each bar will vibrate in a horizontal plane. To the first order of small quantities the points H and K remain fixed, and the bending moment at all transverse sections of the wire is constant. To prove this, let F be the force applied to each bar. Consider the equilibrium of the portion BHK_1, of the system. If the displacement of B is so small that F is normal to the axis of AB and the

displacement of K_1 is small compared with the half-length, a, of AB, we have, in the usual way,

$$S = 0,$$

and
$$M - Fa = 0,$$

so that the bending moment is constant; the trace of the neutral surface for the wire in the plane of bending will therefore be the arc of a circle of radius R, for

$$M = \frac{EI}{R},$$

where E is Young's modulus for the material of the wire, I is the 'moment of inertia' of the area of a cross-section of the wire about an axis through its 'centre of gravity' and perpendicular to the plane of bending.

If ϕ is the angle through which each bar has been displaced

$$R = \frac{l}{2\phi},$$

where l is the length of the wire. When the forces applied to the bars are removed, the equation of motion for either bar is

$$J\ddot{\phi} + M = 0,$$

or
$$J\ddot{\phi} + \frac{2EI}{l}.\phi = 0,$$

where J is the moment of inertia of a bar about its axis of rotation. Hence T_1, the period, is given by

$$T_1 = 2\pi\sqrt{\frac{Jl}{2EI}}.$$

For a circular wire $I = \frac{1}{4}\pi r^4$, where r is its radius, so that

$$E = \frac{8\pi Jl}{T_1^{\,2} r^4}.$$

Suppose now that the bars are unhooked and one is held in a clamp so that the wire is vertical and the other bar is free to vibrate in a horizontal plane. Its period T_2 is expressed by

$$T_2 = 2\pi\sqrt{\frac{2Jl}{\pi n r^4}},$$

where n is the rigidity of the material of the wire [cf. p. 292].

$$\therefore\ n = \frac{8\pi Jl}{T_2^{\,2} r^4}.$$

Hence
$$\frac{E}{n} = \frac{T_2^2}{T_1^2},$$

and
$$\sigma = \frac{E}{2n} - 1, \qquad \text{[cf. p. 306]}$$

where σ is Poisson's ratio. Hence E, n, and σ are readily determined by this method.

[N.B. Since both $\frac{E}{n}$ and σ depend only on the ratio of the squares of the periods, the dimensions of the wire, etc., are not required. Also, it may be noted, that since $0 < \sigma < 0{\cdot}5$, we have when $\sigma = 0$, $2n = E$, and when $\sigma = 0{\cdot}5$, $3n = E$.

To obtain good results with this apparatus the amplitude of vibration of the bars must be small in order that the bending moment may be constant; the motion is then simple harmonic. In some of his experiments Searle used an amplitude not exceeding 1° and yet was able to time 100 vibrations.

To avoid giving the bars any translatory motion their ends are brought slightly together by means of a loop of cotton, and when the whole system is stationary, the cotton is burnt. The period is then found; do not forget the fiducial mark!

To prevent straining the wire while fitting up the apparatus both hands should be used in carrying the bars so that the wire is never subjected to a large bending moment, and when attaching the strings to the hooks, the bars should rest on a board which is afterwards removed. A suitable wooden frame for supporting the apparatus in the two parts of the experiment is shown in elevation and in plan in Fig. 8·24(d) and (e).

The strain-energy in a bent beam.—Let us now consider the strain-energy arising from the bending of a uniform beam of isotropic and homogeneous material and originally straight. For our present purpose the shape of the cross-section of the beam is immaterial. A short length of the beam in its unstrained condition is shown in Fig. 8·25(a). When the bending is produced by a couple, i.e. it is pure, let the portion of the beam under consideration assume the shape shown in Fig. 8·25(b). The length δx of the neutral axis N_1N_2 remains unaltered. Consider, however, an element at height z above the neutral axis. Let R be the radius of curvature of the neutral axis, δA the cross-section of the element in a plane normal to that of the diagram and δz its thickness in the plane of the diagram. If $\delta\theta$ is the angle indicated, the strain in the particular element considered is given by

$$\epsilon = \frac{(R + z)\,\delta\theta - R\,\delta\theta}{R\,\delta\theta} = \frac{z}{R}.$$

Hence the strain-energy, cf. p. 317, associated with the element considered is given by

$$\delta W = \tfrac{1}{2}p\epsilon(\delta A . \delta x) = \tfrac{1}{2}E . \frac{z^2}{R^2} . \delta A . \delta x.$$

since $p = \epsilon E$.

Integrating over the whole cross-section, we find that the strain-energy in the portion of the rod whose neutral axis is N_1N_2 is

$$\tfrac{1}{2}\delta x . \frac{E}{R^2}\int z^2\, dA = \tfrac{1}{2}\delta x . \frac{EI}{R^2},$$

where I is the 'moment of inertia' for the cross-section considered.

Since $M = \dfrac{EI}{R}$, the above expression may be written $\dfrac{M^2}{2EI} . \delta x$.

$$\therefore\ W = \frac{1}{2E}\int_0^l \frac{M^2}{I}\, dx,$$

which, for a uniform beam, becomes

$$W = \frac{1}{2EI}\int_0^l M^2\, dx.$$

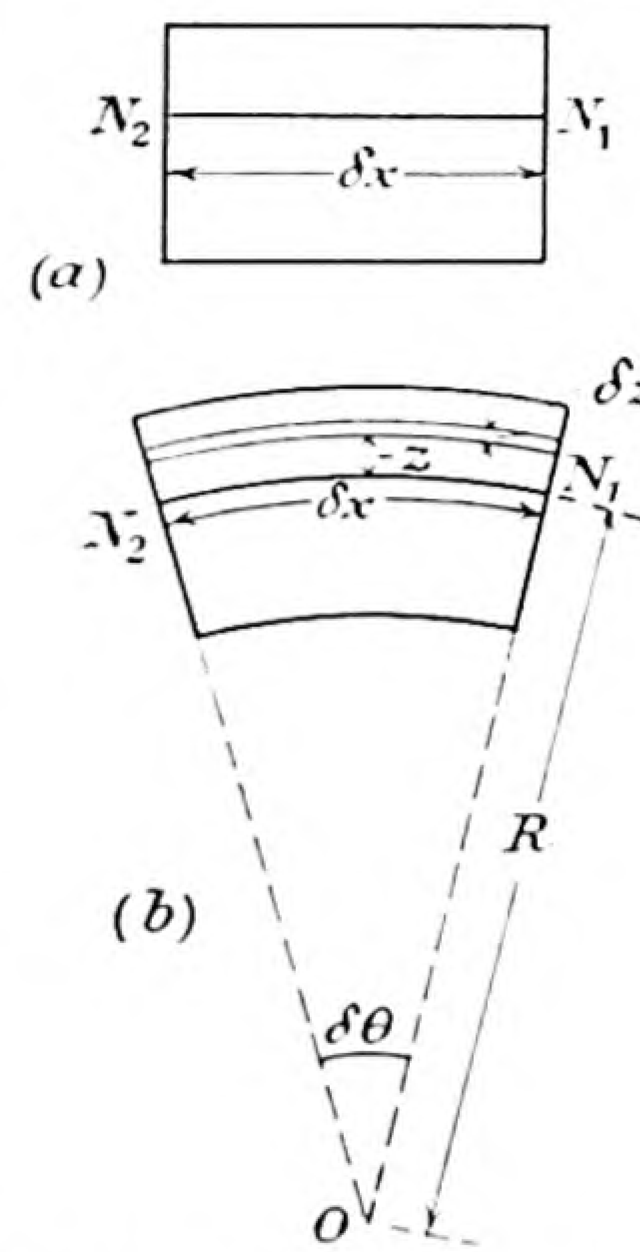


FIG. 8·25.—Strain-energy in a bent beam.

Example.—(*a*). For a uniform beam of length l clamped at one end, free at the other, and having a weight w per unit length, the bending moment at a distance x from the clamped end is given by

$$M = \tfrac{1}{2}w(l - x)^2, \qquad \text{[cf. p. 339]}$$

$\therefore$ Energy stored in the beam is given by

$$W = \frac{1}{2EI}\int_0^l M^2\, dx = \frac{1}{2EI}\int \tfrac{1}{4}w^2(l - x)^4\, dx$$

$$= \frac{w^2}{40EI}\Big[-(l - x)^5\Big]_0^l = \frac{w^2l^5}{40EI}.$$

(*b*). For an encastré beam, cf. p. 354, if $2a$ is the length of the beam,

$$M = \tfrac{1}{2}wx^2 - wax + \tfrac{1}{3}wa^2.$$

$$\therefore\ W = \frac{1}{2EI}\int_0^{2a}(\tfrac{1}{2}wx^2 - wax + \tfrac{1}{3}wa^2)^2\, dx$$

$$= \frac{1}{45} . \frac{w^2a^5}{EI}.$$

(*c*) For a light cantilever of length l and carrying a load W_0 at its free end, cf. p. 338,

$$M = W_0(l - x).$$

$$\therefore\ W = \frac{1}{2EI}\int_0^l W_0^{\,2}(l - x)^2\, dx = \frac{1}{6EI} . W_0^{\,2}l^3.$$

(*d*) Let us now return to (*a*) and calculate the strain-energy from first principles. Since, in this instance, cf., p. 362,

$$EIy = \tfrac{1}{24}w(l - x)^4 + \tfrac{1}{6}wl^3x - \tfrac{1}{24}wl^4,$$

the work done in deforming the bar is

$$\tfrac{1}{2}\int_0^l wy\,dx = \frac{1}{2}\cdot\frac{w^2}{EI}\left[-\tfrac{1}{5}\cdot\tfrac{1}{24}(l-x)^5 + \tfrac{1}{12}l^3x^2 - \tfrac{1}{24}l^4x\right]_0^l = \frac{w^2l^5}{40EI}$$

(*e*) Finally, let us consider the strain-energy in a uniformly bent beam If l is the length of that portion of the beam which is uniformly bent, cf. Fig. 8·26(*a*), the strain-energy in this portion of the beam is given by

$$W = \frac{1}{2EI}\cdot\int_0^l M^2\,dx = \frac{M^2}{2EI}\cdot l.$$

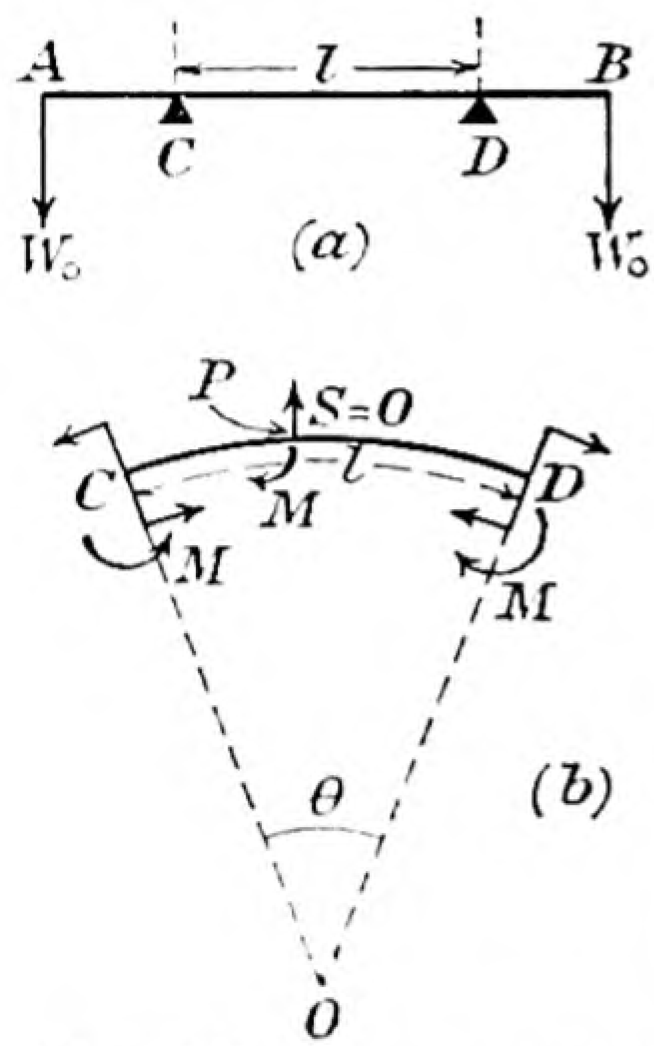


FIG. 8·26.—Strain-energy in a uniformly bent beam.

To derive this expression from first principles we may proceed as follows. At any section P, Fig. 8·26(*b*), in the span CD the bending moment is M and is constant; it is clockwise. The external couples at the ends of the span each have magnitude M but that at C is anticlockwise while that at D is clockwise. Hence the work done in deforming this portion of the beam provided, as usual, that the elastic limit is not exceeded, is

$$2[\tfrac{1}{2}M(\tfrac{1}{2}\theta)] = \tfrac{1}{2}M\theta = \tfrac{1}{2}M\cdot\frac{l}{R} = \frac{1}{2}\frac{M^2}{EI}\cdot l,$$

as before.

The resilience of bent beams.—Since the work done in deflecting a horizontal beam of any kind supporting a concentrated load W is

$$\tfrac{1}{2}W \times \text{(deflexion at the load)},$$

this expression also measures the resilience, cf. p. 322, of the beam.

Now if a light beam of length $2a$, breadth b and depth d, and supported at both ends carries a concentrated load W, the deflexion at the centre is, cf. p. 349, $\frac{1}{6}\cdot\frac{W}{EI}\cdot a^3$.

$$\therefore\ \text{Resilience} = \tfrac{1}{2}W\cdot\frac{1}{6}\cdot\frac{W}{EI}\cdot a^3$$

$$= \frac{W^2a^3}{Ebd^3}. \qquad [\because\ I = \tfrac{1}{12}bd^3].$$

To obtain an expression for the resilience of any horizontal beam, let δx be a short length of the beam, M the bending moment at any cross-section of this element and $\delta\beta$ the difference in the slopes at

its extremities. Then the strain energy for this portion of the beam is $\frac{1}{2}\mathrm{M}.\delta\beta$. Hence for a finite length of the beam the strain energy is

$$\tfrac{1}{2}\int \mathrm{M}.d\beta,$$

which, with the usual notation, may be written

$$\tfrac{1}{2}\int \mathrm{M}.\frac{d\beta}{dx}.dx = \tfrac{1}{2}\int \mathrm{M}\,\frac{d^2y}{dx^2}\,dx = \tfrac{1}{2}\int \frac{\mathrm{M}^2}{\mathrm{EI}}\,dx$$

$$= \frac{1}{2\mathrm{EI}}\int \mathrm{M}^2\,dx,$$

if the flexural rigidity, EI, is constant.

This equation enables us to calculate the resilience of any beam when its bending moment diagram is available.

Beam deflexion calculated from resilience.—Let a light horizontal beam of length $(a + b)$ carry a concentrated load W at $x = a$. Then since the reaction at $x = 0$ is $\mathrm{W}\left(\dfrac{b}{a + b}\right)$, the bending moment at any section where $0 < x < a$, is given by

$$\mathrm{W}\left(\frac{b}{a + b}\right)x + \mathrm{M} = 0,$$

or

$$\mathrm{M} = -\,\frac{b\mathrm{W}}{(a + b)}.x.$$

The strain energy of this portion of the beam is

$$\frac{1}{2\mathrm{EI}}\int_0^a \left(\frac{b\mathrm{W}}{a + b}.x\right)^2 dx = \frac{1}{6\mathrm{EI}}\left(\frac{b\mathrm{W}}{a + b}\right)^2.a^3.$$

Hence for the whole beam its strain energy, or resilience, is

$$\frac{1}{6\mathrm{EI}}\left(\frac{\mathrm{W}}{a + b}\right)^2.(b^2a^3 + a^2b^3) = \frac{1}{6\mathrm{EI}}.\mathrm{W}^2a^2b^2.$$

But this is also given by $\frac{1}{2}(y_{x=a}).\mathrm{W}$.

$$\therefore\ y_{x=a} = \frac{\mathrm{W}a^2b^2}{3\mathrm{EI}}.$$

[This should be verified by the usual method for calculating deflexions; the advantage of the resilience method will then be appreciated more fully.]

The proof resilience for the material of a bent beam.—The ***proof resilience***, cf. p. 322, for the material of a wire, when the

stress in the material is everywhere constant, has been shown to be $\frac{1}{2}\frac{p_0^2}{E}$. For beams the proof resilience is $\alpha . \frac{p_0^2}{E}$, where $0 < \alpha < 0{\cdot}5$. The coefficient α depends upon the manner in which the beam is loaded; the elastic limit is only reached at parts of the beam for the stress is zero along the neutral axis and increases towards its greatest value at the upper or lower surface of the beam.

Closely coiled helical or cylindrical springs.—Suppose that a uniform wire of isotropic material is bent so that its axis becomes

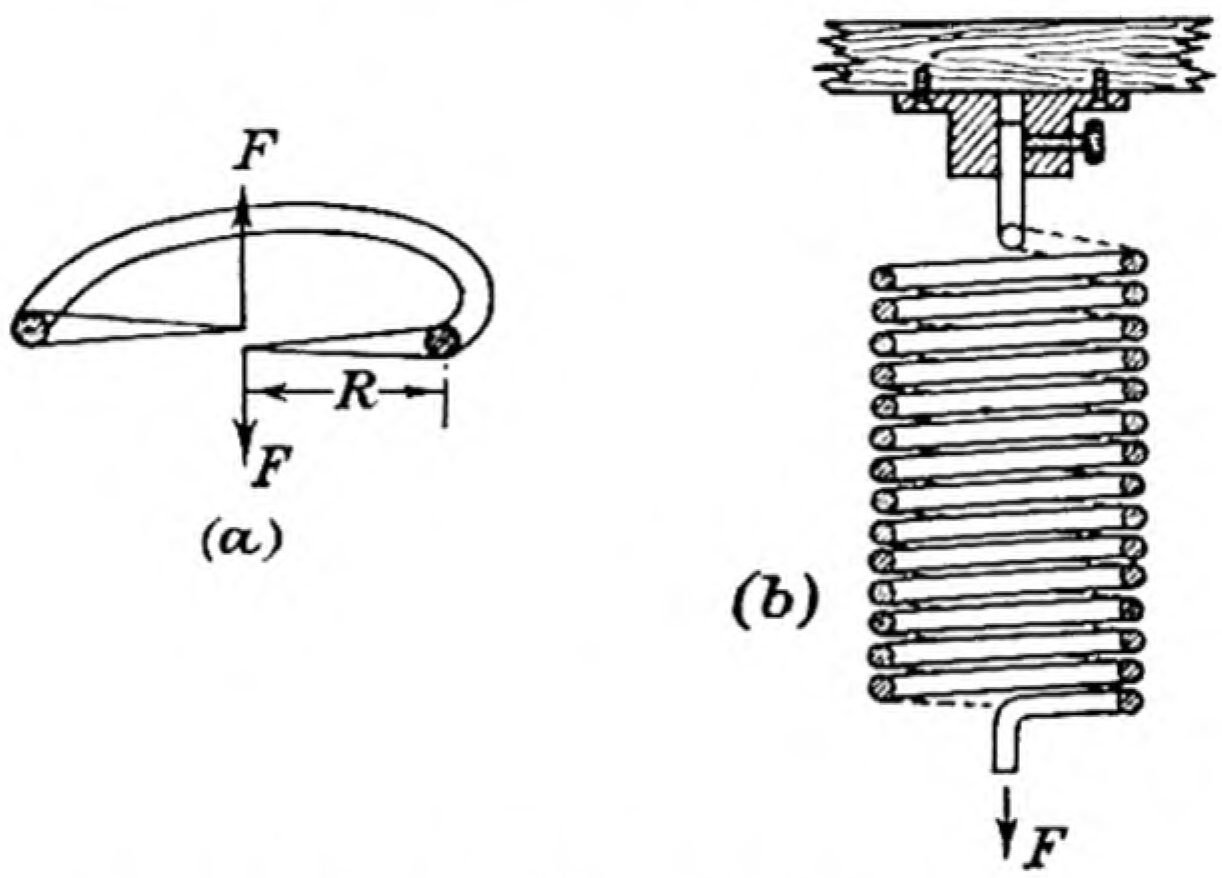


FIG. 8·27.—A closely coiled helical spring.

a semi-circle of radius R; let two arms of length R be attached to the end of the wire as shown in Fig. 8·27(a) and equal forces F applied at their extremities. These forces act through the centre of the circle of which the wire is part and are normal to the plane containing it; they act in opposition. The torque is then FR, and since the length of the wire is πR, the relative angular displacement between the arms, θ, is given by

$$\frac{\pi n a^4 \theta}{2l} = \text{FR}, \qquad \text{[cf. p. 286]}$$

or

$$\theta = \frac{2\text{FR}^2}{na^4},$$

where $2a$ is the diameter of the wire. Hence, the relative vertical displacement, Δz, between the ends of the arms is given by

$$\Delta z = \text{R}\theta = \frac{2\text{FR}^3}{na^4}.$$

Suppose now that there are N complete turns, all neighbouring turns being very close together, cf. Fig. 8·27(b), so that the whole

forms a so-called closely coiled helical spring. Then z, the extension for the whole spring, is such that

$$z = 2\mathrm{N}\,\Delta z = \frac{4\mathrm{NFR}^3}{na^4}.$$

If the spring, in a vertical position, is stretched by a load of mass M attached to one end, the other end being rigidly fixed, then

$$\mathrm{F} = \mathrm{M}g,$$

so that

$$\frac{\mathrm{M}g}{z} = \frac{na^4}{4\mathrm{NR}^3}.$$

From this equation it follows that if we determine $\frac{\mathrm{M}}{z}$, graphically for preference, a value for the rigidity of the material of the wire may be determined.

The expression for $\frac{\mathrm{M}g}{z}$ gives the force per unit depression of the load carried by the spring; it will be denoted by f.

The rigidity may also be obtained from observations on the period of the up-and-down oscillations of a light helical spring carrying a load at its lower end. When the loaded spring is at rest its lower end will be at some definite position. In order to produce an additional extension of the spring, a vertical force equal to f times the displacement must be applied. When this force is removed the load will tend to return to its position of static equilibrium If, at any instant, the displacement from this position is ξ, the upward force acting on the load due to the fact that the spring is extended, will be $(\mathrm{M}g + f\xi)$, while the downward pull will be $\mathrm{M}g$. The resultant of these forces will be $f\xi$ acting upwards and tending to restore the load to its equilibrium position. Thus the force acting on M will be directly proportional to the displacement of M from its rest position, and always directed towards it. The oscillations will therefore be simple harmonic and, if the mass of the spring itself is neglected, are expressed by $\mathrm{M}\ddot{\xi} + f\xi = 0$. The period, T_1, for these vertical displacements is therefore given by

$$\mathrm{T}_1 = 2\pi\sqrt{\frac{\mathrm{M}}{f}} = 2\pi\sqrt{\frac{\mathrm{M}.4\mathrm{NR}^3}{na^4}}.$$

Helical springs; correction for the mass of the spring.—In the investigation, hitherto, of the motion of body attached to the free end of a helical spring the mass of the spring has been considered negligible. When this is no longer possible an approximate correction may be obtained as follows.

First consider the kinetic energy of the system. Let M be the mass of the load, and suppose that at a given instant it is at a distance ξ from its position of static equilibrium. Let l be the total length of wire in the coils of the spring. Then the displacement of a point at distance x from the fixed end of the coil is $\left(\frac{x}{l}\right)\xi$, its velocity therefore being $\left(\frac{x}{l}\right)\dot{\xi}$. If μ is the mass per unit length of the spring, the kinetic energy of a small element δx of the spring, when the displacement at the lower end is ξ, is $\frac{1}{2}\mu\ \delta x\left[\left(\frac{x}{l}\right)\dot{\xi}\right]^2$.

If the spring is uniform in diameter and material, W, the total kinetic energy of the load and spring, is given by

$$W = \tfrac{1}{2}M\dot{\xi}^2 + \tfrac{1}{2}\frac{\mu}{l^2}\dot{\xi}^2\int_0^l x^2\,dx = \tfrac{1}{2}(M + \tfrac{1}{3}m)\dot{\xi}^2,$$

if we write $m = \mu l$, i.e. m is the mass of the spring.

Now consider the potential energy of the system; this is taken to be zero when the spring is loaded and in its position of static equilibrium. In its displaced position the potential energy of the system is

$$V = -Mg\xi - \tfrac{1}{2}mg\xi + \text{extra energy stored in the spring,}$$

the term in m appearing since the mass centre of the spring descends a distance $\frac{1}{2}\xi$.

When the load M is attached let z_0 be the displacement of the lower end of the spring. The energy stored in the spring is

$$\int_0^{z_0} fz\,dz = \tfrac{1}{2}fz_0^{\,2}.$$

When the extension is $z_0 + \xi$, the extra energy stored is

$$\begin{aligned}\tfrac{1}{2}f(z_0 + \xi)^2 - \tfrac{1}{2}fz_0^2 &= fz_0\xi + \tfrac{1}{2}f\xi^2\\ &= Mg\xi + \tfrac{1}{2}f\xi^2, \quad \because\ Mg/z_0 = f.\end{aligned}$$

In calculating this amount of energy the mass of the spring has been neglected. The spring is stretched by its own weight and the extension due to this may be taken as one-half that due to a mass m attached to the spring, i.e. the effective load is $(M + \frac{1}{2}m)$ in so far as the additional stored energy is concerned.

$\therefore$ Extra energy stored in the spring when the extension is ξ is

$$(M + \tfrac{1}{2}m)g\xi + \tfrac{1}{2}f\xi^2.$$

Since the total energy of the vibrating system is constant

$$W - Mg\xi - \tfrac{1}{2}mg\xi + \text{extra energy stored} = \text{const.},$$

i.e. $$\tfrac{1}{2}(M + \tfrac{1}{3}m)\dot{\xi}^2 - (M + \tfrac{1}{2}m)g\xi + (M + \tfrac{1}{2}m)g\xi + \tfrac{1}{2}f\xi^2 = \text{const.}$$

Differentiating with respect to time, we obtain,

$$(M + \tfrac{1}{3}m)\ddot{\xi} + f\xi = 0.$$

Thus the motion is simple harmonic, with a period given by

$$T_1 = 2\pi\sqrt{\frac{M + \frac{1}{3}m}{f}} = 2\pi\sqrt{\frac{(M + \frac{1}{3}m)4NR^3}{na^4}}.$$

Experimental study of a closely coiled helical spring supporting a bar of variable moment of inertia about the axis of the spring.—Fig. 8·28(a) shows a closely coiled helical spring

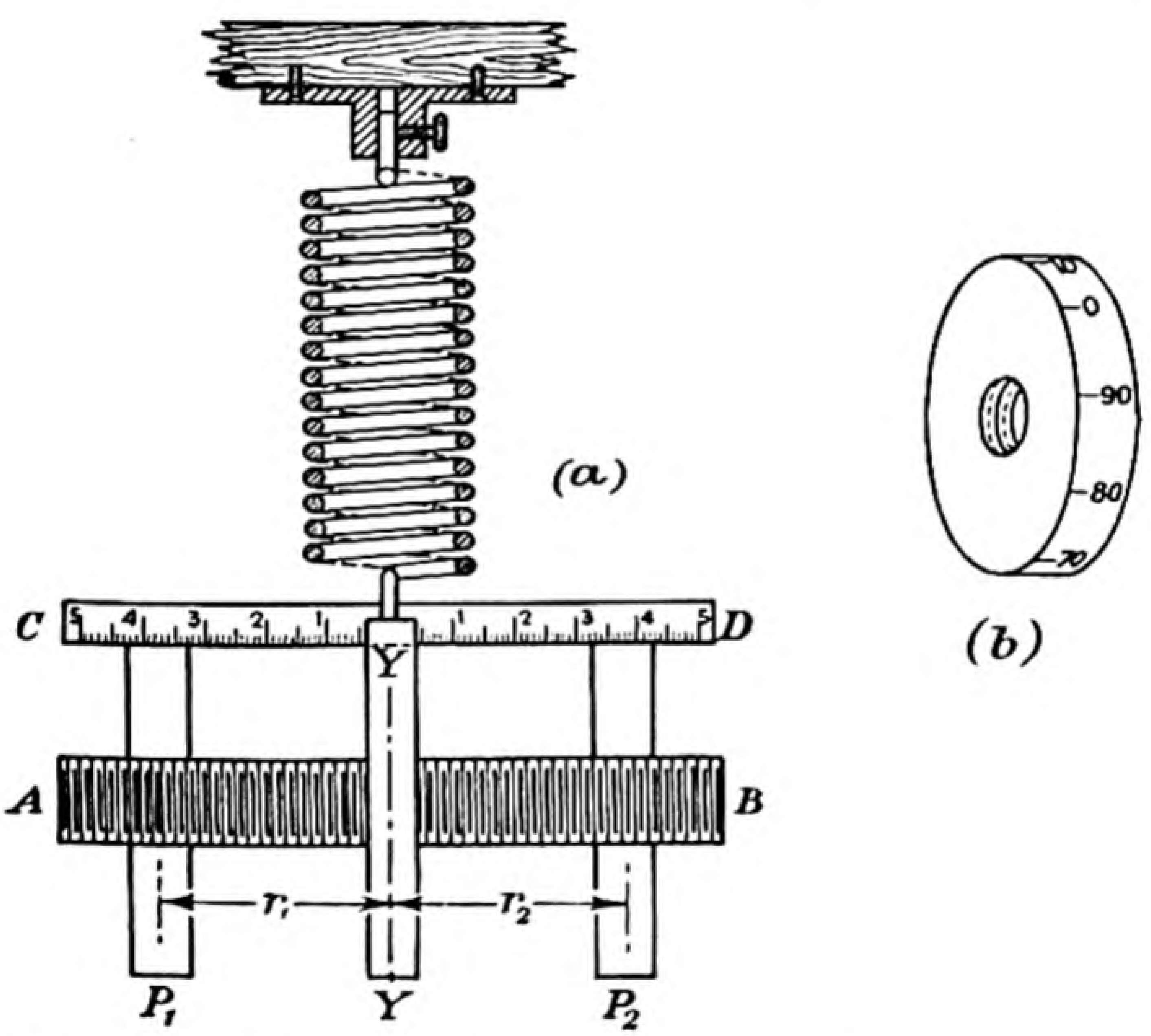


FIG. 8·28.—A helical spring with a variable inertia system.

rigidly fixed at its upper end and carrying at its lower end a horizontal screw AB along which two equal thick cylindrical brass discs may be moved. One of these discs is shown in Fig. 8·28(b). The pitch of the screw is 1 mm. and the periphery of each disc is divided into one hundred equal divisions to help in locating the position of the discs with respect to the vertical axis of the spring, these positions being stated with reference to CD, a horizontal scale in cm., etc.

(*a*) ***Vertical oscillations.*** In this instance the periodic time, T_1, for small vertical oscillations of the system is the same for all symmetrical positions of the brass discs; if a value for T_1 is determined experimentally, a value for n, the rigidity of the material of the wire, may be calculated.

(*b*) ***Angular oscillations.*** Let I_0 be the moment of inertia of AB, the scale CD and the metal disc YY to which AB and CD are soldered, about a vertical axis through the coil and centre of gravity of the suspended system. Let m_1 and m_2 be the masses of the discs P_1 and P_2, and I_1 and I_2 their moments of inertia about a vertical axis through their centres of gravity. Then J, the moment of inertia of the suspended body about the axis of rotation, is given by

$$J = I_0 + (I_1 + m_1 r_1{}^2) + (I_2 + m_2 r_2{}^2),$$

by the theorem of parallel axes [cf. p. 70], where r_1 and r_2 are the distances of the centres of gravity of the discs from the axis of rotation. If the discs have been made accurately, $m_1 = m_2 = m$ (say), and $r_1 = r_2 = r$; therefore $I_1 = I_2 = I$ (say). Then $J = I_0 + 2(I + mr^2)$.

Applying a correction, α, for the finite mass of the spring, its period T_2 for small angular oscillations is given by

$$T_2 = 2\pi\sqrt{\frac{I_0 + 2(I + mr^2) + \alpha}{b_2}},$$

where b_2 is the couple per unit twist (one radian) at the lower end of the spring. The above equation may be written

$$T_2{}^2 = 4\pi^2(J_0 + \beta r^2) \div b_2,$$

where J_0 and β are constants, whose values are at once apparent. Hence if r is varied and the time of oscillation found on each occasion, corresponding values of r^2 and $T_2{}^2$ when plotted will lie on a straight line whose slope is $(4\pi^2\beta \div b_2)$ and which makes an intercept $(4\pi^2 J_0 \div b_2)$ on the y-axis. If these are measured J_0 and b_2 may be found. But† $b_2 = \pi E a^4 \div 4l$, where $2a$ is the diameter of the wire, so that if the dimensions of the wire forming the spring are known, a value for Young's modulus for the material of the wire may be calculated.

A helical spring (not closely coiled) with an axial pull.—In Fig. 8·29(*a*) there is shown a helical spring made from a wire of uniform cross-sectional area, πa^2, and subjected to an axial pull F; the spring has N turns, each of diameter 2R. Let α be the inclination of the axis of the coils to the circular sections of the cylinder on

† Wilberforce, *Phil. Mag.*, **38**, 386, 1894.

which the coils may be supposed to lie; cf. Fig. 8·29(b). To determine the stresses at a section B of the coil, let S be the force downwards and M the couple exerted on the portion of the coil above B by the portion below. The couple M is represented by a vector $\vec{\mathrm{M}}$

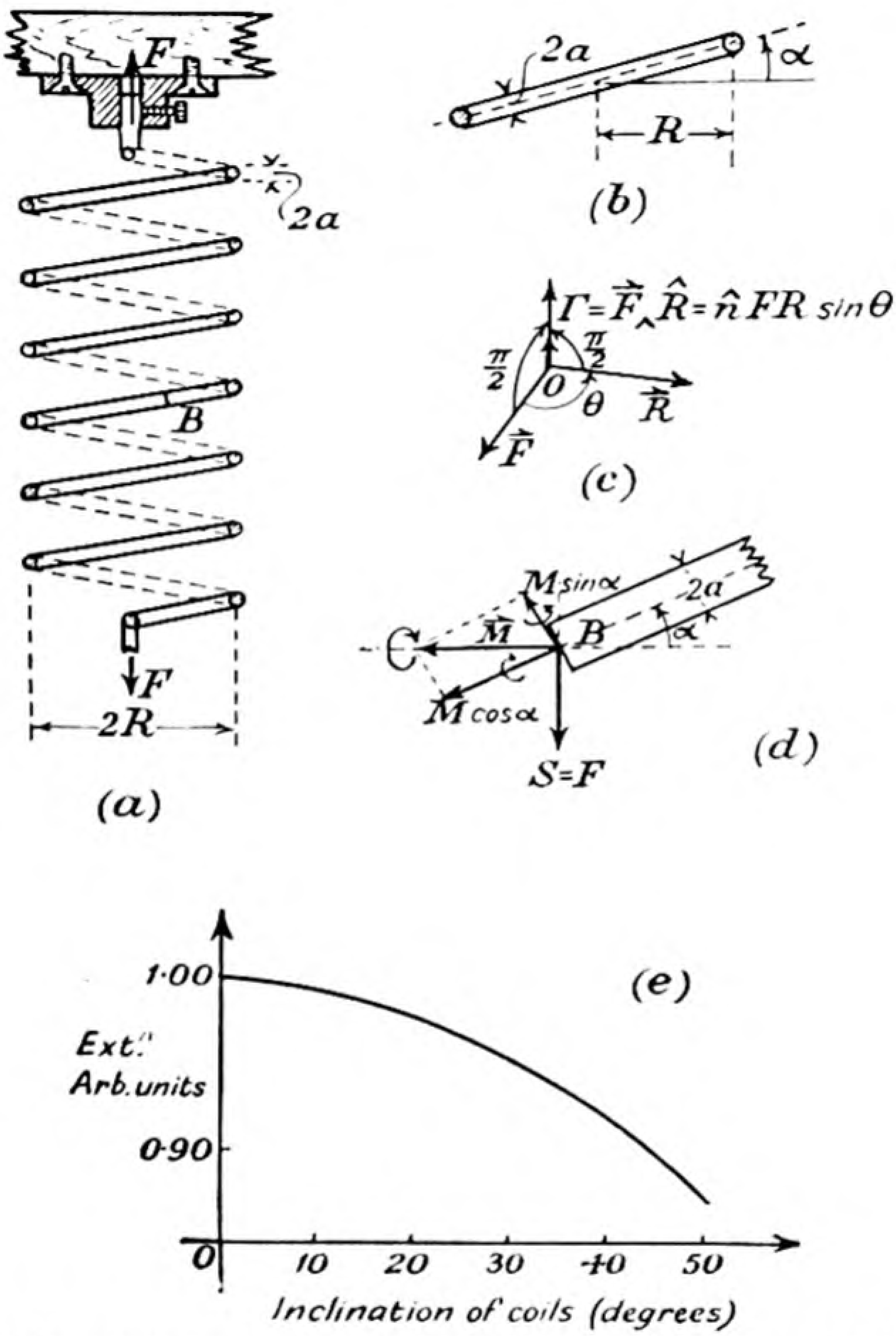


FIG. 8·29.—A helical spring (*not* closely coiled) with an axial pull.

in the vertical plane through which the axis of the wire at the point B passes. [The relation between the components of a couple and the vector representing it is shown in Fig. 8·29(c), where

$$\vec{\mathrm{M}} = \vec{\mathrm{F}}_\wedge \vec{\mathrm{R}}.]$$

For equilibrium,

$$\mathrm{S} = \mathrm{F}, \quad \text{and} \quad \mathrm{FR} = \mathrm{M}.$$

Thus M is constant at all sections of the wire and $\vec{\mathrm{M}}$ can be resolved

into components along the axis of the wire and at right angles to it; they are shown in Fig. 8·29(d), their magnitudes being M cos α and M sin α respectively. The first, M cos α, gives rise to a twist in the wire, while the second, M sin α, is the bending moment.

Now let W be the work done by the stretching forces in causing the spring to pass from a state of zero stress to its actual state. Then

$$\mathrm{W} = \int_0^z \mathrm{F}\,dz,$$

where z is the vertical distance through which the lower end of the spring descends, i.e. it is the extension of the spring. This energy is stored in the spring and, cf. pp. 383 and 320, amounts to

$$\tfrac{1}{2}\int \frac{\mathrm{M}^2}{\mathrm{EI}}\,ds + \int \frac{\Gamma^2}{\pi n a^4}\,ds,$$

$$\left[\because\ \frac{\Gamma}{\theta} = \frac{\pi n a^4}{2l},\ \text{and} \int \Gamma\,d\theta \quad \text{is the work done by a couple}\right]$$

where δs is an element of the spring and the integrations cover its whole length. Since M and Γ are constant, we have

$$\int_0^z \mathrm{F}\,dz = \tfrac{1}{2}\,\frac{\mathrm{M}^2}{\mathrm{EI}}\,l + \frac{\Gamma^2}{\pi n a^4}\,l,$$

where l is the total length of wire in the spring, i.e. $l = 2\pi\mathrm{NR}$. Using

$$\mathrm{M} = \mathrm{FR}\sin\alpha \quad \text{and} \quad \Gamma = \mathrm{FR}\cos\alpha,$$

this last equation becomes

$$\int_0^z \mathrm{F}\,dz = \tfrac{1}{2}\,\frac{\mathrm{F}^2\mathrm{R}^2 l \sin^2\alpha}{\mathrm{EI}} + \frac{\mathrm{F}^2\mathrm{R}^2 l\cos^2\alpha}{\pi n a^4}.$$

Differentiating both sides with respect to z, which depends upon F, we find

$$\mathrm{F} = \left[\frac{\mathrm{FR}^2 l\sin^2\alpha}{\mathrm{EI}} + 2\,\frac{\mathrm{FR}^2 l\cos^2\alpha}{\pi n a^4}\right]\frac{d\mathrm{F}}{dz}.$$

$$\therefore\ \frac{dz}{d\mathrm{F}} = \mathrm{R}^2 l\left[\frac{\sin^2\alpha}{\mathrm{EI}} + \frac{2\cos^2\alpha}{\pi n a^4}\right].$$

If α may be assumed constant, we find by integration, since when $\mathrm{F} = 0$, $z = 0$,

$$z = \mathrm{FR}^2 l\left[\frac{\sin^2\alpha}{\mathrm{EI}} + \frac{2\cos^2\alpha}{\pi n a^4}\right]$$

$$= \frac{\mathrm{FR}^2 l}{\pi a^4}\left[\frac{4\sin^2\alpha}{\mathrm{E}} + \frac{2\cos^2\alpha}{n}\right]. \qquad [\because\ \mathrm{I} = \tfrac{1}{4}\pi a^4.]$$

The extension, thus calculated, is due to the bending and twisting of the wire; the extension due to the shearing forces in the wire has been neglected and somewhat laborious calculations show that for thin wires the extension caused by the shearing forces is negligible in comparison with that due to the twisting of the wire.

If $\frac{E}{n} = \frac{5}{2}$, a ratio typical of many metals,

$$z = \frac{FR^2 l}{\pi a^4 n}\left[\frac{8 \sin^2 \alpha}{5} + 2 \cos^2 \alpha\right]$$

$$= \frac{FR^2 l}{\pi a^4 E}[4 \sin^2 \alpha + 5 \cos^2 \alpha]$$

$$= \frac{FR^2 l}{\pi a^4 E}[4 + \cos^2 \alpha].$$

Fig. 8·29(e) is a curve to illustrate how the extension of a helical spring, whose material is such that $2E = 5n$, varies with the inclination, α, of the coils. The extension for $\alpha = 0$ is taken as unity, i.e. the curve is really

$$y = [4 + \cos^2 x].$$

Springs of zero length.—When a helical spring is so wound that its extension is equal to the distance between the points to which it is attached we have a spring of zero length, if we define the initial length as the actual physical length minus the elongation. Mention has been made already, cf. p. 194, of such a spring and the method for winding one is shown in Fig. 8·30. B is a flat bar of brass with a hole drilled across it at C. The wire used in making the spring is under constant tension and passes through this hole in order that it may be wound on a spindle held in the chuck of a slow motion lathe. B is held flat against the spring its inclination to the axis of the spindle being ϕ. As the wire comes out of the hole at A it must bend in order to get it in line with other turns in the spring. This causes the turns of the spring to press against each other and so give to the spring its desired characteristic; the tension in the wire and the value of ϕ can only be found by trial and error.

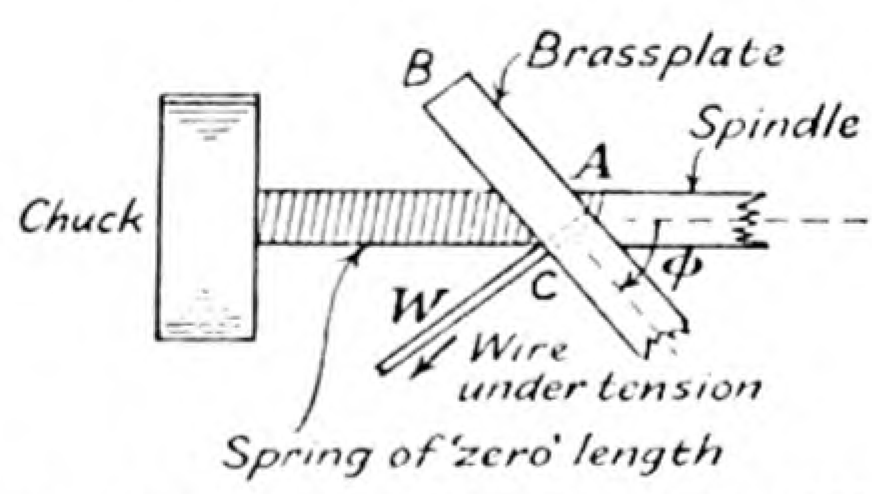


Fig. 8·30.—A spring of 'zero length'.

In actual practice all such springs are wound to have a negative length and then a short piece of straight wire is added to bring the initial length of the spring to zero.

On the bending of long columns; the existence of a critical load.—Before discussing the stability of a long vertical column of

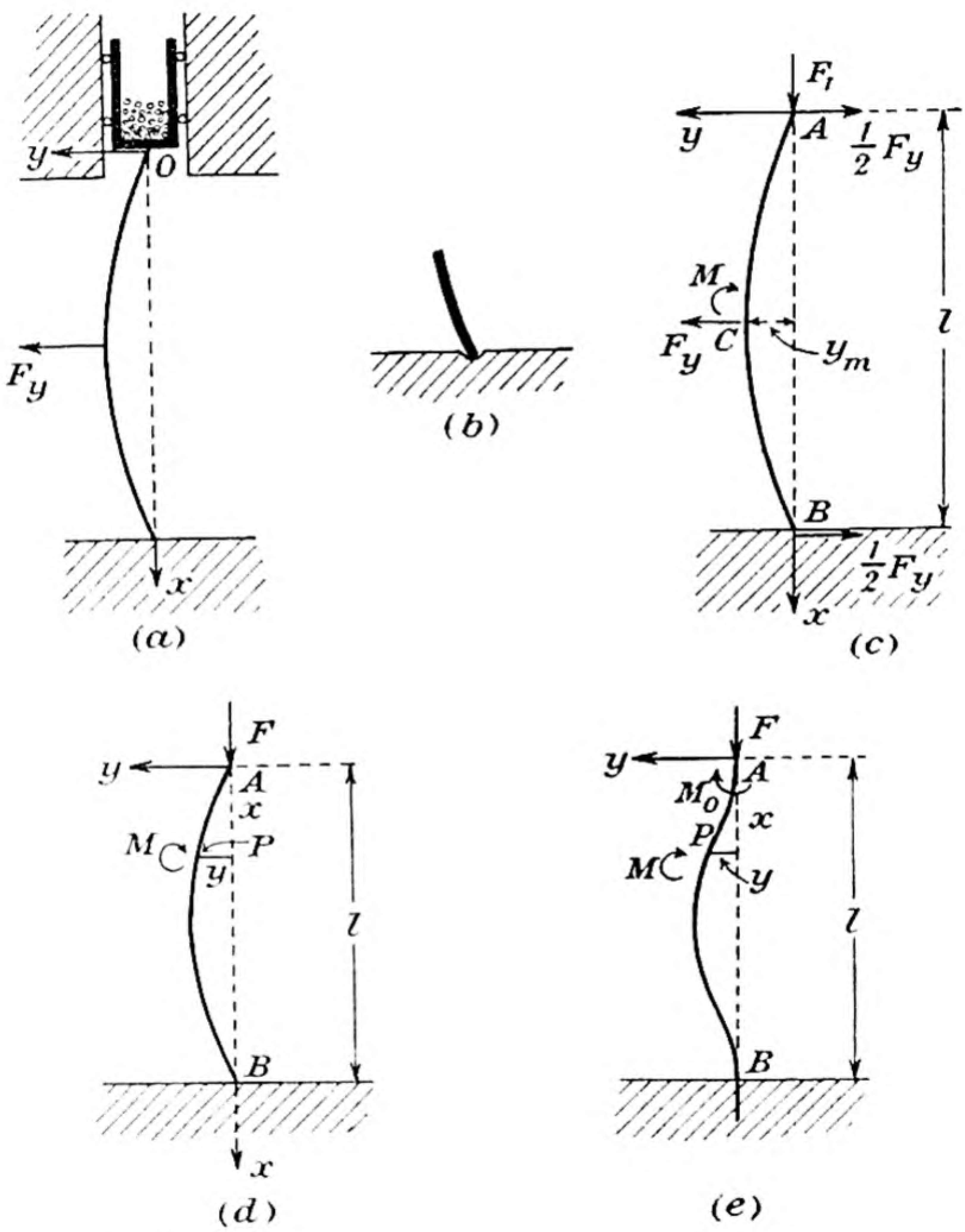


FIG. 8·31.—The bending of a long column; the existence of a critical load.

isotropic material, the following experiment will be found instructive. A long straight strip of steel is placed in a vertical position and loaded, the load being between guides so that it descends vertically, as indicated in Fig. 8·31(a). [Lead shot or mercury may be placed in an iron

bucket to vary the load.] The ends of the strip should be rounded and allowed to rest in grooves, cf. Fig. 8·31(b), so that the strip is free to bend. Suppose that the initial load is insufficient to bend the steel strip and that a lateral force F_v is applied at the centre in order to produce a definite deflexion of the strip. When this horizontal force is removed the strip will become straight again. The experiment is continued in the same manner with gradually increasing loads, until a stage is reached where the strip remains deflected when the lateral force is removed. At this stage it will also be found that if the deflexion is increased, the strip will remain bent for the same value of the load on top—we assume that the limit of perfect elasticity has not been exceeded. The load when this occurs is termed the ***critical load.*** If the load is increased beyond this, the amount of deflexion will increase, and the strip will take a permanent set, or collapse.

To account for the existence of this critical load, let a vertical force F_1 be applied to a strip or column AB, Fig. 8·31(c), together with a lateral force F_v at its mid-point. Suppose l is the length of the column and that y_m is the deflexion at the centre, C, the direction of coordinate axes being indicated. In order for the column to be in equilibrium, one of the conditions requires that the force F_v should be balanced by two horizontal forces $\frac{1}{2}F_v$ acting as shown. Let M be the bending moment at C. Taking moments of forces about C, so that it is not necessary to know the value of the shearing force at this point, we find that in order for the portion AC of the column to be in equilibrium, the equation

$$F_1 y_m + \frac{F_v}{2}.\frac{l}{2} + M = 0,$$

i.e.
$$F_1 y_m + \tfrac{1}{4}F_v l + EI[y'']_{x=l/2} = 0,$$

must be satisfied.

When F_v is gradually reduced to zero and F_1 is increased to the critical load F, the above equation becomes

$$Fy_m + EI[y'']_{x=l/2} = 0.$$

Suppose now that the load is increased so that the force applied to the upper end is $(F + f)$, where f is positive. Let a lateral force Z be applied at C in order to preserve the deflexion y_m. The force Z is considered positive if it is in the direction of F_v. Then

$$(F + f)y_m + \tfrac{1}{4}Zl + EI[y'']_{x=l/2} = 0.$$

But
$$Fy_m + EI[y'']_{x=l/2} = 0.$$

Hence, by subtraction,

$$fy_m + \tfrac{1}{4}Zl = 0,$$

from which it follows that Z must be negative. Thus, when Z is not operative, the deflexion of the beam will increase.

Again, let us suppose that when the vertical force is $(F + f)$, the deflexion at the centre becomes $\bar{y}_m$, where $\bar{y}_m > y_m$, so that the curvature at the centre increases. Then

$$(F + f)\bar{y}_m + EI[\bar{y}'']_{x=l/2} = 0.$$

$$\therefore\ f\bar{y}_m + EI[\bar{y}'' - y'']_{x=l/2} = 0.$$

But this is impossible since $[\bar{y}'' - y'']$ and all other quantities in this equation are positive. Hence the beam will continue to bend and finally break.

Thus this theory supports the facts which the experiment described at the beginning of this paragraph revealed.

The bending of long columns; Euler's theory.—Consider a vertical column of isotropic material, whose length is great compared with its dimensions in directions at right angles to it. Let its cross-section be uniform, the column itself being initially upright. It will be assumed that its ends are rounded so that it is free to bend along its whole length. Suppose that a vertical load equal to the critical load F is applied to the upper end of the column, and that the beam is slightly bent, cf. Fig. 8·31(*d*). Then at every transverse section of the column there will be a definite bending moment. Let us consider the equilibrium of a portion AP of the column, where P is at a distance x from A, the deflexion at the point P being y. Taking moments of forces about P, we have

$$EIy'' + Fy = 0.$$

$$\therefore\ y = A\cos\left[\sqrt{\frac{F}{EI}}\,x + \alpha\right],$$

where A and α are constants to be determined. Now at $x = 0$, $y = 0$.

$$\therefore\ [y]_{x=0} = 0 = A\cos\alpha.$$

$$\therefore\ \alpha = \tfrac{1}{2}\pi, \quad \text{or} \quad A = 0.$$

[Here, as elsewhere, we shall reject the other values for α, since these correspond to the existence of points of inflexion in the beam.] The condition $A = 0$ is impossible for we have assumed the column to be bent.

Also y' is zero when $x = l/2$. Hence, since

$$y' = -A\sqrt{\frac{F}{EI}}\sin\left[\sqrt{\frac{F}{EI}}\,x + \alpha\right],$$

$$[y']_{x=l/2} = 0 = -A\sqrt{\frac{F}{EI}}\sin\left[\sqrt{\frac{F}{EI}}\cdot\frac{l}{2} + \alpha\right].$$

$$\therefore\ \sqrt{\frac{F}{EI}}.\frac{l}{2} = -\alpha.$$

But
$$\alpha = \tfrac{1}{2}\pi.$$

$$\therefore\ F = \frac{\pi^2 EI}{l^2}.$$

This gives the end-load which is sufficient to keep the beam bent when the initial curvature has been produced. If the vertical force due to the load exceeds the critical load F, the column will give way. It must be emphasized, however, that the expression obtained for the critical load is only valid if the column is long and narrow, its material isotropic, and the loading axial.

[The value for F may also be obtained by using the fact that the deflexion is zero when $x = l$.]

Suppose now that the ends of the column are fixed so that the tangents to it at these points are always vertical, cf. Fig. 8·31(e). When the critical load, F, is applied let M_0 be the external couple acting on the column at its upper end in order to fulfil the condition that the tangent at this point is vertical. [An equal but opposite couple must exist at the lower end.] The differential equation, whose solution will give the form of the neutral surface, is

$$EIy'' + Fy + M_0 = 0.$$

This is obtained by taking moments of forces about P, the point (x, y), in the beam where the bending moment is M or EIy''. The solution to the above equation is

$$y = A\left\{\cos\left[\sqrt{\frac{F}{EI}}\,x + \alpha\right]\right\} - \frac{M_0}{F},$$

where A and α are constants to be determined.

At $x = 0$, $y = 0$,

$$\therefore\ 0 = \{A\cos\alpha\} - \frac{M_0}{F}.$$

$$\therefore\ \cos\alpha = \frac{M_0}{AF}.$$

At $x = l$, $y = 0$,

$$\therefore\ 0 = A\left\{\cos\left[\sqrt{\frac{F}{EI}}\,l + \alpha\right]\right\} - \frac{M_0}{F}.$$

$$\therefore\ \cos\left(\sqrt{\frac{F}{EI}}\,l + \alpha\right) = \frac{M_0}{FA} = \cos\alpha.$$

$$\therefore\ \sqrt{\frac{F}{EI}}\,l = 2\pi \qquad \text{or} \qquad F = \frac{4\pi^2 EI}{l^2},$$

i.e. the critical load is four times as great as in the previous instance, or the column has four times the ability to resist thrust. The above expression shows that if a strut is fixed at both ends the load which it will stand without bending is the same as that for a strut of half the length but hinged at both ends.

Now at $x = \frac{1}{2}l$, $y' = 0$,

$$\therefore\ 0 = -A\sqrt{\frac{F}{EI}}\sin\left(\sqrt{\frac{F}{EI}}\frac{l}{2} + \alpha\right).$$

$$\therefore\ \alpha = -\sqrt{\frac{F}{EI}}\frac{l}{2} = -\pi.$$

Hence $\qquad M_0 = FA\cos\alpha = -FA.$

Thus the equation to the neutral surface is

$$y = A\left[\left\{\cos\left(\frac{2\pi}{l}x - \pi\right)\right\} + 1\right].$$

At $x = l/2$, the deflexion is a maximum, viz. y_m.

$$\therefore\ [y]_{x=l/2} = y_m = 2A.$$

$$\therefore\ A = \tfrac{1}{2}y_m.$$

Now at a point of inflexion, $y'' = 0$. In the present instance this occurs when $x = l/4$; and, by symmetry, there must also be a point of inflexion when $x = \frac{3}{4}l$.

The maximum height of a column or tree.—Since a column is buckled when the load it carries on top exceeds a certain critical value depending upon the manner in which the beam is supported and the critical load decreases as the height of the column increases, it follows that a pole of given cross-section would, if high enough, tend to bend under its own weight, i.e. the height of a pole is limited. In the same way it may be shown that the height of a tree is limited—in the theory it is necessary to take account of the fact that the tree tapers: the solution of the differential equation in such an instance involves the use of Bessel functions, and is therefore beyond the scope of this book. The maximum height of a uniform column, or pole, may be determined as follows.

When a column is fixed at one end let W be the weight of the load at its upper and free end when instability tends to set in; the loaded column is shown in Fig. 8·32, the displacement at the free end being a. The column will then be in equilibrium under the action of W, and its own flexural resisting forces and the reactions at its supported end. With the notation indicated in the diagram, the reaction at the ground consisting of a force W directed upwards and an anti-clockwise couple of moment Wa, the bending moment at P is given

by $M + Wy = Wa$, i.e. $EIy'' - W(a - y) = 0$. If we call $z = a - y$, we have $z' = -y'$ and $z'' = -y''$ so that the above equation becomes $EIz'' + Wz = 0$. The solution to this equation will give the shape of the bent column. We have, at once,

$$z = A\cos\left(\sqrt{\frac{W}{EI}}\,x + \alpha\right),$$

where A and α are constants.

Now at $x = 0$, $y' = 0 = z'$, i.e. $\alpha = 0$. At $x = l$, $y = a$, i.e. $z = 0$. Hence

$$\cos\left(\sqrt{\frac{W}{EI}}\,l\right) = 0.$$

$$\therefore \sqrt{\frac{W}{EI}}\,l = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \text{ etc.}$$

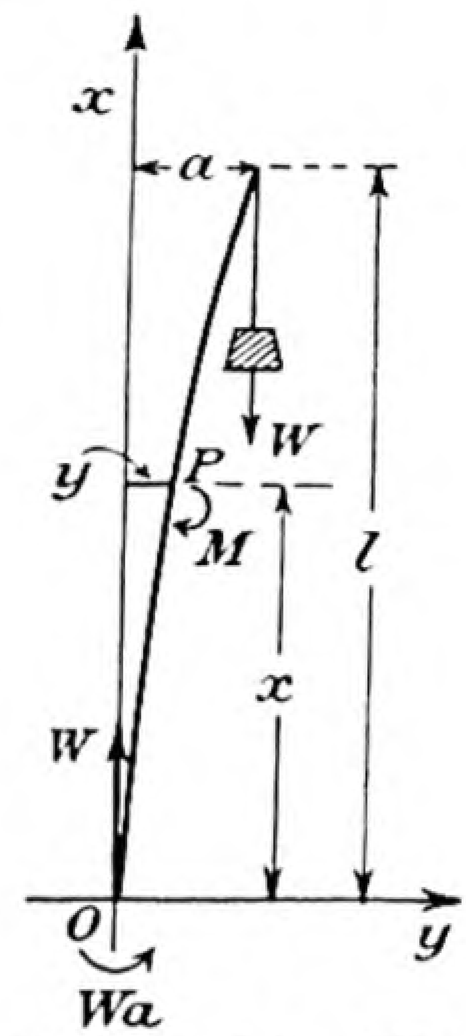


FIG. 8·32.—The bending of a long column.

Taking the first of these values we get $W = \frac{\pi^2 EI}{4l^2}$. This gives the weight of the smallest load which will cause the beam to collapse and for a long column the compressive stress which this load produces is considerably below the elastic limit for compressive stresses.

If the cross-section of the column is a circle of radius r, $I = \frac{1}{4}\pi r^4$. Thus the weight of the load which a vertical column can support without instability occurring is directly proportional to the fourth power of the radius and inversely proportional to the square of the length of the column.

It remains for us to find a value for A. From the end condition $x = 0$, $z = a$, and using the fact already established that α is zero, we have $a = A$. Thus

$$z = a\cos\sqrt{\frac{W}{EI}}\,x,$$

or

$$y = a\left[1 - \cos\sqrt{\frac{W}{EI}}\,x\right].$$

This shows that the shape of the column when slightly bent is sinusoidal.

Since a column cannot support a load whose weight exceeds a certain value and because this weight decreases as the length of the column increases, it follows that a column or rod of given cross-section would, if sufficiently high, begin to bend under its own

weight. If W is this weight and it is assumed as a first approximation that the problem is the same as if the weight were applied at the centre of the column, then bending will set in when

$$W = \frac{\pi^2 EI}{4(l/2)^2} = \frac{\pi^2 EI}{l^2}.$$

To illustrate this problem let us consider an oak 'tree' of uniform circular section, radius 8 cm. Then $E = 1{\cdot}3 \times 10^{11}$ dyne.cm.$^{-2}$, and the density of oak may be taken as $0{\cdot}8$ gm.cm.$^{-3}$. Then $W = g\rho l A$, so that (assuming $\pi^2 = 10$ and $g = 1000$ cm.sec.$^{-2}$)

$$1000 \times 0{\cdot}8 \times l \times A = \frac{10 \times 1{\cdot}3 \times 10^{11} \times A \times 8^2}{4l^2}.$$

Thus the greatest length is

$$l = \sqrt[3]{2{\cdot}6 \times 10^{10}} \text{ cm.} \fallingdotseq 30 \text{ metres.}$$

Electrical resistance strain gauges.—The electrical resistance strain gauge is a device for measuring superficial strains in flat or curved objects. It is distinguished by its small size, simplicity of application and its ability to cope with slow or rapid changes of strain. Before the introduction of the resistance strain gauge, surface strain measurements were limited for the most part to experiments on test bars, useful in many respects but of little value for determining the actual surface strains occurring in complete articles and structures.

The electrical resistance strain gauge, introduced in 1939 by SIMMONS and RUGE and subsequently developed at the National

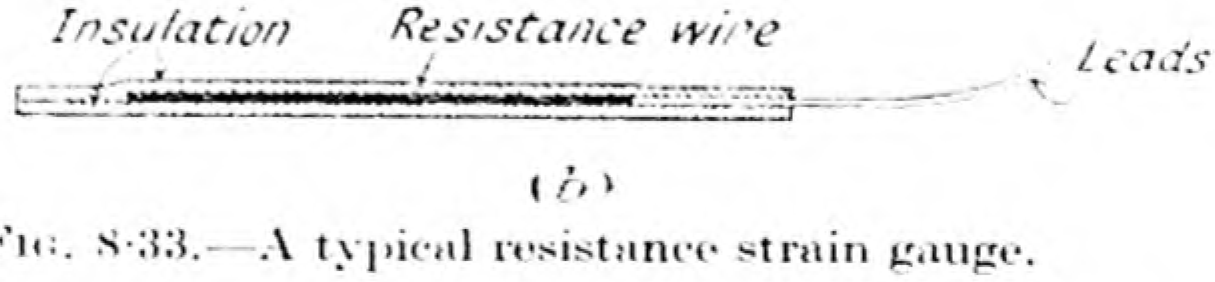


FIG. 8·33.—A typical resistance strain gauge.

Physical laboratory, depends on the fact that a metal wire, if extended or compressed, undergoes a change of resistance bearing a definite relationship to the change in length. The usual form of resistance strain gauge consists of a single length of very fine resistance wire wound in close zig-zag form to give a series of parallel lengths in the form of a grid, as shown in Fig. 8·33(a). The wire

is laid between sheets of tissue paper and the whole impregnated with an insulating medium. The element so formed is cemented to the surface under test, cf. Fig. 8·33(b). To do this satisfactorily the underside of the gauge is treated uniformly with cellulose lacquer and this is moistened with acetone and then applied quickly to the test surface which should be slightly roughened and free from grease. The gauge is left in position several days before use.

If a wire is made from a material with a resistivity χ, then its resistance R is given by

$$\mathrm{R} = \chi\left(\frac{l}{\pi a^2}\right),$$

where l is its length and a its radius of cross-section. Differentiating logarithmically, we get, if χ is constant,

$$\frac{\Delta \mathrm{R}}{\mathrm{R}} = \frac{\Delta l}{l} - 2\left(\frac{\Delta a}{a}\right).$$

But Poisson's ratio, σ, is $-\left(\dfrac{\Delta a}{a}\right) \div \dfrac{\Delta l}{l}$, so that

$$\frac{\Delta \mathrm{R}}{\mathrm{R}} = (1 + 2\sigma)\,\frac{\Delta l}{l}.$$

The ***strain sensitivity factor*** or ***gauge factor*** is defined as

$$\frac{\Delta \mathrm{R}}{\mathrm{R}} \div \frac{\Delta l}{l} = 1 + 2\sigma \fallingdotseq 1{\cdot}6 \qquad [\because\ \sigma \to 0{\cdot}3].$$

In practice it is found that this factor is about 2·1, since the resistivity of the material depends upon the stress. The above equation may be written

$$\text{Strain} = \frac{\Delta l}{l} = \frac{\Delta \mathrm{R}}{\mathrm{R}} \times \frac{1}{\text{strain sensitivity of the gauge}}.$$

Unfortunately the resistance of a strain gauge cannot be made independent of temperature so that with every gauge a 'compensating' gauge is used. This is mounted in exactly the same way as the test surface and so is subjected to the same thermal influences as the test gauge but is free from strain. By connecting the two gauges in adjacent arms of a Wheatstone bridge circuit, the variation of resistance with temperature changes is neutralized.

One convenient bridge circuit is shown in Fig. 8·34(a). On account of the Joule effect the current through the gauges should seldom exceed 5 mA. M is the main gauge and C the compensating gauge; these are arranged so that when the bridge is balanced the current in each gauge is the same. R_1 and R_2 are standard resistances arranged as shown in conjunction with a slide wire resistance

AB. G is a sensitive galvanometer. Suppose that when the bridge is balanced, M being free from strain, the sliding contact is at O. Let P be the resistance of R_1 and the portion AO of the bridge wire; let Q be R_2 plus the resistance of BO. Then if M and C denote the resistances of these components

$$\frac{M}{C} = \frac{P}{Q}.$$

When M is subjected to strain, let its resistance become $M + \Delta M$

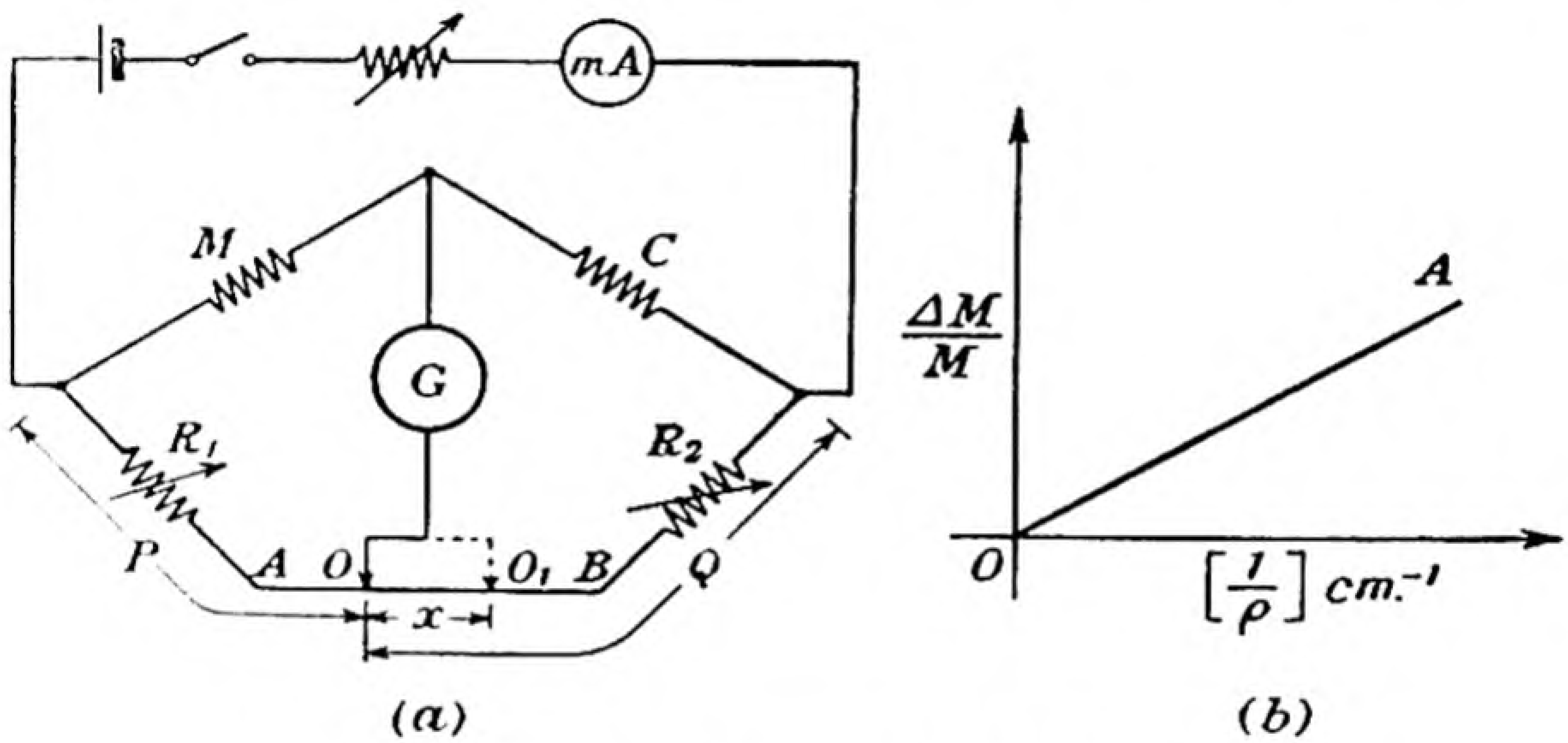


FIG. 8·34.—Bridge for use with a strain gauge.

and let the sliding contact be shifted to O_1 ($OO_1 = x$) to balance the bridge. If r is the resistance per unit length of the bridge wire

$$\frac{M + \Delta M}{C} = \frac{P + rx}{Q - rx},$$

i.e.

$$\frac{M + \Delta M}{M} = \frac{1 + \dfrac{rx}{P}}{1 - \dfrac{rx}{Q}},$$

and if rx is small compared with P and Q, this equation gives

$$\frac{\Delta M}{M} = rx\left(\frac{1}{P} + \frac{1}{Q}\right).$$

Thus $\dfrac{\Delta M}{M}$ is proportional to x, i.e. with sufficient accuracy the slide wire can be calibrated linearly in terms of the strain $\times$ strain sensitivity.

In some applications it is possible to double the effective sensitivity

by mounting the compensating gauge in a position in which the strain is equal, but of opposite sign, to that of the main gauge.

Determination of Poisson's ratio by means of resistance strain gauges.—For this experiment three gauges are needed: one serves as a compensator while the others are placed respectively with the wires parallel to and at right angles to the length of a uniform bar. In this way the strains in the two directions may be compared. The bar should be bent uniformly (cf. p. 357) so that the curvature in the plane of bending can be calculated from measurements on the deflexion at the centre of the bar, etc.

Let ρ be the radius of curvature of the neutral axis and let z be the distance between the gauge and the neutral axis. If a length l on the surface of the bar becomes $l + \Delta l$ under these conditions

$$\frac{\Delta l}{l} = \frac{z}{\rho}. \qquad \text{[cf. p. 345.]}$$

If $\Delta \mathrm{M}$ is the change in the resistance M of the gauge concerned

$$\frac{\Delta \mathrm{M}}{\mathrm{M}} = k\left(\frac{z}{\rho}\right),$$

where k is a constant. Thus $\dfrac{\Delta \mathrm{M}}{\mathrm{M}}$ is a linear function of the curvature $\left(\dfrac{1}{\rho}\right)$. This relationship is shown by the straight line OA, Fig. 8·34(b). From the observations made in the plane of bending, this graph can be constructed.

To determine the radius of anticlastic curvature, observations are made on the appropriate gauge and it is usual to assume that for this gauge the fractional change to its resistance is related to the curvature by the same equation as that which applies to the other gauge. A mean value for Poisson's ratio is then found in the usual way.

The elastic deformation of a thin circular ring subjected to a longitudinal pull.—*Approximate theory.* A thin circular ring subjected to a pull (or thrust) through its centre has, at any radial section, a bending moment, a shearing force and a direct pull (or thrust). To estimate the bending moment the curvature of the ring will be neglected and, rules, strictly applicable to beams initially straight, will be used.

Fig. 8·35(a) represents a circular ring of radius R subjected to a pull F along $\mathrm{A_0B_0}$ and although there is bending at the various sections, it is evident from symmetry that the traces of the four sections at $\mathrm{A_0}$, H, $\mathrm{B_0}$ and G pass through the centre O after the pull

has been applied and therefore between H and A_0, for example, the total bending is zero.

Consider therefore a section XX, defined by the angle ϕ. Let M be the bending moment at this section and M_0 that at the vertical section through H. Now consider the portion of the ring between H and X. The external forces on it due to the rest of the ring are

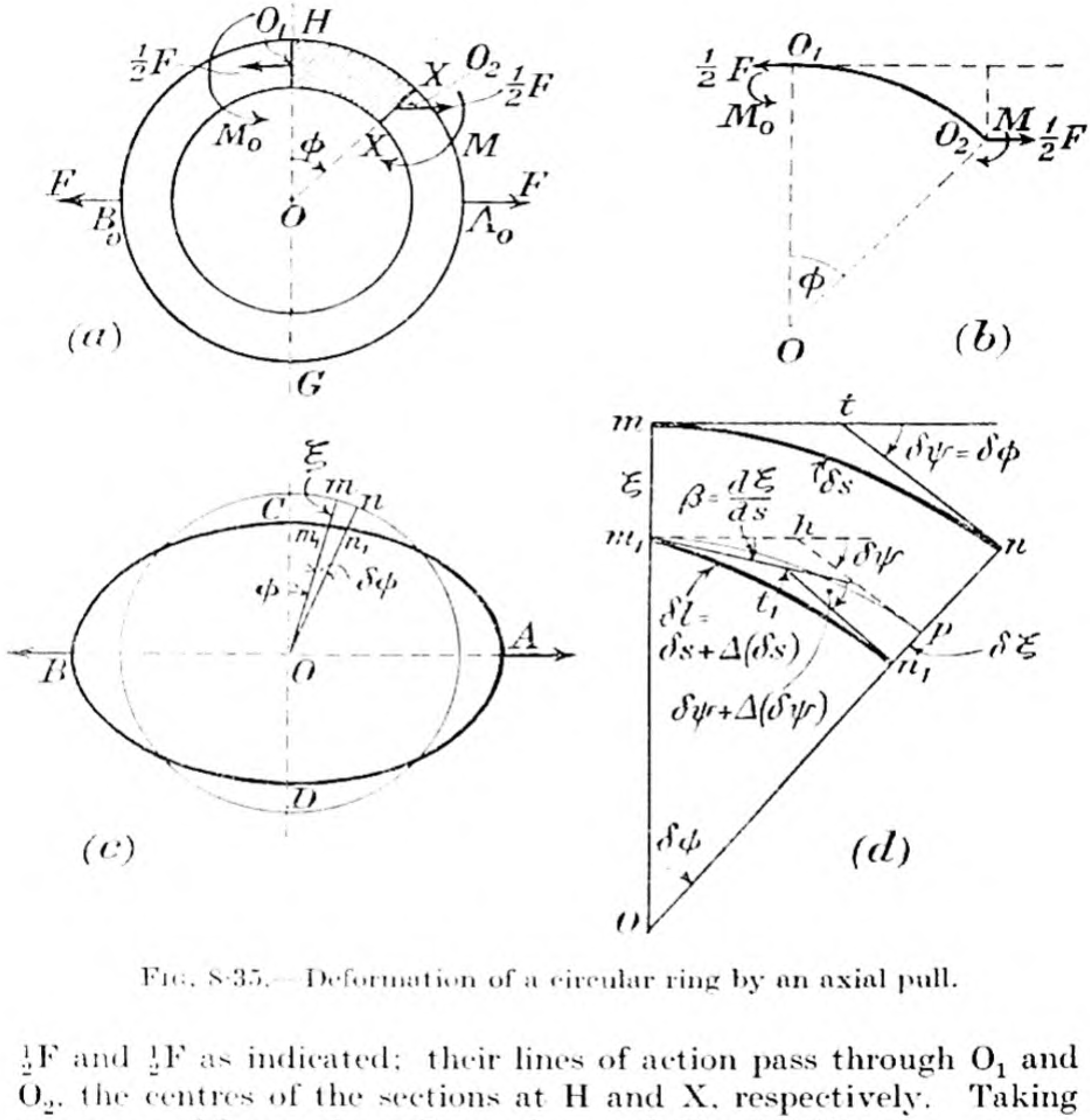


FIG. 8·35.—Deformation of a circular ring by an axial pull.

$\frac{1}{2}$F and $\frac{1}{2}$F as indicated; their lines of action pass through O_1 and O_2, the centres of the sections at H and X, respectively. Taking moments of forces about O_2, we have, cf. Fig. 8·35(b),

$$\tfrac{1}{2}\mathrm{FR}(1 - \cos\phi) + \mathrm{M}_0 = \mathrm{M}.$$

If δs is the axial length of an element of the beam and $\delta\psi$ the angle between the tangents at its extremities, by treating the ring as a straight beam, we have $\dfrac{d\psi}{ds} = \dfrac{\mathrm{M}}{\mathrm{EI}}$, where E is Young's modulus for its material and I the second areal moment for its cross-section.

Since the total bending between H and A_0 is zero,

$$0 = \int_{\phi=0}^{\phi=\frac{1}{2}\pi} d\psi = \frac{1}{4}\oint \frac{M}{EI}.ds = \int_0^{\frac{1}{2}\pi} \frac{M}{EI}.R\,d\phi$$

$$= \frac{R}{EI}\int_0^{\frac{1}{2}\pi} [M_0 + \tfrac{1}{2}FR(1 - \cos\phi)]\,d\phi.$$

$$\therefore\ 0 = \tfrac{1}{2}\pi M_0 + \tfrac{1}{2}FR\left(\frac{\pi}{2} - 1\right), \quad \text{or} \quad M_0 = \frac{FR}{\pi}(1 - \tfrac{1}{2}\pi).$$

$$\therefore\ M = FR\left[\frac{1}{\pi} - \tfrac{1}{2}\cos\phi\right].$$

Now let ACBD, Fig. 8·35(c), be the central line of the ring after deformation. In order to obtain the variation of the curvature of this line due to the bending we proceed as follows. Suppose that mn is an element of the ring of length δs and subtending an angle $\delta\phi$ at O; when this element is displaced it will be assumed to have the same curvature as m_1n_1, the element of the deformed ring cut off by the radii Om and On. If ξ is the radial displacement of m, viz, mm_1, that of n is $\xi + \delta\xi$. The initial curvature of the element considered is given by

$$\lim.\frac{\delta\psi}{\delta s} = \frac{1}{R}.$$

The radius of curvature, ρ, of the same element after bending is given by

$$\frac{1}{\rho} = \lim.\frac{\delta\psi + \Delta(\delta\psi)}{\delta s + \Delta(\delta s)},$$

in which $\delta\psi + \Delta(\delta\psi)$ denotes the angle between the cross-sections at m_1 and n_1 of the deformed bar, and $\delta s + \Delta(\delta s) = m_1n_1$.

To evaluate this fraction, with centre O, Fig. 8·35(d), and radius Om_1 describe a circular arc to cut On in p. Let m_1h and ph be parallel to mt and nt, the tangents at m and n respectively. Let $\beta = h\hat{m}_1t_1$ and this is the angle between the arcs m_1p and m_1n_1; this is very nearly equal to $p\hat{m}_1n_1$. From the Δpm_1n_1 (not actually shown), we have $m_1\hat{p}n_1 = \frac{1}{2}\pi - \frac{1}{2}\delta\phi$, and hence

$$\frac{\sin\beta}{\delta\xi} = \frac{\sin m_1\hat{p}n_1}{\delta l}.$$

where $\delta l = \delta s + \Delta(\delta s)$. Since β is small, we have

$$\beta = \frac{d\xi}{dl}\cos\tfrac{1}{2}\delta\phi = \frac{d\xi}{dl} \simeq \frac{d\xi}{ds}.$$

Now consider the external angles of the quadrilateral $m_1t_1n_1$O. We have

$$\left[\tfrac{1}{2}\pi + \frac{d\xi}{ds}\right] + [\delta\psi + \Delta\,\delta\psi] + \left[\tfrac{1}{2}\pi - \left(\frac{d\xi}{ds} + \frac{d^2\xi}{ds^2}\delta s\right)\right] + [\pi - \delta\phi] = 2\pi.$$

Hence, since $\delta\phi = \delta\psi$,

$$\Delta(\delta\psi) = \frac{d^2\xi}{ds^2}(\delta s).$$

Also, from the diagram, in so far as m_1n_1 may be considered equal to m_1p,

$$\frac{\delta s}{\mathrm{R}} \doteqdot \frac{\delta s + \Delta(\delta s)}{\mathrm{R} - \xi} = \frac{\Delta(\delta s)}{-\xi}.$$

$$\therefore \Delta(\delta s) = -\frac{\xi}{\mathrm{R}}\delta s = -\xi\,\delta\phi.$$

Hence $$\frac{1}{\rho} = \lim. \frac{\delta\phi + \dfrac{d^2\xi}{ds^2}\delta s}{\delta s\left(1 - \dfrac{\xi}{\mathrm{R}}\right)} \simeq \frac{d\phi}{ds}\left(1 + \frac{\xi}{\mathrm{R}}\right) + \frac{d^2\xi}{ds^2}.$$

Now $\dfrac{\delta\phi}{ds} = \dfrac{1}{\mathrm{R}}$ and, cf. p. 346, $\dfrac{1}{\rho} - \dfrac{1}{\mathrm{R}} = \dfrac{\mathrm{M}}{\mathrm{EI}}$, where

$$\mathrm{M} = \mathrm{FR}\left[\frac{1}{\pi} - \tfrac{1}{2}\cos\phi\right].$$

Thus $$\frac{1}{\mathrm{R}^2}\frac{d^2\xi}{d\phi^2} + \frac{\xi}{\mathrm{R}^2} = \frac{\mathrm{FR}}{2\mathrm{EI}}\left[\frac{2}{\pi} - \cos\phi\right].$$

A general solution of this equation is

$$\xi = \mathrm{A}\cos\phi + \mathrm{B}\sin\phi + \frac{\mathrm{FR}^3}{\pi\mathrm{EI}} - \frac{\mathrm{FR}^3}{4\mathrm{EI}}.\phi\sin\phi.$$

The constants A and B will be determined by the conditions of symmetry, viz. $\dfrac{d\xi}{d\phi} = 0$, for $\phi = 0$ and $\phi = \frac{1}{2}\pi$

$$\therefore \mathrm{B} = 0 \quad \text{and} \quad \mathrm{A} = -\frac{\mathrm{FR}^3}{4\mathrm{EI}}.$$

$$\therefore \xi = \frac{\mathrm{FR}^3}{4\mathrm{EI}}\left[\frac{4}{\pi} - \phi\sin\phi - \cos\phi\right].$$

Since there is a decrease in the length of a radius vector when ξ is positive, we have:—

(a) Decrease in *vertical* diameter $= 2[\xi]_{\phi=0} = \dfrac{\mathrm{FR}^3}{2\mathrm{EI}}\left[\dfrac{4}{\pi} - 1\right]$,

(b) Increase in *horizontal* diameter $= -2[\xi]_{\phi=\frac{1}{2}\pi} = -\dfrac{\mathrm{FR}^3}{2\mathrm{EI}}\left[\dfrac{4}{\pi} - \dfrac{\pi}{2}\right]$.

The angle turned through by the tangent at a point on the ring is, to a first approximation, given by

$$\psi = \frac{1}{\mathrm{R}}\frac{d\xi}{d\phi} = \frac{\mathrm{FR}^2}{4\mathrm{EI}}(-\phi\cos\phi).$$

Neglecting the minus sign, this is a maximum when $\phi = \cot \phi$, i.e. $\phi = 49°$.

Then
$$[\psi]_{\phi=49°} = \frac{0{\cdot}561FR^2}{4EI}.$$

Sucksmith's ring-balance method for the rapid determination of small varying forces.—This method, originally designed for measuring the force on a non-ferrous material when it lies in a strong magnetic field, is due to SUCKSMITH†. It depends upon the

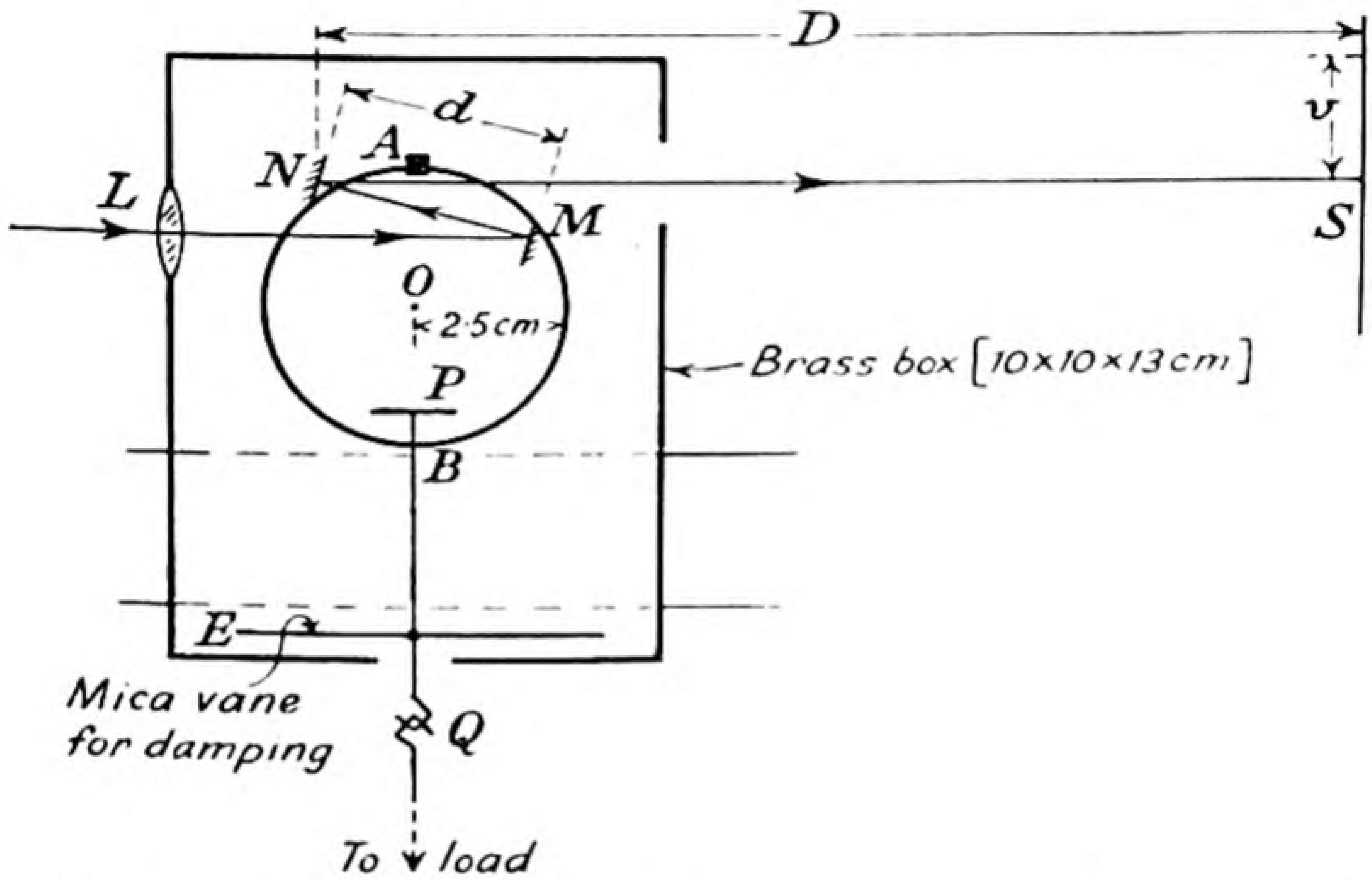


FIG. 8·36.—Sucksmith's ring-balance for the rapid determination of small varying forces.

deformation of a circular ring of strip phosphor bronze when this is subjected to a pull along a diameter; the ring lies in a vertical plane and is fixed at its uppermost point A, Fig. 8·36. Two small plane mirrors M and N are attached to the ring at points where the angular deflexions of the ring are a maximum, cf. above. L is a converging lens and the light transmitted by it, after reflexion from the mirrors, is focused on a scale S. If d and D are the distances indicated, the deflexion v of the spot of light on S is given by

$$v = \psi_m[4D + 2d], \qquad \text{[Cf. p. 371]}$$

where
$$\psi_m = [\psi]_{\phi=49°} = \frac{0{\cdot}561FR^2}{4EI}.$$

Instead of making use of this formula to obtain a value for the force F, the balance may be calibrated by placing, in turn, standard masses on the pan P which is fixed rigidly to the ring; the displacement, as theory suggests, is found to be a linear function of the load.

† *Phil. Mag.*, **8**, 158, 1929.

The whole assemblage of ring and mirrors is enclosed in a brass box, which serves as a constant temperature enclosure, and E is a mica vane to damp the motion of the ring so that small slowly varying forces may be measured. Q is a hook to support any load which may be used in conjunction with the balance. An example of the use of this balance is given on p. 470.

Continuous beams; Clapeyron's theorem of the three moments.—This theorem is due to CLAPEYRON (1875) and concerns a continuous beam uniformly loaded and resting on a number of

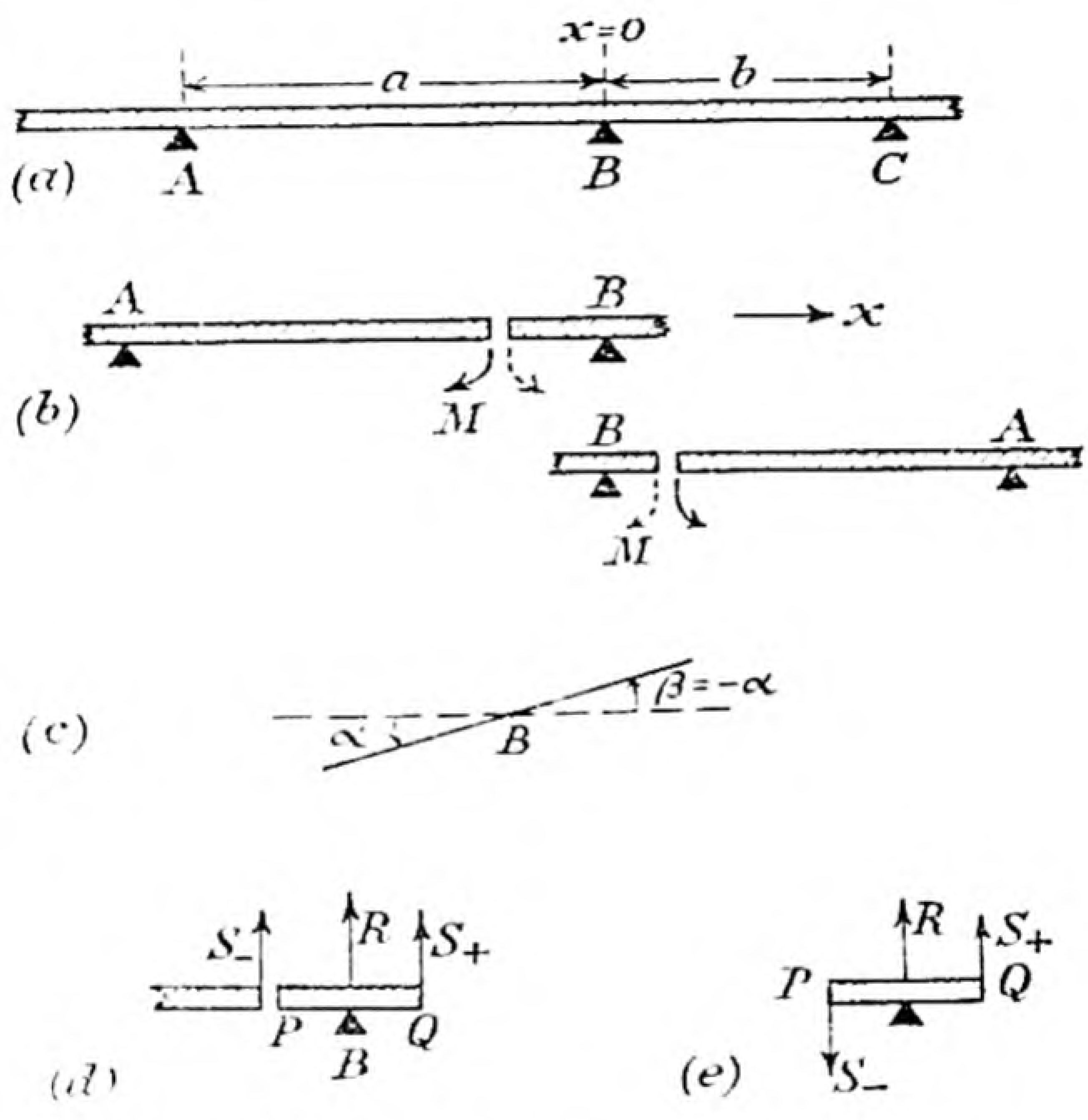


FIG. 8·37.—Continuous beams; Clapeyron's theorem of the three moments.

supports at the same level. It gives us a relation between the values of the bending moments at any three consecutive supports such as A, B and C, Fig. 8·37(a).

Let AB $= a$, BC $= b$ and take the origin at B and the positive direction of x that of BC. Then, with the usual notation, in the segment BC we have $\mathrm{EI}y^{\mathrm{IV}} = w$, so that

$$\mathrm{M} = \mathrm{EI}y'' = \tfrac{1}{2}wx^2 + \mathrm{A}x + \mathrm{M_B}, \qquad . \qquad . \qquad \text{(i)}$$

where $\mathrm{M_B}$ is the bending moment at B and A is an integration constant to be determined. Hence

$$\mathrm{EI}y' = \tfrac{1}{6}wx^3 + \tfrac{1}{2}\mathrm{A}x^2 + \mathrm{M_B}x + \mathrm{EI}\alpha, \qquad . \qquad . \qquad \text{(ii)}$$

where α is the value of $[y']_{x=0}$. Integrating again, we get

$$EIy = \tfrac{1}{24}wx^4 + \tfrac{1}{6}Ax^3 + \tfrac{1}{2}M_B x^2 + EI\alpha x, \quad . \quad . \quad \text{(iii)}$$

the integration constant being zero.

Now the conditions at C, i.e. $x = b$, give us

$$M_C = EI[y'']_{x=b} = \tfrac{1}{2}wb^2 + Ab + M_B, \quad . \quad . \quad \text{(iv)}$$

and since the deflexion at C is zero,

$$0 = \tfrac{1}{24}wb^4 + \tfrac{1}{6}Ab^3 + \tfrac{1}{2}M_B b^2 + EI\alpha b$$

i.e.

$$0 = \tfrac{1}{24}wb^3 + \tfrac{1}{6}Ab^2 + \tfrac{1}{2}M_B b + EI\alpha. \quad . \quad . \quad \text{(v)}$$

Eliminating A from equations (iv) and (v) we have

$$bM_C + 2bM_B - \tfrac{1}{4}wb^3 = -6EI\alpha. \quad . \quad . \quad \text{(vi)}$$

Now consider the segment BA, the origin at B but the positive direction of x reversed. Fig. 8·37(b) shows the bending moments at a section in the segment before and after this reversal occurs. The sign of the bending moment M remains unchanged and if β is the slope of the beam at B we find

$$aM_A + 2aM_B - \tfrac{1}{4}wa^3 = -6EI\beta. \quad . \quad . \quad \text{(vii)}$$

But $\beta = -\alpha$, so that by addition of equations (vi) and (vii), we find

$$aM_A + 2(a + b)M_B + bM_C = \tfrac{1}{4}w(a^3 + b^3),$$

which expresses ***Clapeyron's theorem of the three (bending) moments.***

To find the thrust on the beam at the support B, for example, we may proceed as follows. Let R be the reaction at B and consider the forces on an element PQ of the beam, cf. Fig. 8·37(d), where, in the original system of coordinates, P is on the negative and Q on the positive side of B. Let S_+ be the shearing force immediately to the right of B, i.e. at Q; then S_- is the shearing force on a section immediately to the left of B. Thus the forces on the element PQ are as in Fig. 8·37(e).

$$\therefore\ R = S_- - S_+.$$

Now, cf. p. 335, $S = \dfrac{dM}{dx}$. Hence from equation (i)

$$S_+ = (wx + A)_{x=0} = A$$

$$= \frac{M_C - M_B}{b} - \tfrac{1}{2}wb. \qquad \text{[cf. (iv.)]}$$

To find S_- let us imagine the beam turned round about a vertical

axis through B, i.e. P is now on the positive side of B. Then the shearing force S at P is given by

$$S = \frac{M_A - M_B}{a} - \tfrac{1}{2}wa.$$

But according to our sign convention $S = -S_-$, so that

$$S_- = \tfrac{1}{2}wa - \frac{M_A - M_B}{a}.$$

$$\therefore\ R = S_- - S_+ = \tfrac{1}{2}w(a + b) + M_B\left(\frac{1}{a} + \frac{1}{b}\right) - \frac{M_A}{a} - \frac{M_C}{b}.$$

EXAMPLES VIII

8·01. A hollow metal tube 1·20 cm. in internal and 1·60 cm. in external diameter, whose length is 100 cm., is supported at the ends and loaded with 10·0 kg. in the middle. If the sag at the middle is 0·496 cm., calculate a value for Young's modulus for the material of which the tube is made. [$1{\cdot}87 \times 10^{12}$ dyne.cm.$^{-2}$]

8·02. Derive an expression for the bending moment of a beam. A uniform beam of weight W and of length l is clamped horizontally at one end. Find an expression for the depression at the free end.

In the case of a similar beam, but of negligible weight, what weight must be placed at the free end to produce the same depression? (G) [$\frac{3}{8}$W]

8·03. It is desired to cut a rectangular beam from a cylindrical tree whose diameter is $2a$. The beam is to carry a central load and its own weight may be neglected. Show that the maximum stress in the beam will be a minimum if the width is $2a \div \sqrt{3}$.

8·04. Describe and explain how you would determine Young's modulus for a piece of glass about 1 metre long and of rectangular cross-section about 2 cm. by 0·2 cm.

Derive the formula required in the calculation. (G)

8·05. Derive the expression for the bending moment of a beam bent into the form of an arc of radius R.

A light rod is held horizontally at one end and carries a load at the other. It is of circular section but of varying radius. Find an expression for the stress at the surface of the rod in terms of the load, radius, r, of the section and distance, d, from the loaded end.

Show that this surface stress is constant all along the rod provided that r is proportional to $d^{\frac{1}{3}}$, and discuss a possible application of this result. (G)

8·06. Show that at any cross-section of a bent beam the stress has a value $p = \frac{M}{I}z$, where z is the distance of the small element considered from the neutral axis.

For a rod of circular section show that at a given section of radius a, the maximum longitudinal stress is

$$p_{\text{max.}} = \frac{4M}{\pi a^3}.$$

If the cross-section is in the form of a ring, outer and inner diameters D and d respectively, show that the maximum stress is

$$\frac{32\text{MD}}{\pi(\text{D}^4 - d^4)}.$$

8·07. A beam 20 ft. long is supported horizontally at distances 5 ft. from each end and carries two loads each of mass 1 ton at distance 4 ft. and 12 ft. from the left-hand end. Assuming

$$\text{E} = 12 \times 10^3 \text{ ton.-wt.in.}^{-2} \text{ and I} = 200 \text{ in.}^4,$$

calculate the deflexions at the ends of the beam and at its centre; also the slope of the beam at the supports.

8·08. A horizontal beam 10 ft. long, 1 ft. deep and 4 in. wide rests on supports at its ends and carries a mass at its centre. Young's modulus for the material of the beam may be taken as

$$2 \times 10^6 \text{ lb.-wt.in.}^{-2}.$$

If the maximum deflexion of the beam is $\frac{1}{3}$ in. find a value for the maximum longitudinal stress within the beam. [Neglect the weight of the beam.] [55·6 lb.-wt.in.$^{-2}$]

8·09. Explain the term ***second areal moment.***

A horizontal cantilever has a load W applied at its free end. Assuming the weight of the beam is negligible, obtain an expression for the depression of its free end.

If the cross-section of the beam is rectangular, with sides of length a and b, and if the maximum depressions of the end of the beam for a given load are y_a and y_b respectively, when a and b are vertical, show that

$$\frac{y_a}{y_b} = \frac{b^2}{a^2}.$$

Calculate y_a and y_b for a bar 50 cm. long loaded with a mass of 500 gm., when $a = 2·0$ cm. and $b = 0·50$ cm. and Young's modulus for the material is 21×10^{11} dyne.cm.$^{-2}$. [0·029 cm., 0·47 cm.]

8·10. A vertical tube made of mild steel, for which Young's modulus may be taken as 2×10^{12} dyne.cm.$^{-2}$, has an external diameter of 20 cm. and its walls are 2·0 cm. thick. Its height above ground is 4·0 metres and it is subjected at the upper end to a horizontal pull of 500 kg.-wt. Calculate values for the maximum longitudinal stress at the ground level and the deflexion at the top. [1·36 $\times$ 10^3 kgm.-wt.cm.$^{-2}$, 3·54 cm.]

8·11. Define ***modulus of elasticity*** and give examples.

Obtain an expression for the maximum depression of a beam loaded at one end and fixed horizontally at the other.

A cylindrical rod of iron, AB, radius 0·50 cm. and length 25 cm., is clamped at A so that its axis is horizontal. Calculate the deflexions of the free end of the rod when a force of 3 kg.-wt. is applied horizontally (a) at B, (b) at the point midway between A and B.

[Assume that Young's modulus for iron is $21·0 \times 10^{11}$ dyne.cm.$^{-2}$.]

[(a) 0·467 cm., (b) 0·145 cm.]

8·12. A light beam is clamped horizontally at one end. Derive an expression for the lowering of its mid-point, when it is loaded at the other end. What would be the lowering if the same beam were supported horizontally on knife-edges at its ends and the same load suspended at its middle?

8·13. If a light beam of uniform section deflects 1·0 cm. in a span of

100 cm. under a central load, what will be the slope of the beam at either end? [$3{\cdot}0 \times 10^{-2}$ radian.]

8·14. A light uniform rectangular beam is rigidly fixed at one end in a horizontal position. Obtain an expression for the depression of the free end of the beam produced by a load hung there. What change in the load would be necessary to maintain the same depression of the free end if the linear dimensions of the beam were halved? [Halved]

8·15. Show that for a beam carrying a uniformly distributed load over its whole span, the span must not exceed 24 times the depth of the beam if the bending stress is not to exceed 12 ton.-wt.in.$^{-2}$ and the deflexion is limited to one two-hundredth of the span.

[Assume $E = 1{\cdot}2 \times 10^4$ ton.-wt.in.$^{-2}$.]

8·16. Derive an expression for the bending moment at any section of a uniform bar, originally straight, which is bent into the arc of a circle of large radius ρ, Young's modulus for the material being E.

A wooden bar 100 cm. long and 2·0 cm. × 2·0 cm. in section rests horizontally and symmetrically on two knife-edges 60 cm. apart. Find the elevation of the centre when (*a*) a mass of 2·0 kg. is hung from each end of the bar, (*b*) the extension per cm. length of the most strained filaments is $5{\cdot}0 \times 10^{-4}$. Assume that $E = 1{\cdot}3 \times 10^{11}$ dyne.cm.$^{-2}$.

[(*a*) 0·102 cm., (*b*) 0·0563 cm.]

8·17. A hollow metal tube 1·0 cm. in internal diameter and with walls 0·30 cm. thick is 100 cm. long. It is supported horizontally at the ends and carries a load of 15 kg. at its mid-point. The maximum sag is 0·415 cm. Calculate a value for Young's modulus for the material of the tube. [$2{\cdot}71 \times 10^{12}$ dyne.cm.$^{-2}$.]

8·18. For a light cantilever of length l and carrying a load of weight W at its free end, show that the deflexion at a distance x from the fixed end is

$$y = \frac{W}{EI}\left(\frac{lx^2}{2} - \frac{x^3}{6}\right).$$

What is the curvature of the beam at $x = 0$, $x = \frac{1}{2}l$ and $x = l$?

$$\left[\frac{Wl}{EI},\quad \frac{1}{2}\frac{Wl}{EI},\quad 0\right]$$

8·19. A cantilever 12 ft. long carries a uniformly distributed load of 1 ton per foot over the centre 6 ft. If $E = 12 \times 10^3$ ton-wt.in.$^{-2}$ and $I = 800$ in.^{4}, determine the slope at the free end.

8·20. A weight W suspended from the free end of a light cantilever of length L causes it to bend slightly, so that the depression at a point P, distant $pL(p < 1)$ from the fixed end is y. Calculate the depression which would have resulted at the free end if W had been suspended from P.

Discuss the effect, if any, on the result of the calculation if the cantilever had a weight comparable with W. (G)

[No change.]

8·21. The cross-section of a 'light' cantilever is a circle of radius a. The lever is 5 ft. long and carries a load of mass 4000 lb. The maximum longitudinal stress is 10,000 lb.-wt.in.$^{-2}$. If Young's modulus for the material of the beam is $3{\cdot}0 \times 10^7$ lb.-wt.in.$^{-2}$ find values for a and the deflexion at the free end. [3·13 in., 0·128 in.]

8·22. A cylindrical 'light' cantilever is 40 in. long and 4·0 in. in radius. Young's modulus for the material of the lever may be taken as $2{\cdot}0 \times 10^7$ lb.-wt.in.$^{-2}$. If the load at the free end has a mass of 1·0 ton, find the deflexion at this end and the maximum longitudinal stress in the cantilever. [0·012 in., 0·80 ton-wt.in.$^{-2}$]

8·23. A light rectangular bar is clamped horizontally at one end and a weight is attached to the other end. Derive expressions for (*a*) the vertical depression of the free end of the bar, and (*b*) the period of the oscillations which occur when the weight is slightly depressed and then released.

8·24. A uniform plank of weight W and length $2l$ is supported at its ends and lies in a horizontal plane. It is loaded (*a*) with a weight 2W at a distance $\frac{1}{2}l$ from one end and (*b*) with two equal weights W at distances $\frac{1}{2}l$ from the two ends.

Draw graphs to show how the bending moment varies along the plank in each instance. Find also the maximum values.

8·25. A horizontal beam rigidly fixed at one end is of weight w per unit length. The vertical shearing force at any point at a distance x from a convenient origin is S and bending moment is M. Show that

$$\frac{dS}{dx} = w, \quad \frac{dM}{dx} = F.$$

The bending being assumed to be small, show that

$$EI\frac{d^4y}{dx^4} = w,$$

E denoting Young's modulus of the material and I the second areal moment of the cross-section.

A uniform column stands vertically with one end on the ground, and a weight W rests on the upper end. Assuming again slight bending determine the equation of the axis of the column.

Find the ratio of the length to the diameter in the case of a cylindrical column just able to remain in the form of a single arc under a pressure of 15 ton.-wt.in.$^{-2}$, the value of Young's modulus being

$$3{\cdot}0 \times 10^7 \text{ lb.-wt.in.}^{-2}.$$

8·26. A uniform beam of span l is fixed horizontally at each end. Two equal loads, W, are placed at equal distances h from the ends of the beam. Neglecting the weight of the beam, prove that the greatest deflexion of the beam is $\frac{Wh^2}{24EI}(3l - 4h)$, and that the greatest bending moment is $\frac{Wh^2}{l}$.

8·27. A continuous girder, designed for crossing two equal spans, has a uniform cross-section and mass m per unit length. The girder is launched across the spans and when its centre is almost over the central pier, the advancing end hangs downwards. If the length of each span is a show that the depression of the free end is $\frac{mga^4}{4EI}$, where the symbols have their usual meanings.

8·28. Derive, from first principles, an expression for the depression at the mid-point of a beam of uniform mass per unit length loaded at the centre and supported at the ends on knife-edges at the same level.

In the case of a beam of negligible mass, find the period of small oscillations when the load is slightly displaced vertically from the position of equilibrium and then released. (S)

8·29. Derive an expression for the couple required to bend a beam in terms of the local radius of curvature and other factors involved.

Two cantilevers have the same cross-sectional area, but the cross-section of one is square and that of the other is circular. Both cantilevers are made from the same material and their free ends, loaded by equal weights, show equal displacements. Calculate the ratio of the length of the 'square' cantilever to that of the 'circular' cantilever. (S)

8·30. A light uniform beam of length $3l$ is freely supported at both ends on knife-edges. Calculate the depression of the beam at the point of application of the load (*a*) when a load W is applied centrally, (*b*) when it is applied at a point distant l from one end. (S)

8·31. A light uniform cylindrical rod, 100 cm. long and 1 cm. diameter, is supported symmetrically on two knife-edges, 50 cm. apart, in a horizontal plane. Equal masses of 1 kg. are suspended from the centre and ends of the rod. Calculate (*a*) the depressions at the centre point and at the ends of the rod, (*b*) the strain energy stored in the rod. Young's modulus of the material of the rod is 10×10^{10} dyne.cm.$^{-2}$. (S)

8·32. The extension of a light closely coiled helical spring caused by attaching a 50-gm. mass to its lower end is 3 cm. Obtain an expression for the period of the small vertical oscillations caused by pulling the loaded spring down a short distance and then releasing it. [0·348 sec.]

8·33. A 200-gm. weight when supported by a light closely coiled helical spring makes 50 complete vertical oscillations in 40 seconds. Find the extension of the spring due to increasing the load by 20 gm. Give the theory. [1·59 cm.]

8·34. A hollow mild steel pillar 25 ft. long is fixed at both ends and has an external diameter of 6 in. Obtain a value for the thickness of the metal if the pillar is to carry a load of mass 50 tons, allowing a factor of safety of 6. [Assume Young's modulus for steel to be $3{\cdot}0 \times 10^7$ lb.-wt.-in.$^{-2}$ and take $\pi^2 = 10$.] [0·98 in.]

8·35. A spring balance is constructed from a quartz fibre in the form of a closely coiled helical spring 0·88 cm. in diameter. If the diameter of the fibre is $1{\cdot}65 \times 10^{-3}$ cm. and the sensitivity of the balance $0{\cdot}33 \times 10^5$ cm.gm.-wt.$^{-1}$, what length of fibre is employed, if the modulus of rigidity of fused quartz may be taken as $3{\cdot}0 \times 10^{11}$ dyne.cm.$^{-2}$? How would you determine this modulus experimentally? [47·8 cm.]

CHAPTER IX

THE COMPRESSIBILITY OF LIQUIDS, SOLIDS AND GASES

Early work on the compressibility of matter in the liquid state.—In an elementary account of hydrostatics it is always assumed that liquids are incompressible. The fact that water was not a liquid with zero compressibility was established by CANTON† in 1762. BACON, at an earlier date, had subjected water, completely filling a lead sphere, to pressures greater than atmospheric, but the experiments were frustrated by the fact that the sphere sprang a leak, or else water escaped through the walls of the vessel which were porous. Canton used a glass vessel, containing mercury, shaped like a thermometer, but with an open capillary tube. The level of the mercury in the stem of the instrument at a definite temperature was noted. The vessel was then heated until the mercury just filled it; the open end of the capillary was then sealed and the instrument allowed to cool to its former temperature. It was found that the mercury stood at a higher level in the capillary tube than formerly. To account for this it might be assumed:

(*a*) that the mercury had previously been compressed by the external air, or

(*b*) that the vessel was reduced in size when the pressure inside was less than atmospheric.

The experiment was then repeated with water in the same vessel; the change in level of the water was greater than in the case of mercury. It was therefore established that water was a compressible substance.

OERSTED,‡ in 1822, made a further step in the study of the compressibility of liquids; one form of his apparatus, an example of an instrument known as a piezometer, is shown in Fig. 9·01. It consisted of a glass vessel A, provided with a capillary tube B dipping into mercury contained in a dish C. D was a glass tube, closed at the top, whose lower end dipped into the same dish of mercury. It contained air. T was a mercury-in-glass thermometer. The whole was placed in a strong glass cylinder G furnished with

† Canton, *Phil. Trans. Roy. Soc.*, 1762–4.
‡ Oersted, *Pogg. Ann.*, 9, 603, 1827.

brass end-pieces, the joints between them and the glass being made watertight with some form of wax. G was filled with water. S was a screw plunger by means of which the pressure inside the apparatus could be increased. From the change in volume of the air in the tube D, the change in pressure inside the apparatus could be calculated.

The theory of this experiment is as follows. Let V be the volume of A up to the zero mark 0 on its stem. Let v be the volume per division on the stem B. Suppose that when the pressure is P_1, the mercury stands in B at m; when it is P_2 at n. Then $(m - n)v$ is ***apparently*** the reduction in volume of a volume $(V + mv)$ of liquid when the pressure changes by an amount $(P_2 - P_1)$. The apparent compressibility of the liquid is therefore given by

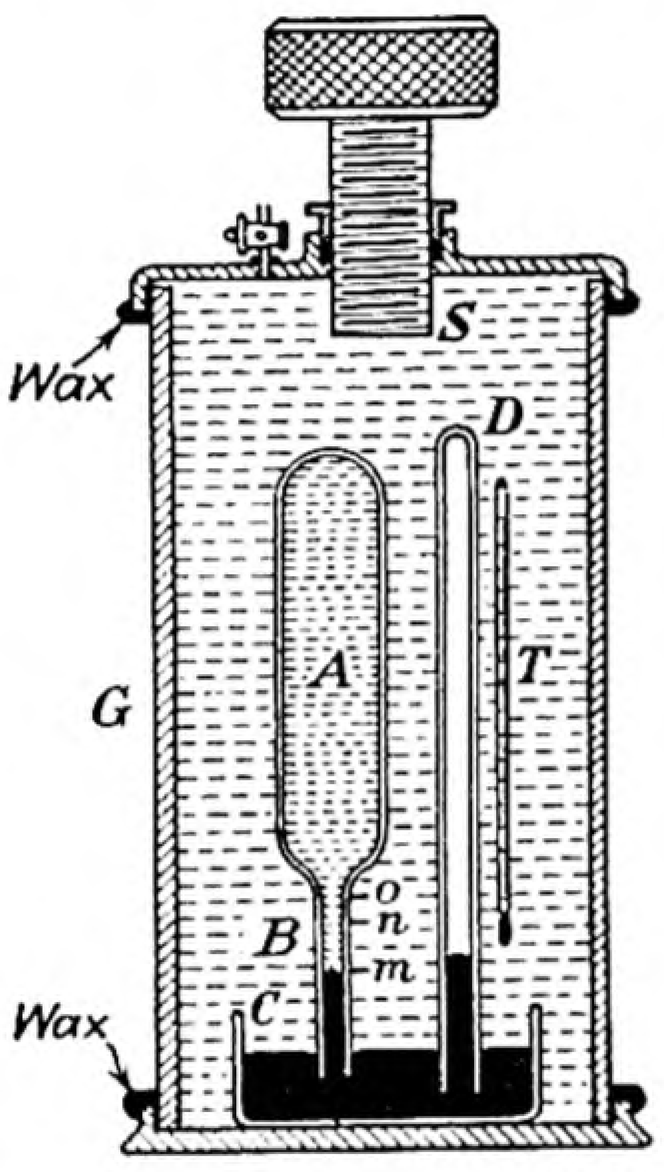


FIG. 9·01.—Oersted's apparatus for studying the compressibility of a liquid.

$$\frac{\text{apparent diminution in volume}}{\text{(original volume)(change in pressure)}}$$

$$= \frac{(m - n)v}{(V + mv)(P_2 - P_1)}.$$

REGNAULT, in 1847, was the first person to obtain accurate values for the compressibilities of several common liquids, including mercury. In interpreting his observations he made use of the theoretical investigations of Lamé [later published in *Leçons sur l'elasticité des corps solides*—cf. p. 189, Paris, 1867]. Before proceeding further it is necessary for us to consider an elementary account of Lamé's work in so far as it is relevant to our present needs.

On the changes in volume occurring when a thick uniform cylindrical shell of isotropic material is subjected to changes in pressure.—Consider a long cylindrical tube of circular cross-section with flat ends subjected to the following increases in pressure; externally P_0 and internally P. Let a and b be the internal and external radii of a cross-section of the tube—cf. Fig. 9·02(a). After the pressures have been applied let a point at distance r from the axis suffer a small displacement ξ. A point originally at distance $r + \delta r$ will be displaced $\xi + \dfrac{\partial \xi}{\partial r}\cdot\delta r$.

$\therefore$ Radial strain

$$= \frac{\text{increase in length of element originally } \delta r}{\text{original length } \delta r} = \frac{\partial \xi}{\partial r},$$

$$= \epsilon_1 \text{ (say)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

Moreover, the circle of radius r originally and circumference $2\pi r$,

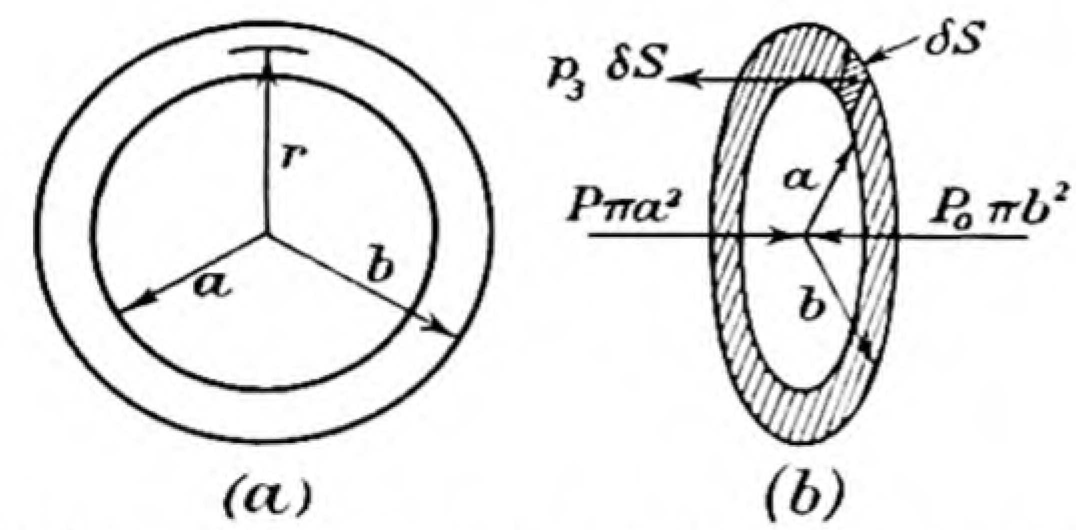


FIG. 9·02.—A thick uniform cylindrical shell subject to internal excess pressure.

has a circumference $2\pi(r + \xi)$ when the pressures are applied. The circumferential strain, ϵ_2, is therefore given by

$$\epsilon_2 = \xi r^{-1} \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

Let ϵ_3 be the longitudinal strain; it is necessarily normal to both ϵ_1 and ϵ_2. Let p_1, p_2 and p_3 be the corresponding stresses. Then, cf. pp. 307–309,

$$p_1 = (\beta + \tfrac{4}{3}n)\epsilon_1 + (\beta - \tfrac{2}{3}n)(\epsilon_2 + \epsilon_3) \quad . \quad . \quad \text{(iii)}$$

To proceed further we have to assume that $\xi = Ar + Br^{-1}$ where A and B are small constants. Lamé established this theoretically in 1867. Then

$$\epsilon_1 = A - Br^{-2}, \qquad \text{and} \qquad \epsilon_2 = A + Br^{-2}.$$

$$\therefore\ p_1 = (\beta + \tfrac{4}{3}n)(A - Br^{-2}) + (\beta - \tfrac{2}{3}n)(A + Br^{-2} + \epsilon_3)$$

$$= 2\beta A + \frac{2n}{3}(A - 3Br^{-2}) + (\beta - \tfrac{2}{3}n)\epsilon_3 \quad . \quad . \quad \text{(iv)}$$

Now when $r = a$, $p_1 = -P$; when $r = b$, $p_1 = -P_0$; the negative signs appear since a stress is considered positive when the material is in a state of tension.

$$\therefore\ -P = 2\beta A + \frac{2n}{3}\left(A - 3\frac{B}{a^2}\right) + (\beta - \tfrac{2}{3}n)\epsilon_3 \quad . \quad \text{(v)}$$

and
$$-P_0 = 2\beta A + \frac{2n}{3}\left(A - 3\frac{B}{b^2}\right) + (\beta - \tfrac{2}{3}n)\epsilon_3 \quad . \quad \text{(vi)}$$

The total force tending to stretch the cylinder is $\pi[a^2P - bP_0]$, cf. Fig. 9·02(*b*); the longitudinal stress p_3 is therefore given by

$$p_3 = \frac{a^2P - b^2P_0}{b^2 - a^2}.$$

But
$$p_3 = (\beta + \tfrac{4}{3}n)\epsilon_3 + (\beta - \tfrac{2}{3}n)(\epsilon_1 + \epsilon_2), \quad \text{[cf. p. 309]}$$
$$= (\beta + \tfrac{4}{3}n)\epsilon_3 + (\beta - \tfrac{2}{3}n)2A. \quad \text{(vii)}$$

From equations (v), (vi) and (vii) we obtain

$$B = \frac{1}{2n}\cdot\frac{a^2b^2}{b^2 - a^2}(P - P_0),$$

$$A = \epsilon_3 = \frac{1}{3\beta}\frac{Pa^2 - P_0b^2}{b^2 - a^2}.$$

When the tube is strained the internal volume will be

$$\pi[(r + \xi)_{r=a}]^2 l(1 + \epsilon_3) = \pi\left(a + Aa + \frac{B}{a}\right)^2 l(1 + \epsilon_3),$$

where l is the initial length of the tube. If V is the internal volume originally and it becomes $V + \varDelta V$ when strained, we have

$$\varDelta V = \pi a^2 l\left[\frac{Pa^2 - P_0b^2}{(b^2 - a^2)\beta} + \frac{b^2}{b^2 - a^2}\cdot\frac{P - P_0}{n}\right],$$

if powers of A, B and ε_3 higher than the first are neglected.
Similarly $\varDelta V_0$, the change in external volume, is given by

$$\varDelta V_0 = \pi b^2 l\left[\frac{Pa^2 - P_0b^2}{(b^2 - a^2)\beta} + \frac{a^2}{b^2 - a^2}\cdot\frac{P - P_0}{n}\right].$$

Practical methods of determining the compressibility of a liquid.—Two special instances, in which use is made of the general result obtained above, arise in experimental determinations of the compressibility of a given liquid.

(*a*) In the first, the pressure is increased so that it is the same inside and outside the cylinder, i.e. $P = P_0$. Then

$$\varDelta V = -\pi a^2 l.\frac{P}{\beta_1} = -\pi a^2 l P\kappa_1,$$

where the suffix (1) is used to show that the quantity thus subscribed refers to the material of the cylinder. The above equation shows that the internal volume of the cylinder diminishes by an amount which is independent of the thickness of the walls of the cylinder.

Early workers in this subject were mistaken, therefore, when they supposed that if the walls of the containing vessel were sufficiently thin, there would be no appreciable change in the internal volume of a cylinder subjected simultaneously to the same increase in pressure both inside and outside.

Suppose now that the above cylinder contains a liquid of bulk modulus β and compressibility κ. Then the true diminution in the

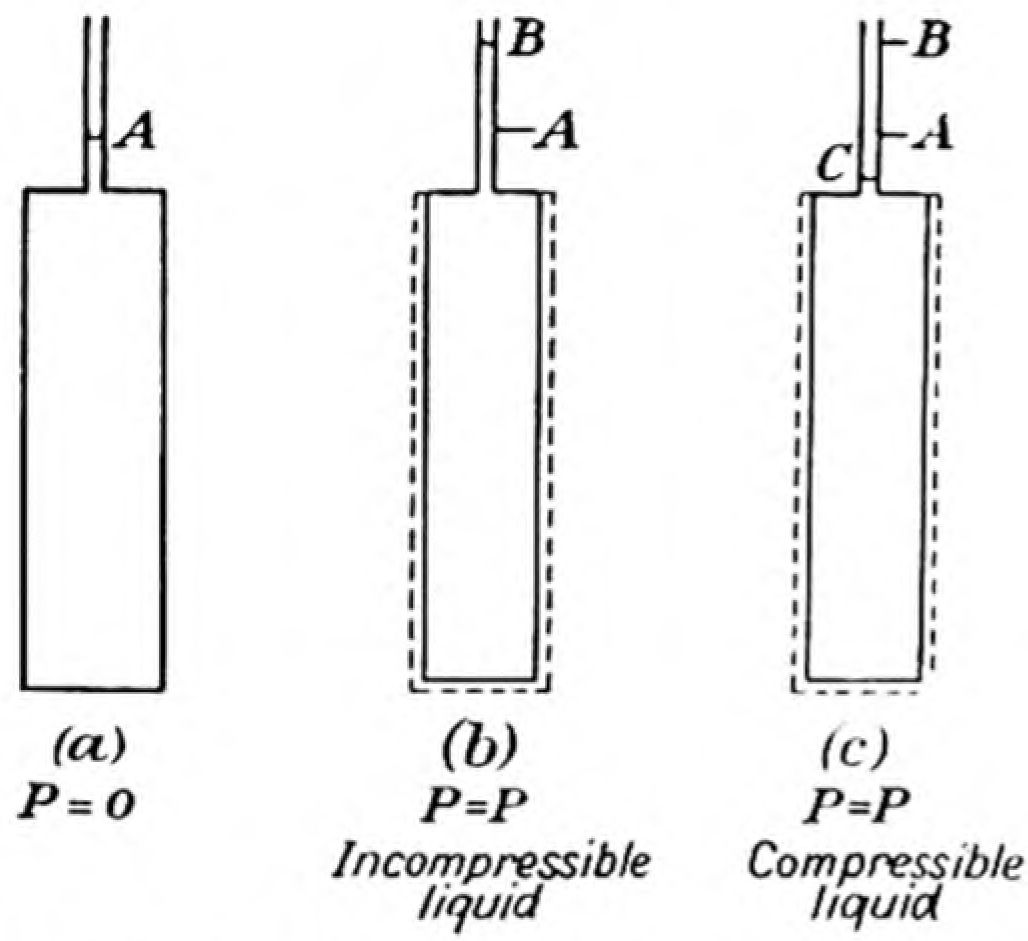


FIG. 9·03.—The apparent compressibility of a liquid.

volume $\pi a^2 l$ of this liquid when subjected to a pressure increase P is $\pi a^2 l \kappa \mathrm{P}$. If a capillary tube is attached to the cylinder and the change of volume observed when the liquid and the containing vessel are subjected to an increase in pressure P both inside and outside, we have to consider how this observed change is related to the change in the internal volume of the cylinder and the true change in the volume of the liquid.

Let Fig. 9·03(a) represent a cylinder with a capillary tube attached, filled with an incompressible liquid, the pressure being zero inside and outside. Let A be the position of the liquid surface in the capillary tube. When the pressure is raised to P everywhere, the liquid will rise to the point B—cf. Fig. 9·03(b). Then

$$\text{Volume BA} = \pi a^2 l \mathrm{P} \kappa_1.$$

Now suppose that the cylinder contains a liquid of compressibility κ, and that this liquid fills the cylinder to the mark A when the pressure is everywhere zero. When the pressure becomes P both outside and inside the apparatus, let the liquid surface be at C in the capillary tube—cf. Fig. 9·03(c). Then

$$\text{Volume BC} = \pi a^2 l \mathrm{P} \kappa.$$

Now the volume AC is the observed change in volume. Hence

$$\text{Volume AC} = \text{Volume BC} - \text{Volume AB} = \pi a^2 l P(\kappa - \kappa_1).$$

Thus $(\kappa - \kappa_1)$ is the quantity which is actually determined by an experiment carried out on these lines, so that it is necessary to find the compressibility of the material of the piezometer before that of the liquid can be obtained.

(b) In the second instance, the pressure is only increased inside the piezometer, when the apparent change in volume of the liquid is the sum of the actual change in the volume of the liquid and that of the interior of the vessel. JAMIN used this method but it will not be discussed here.

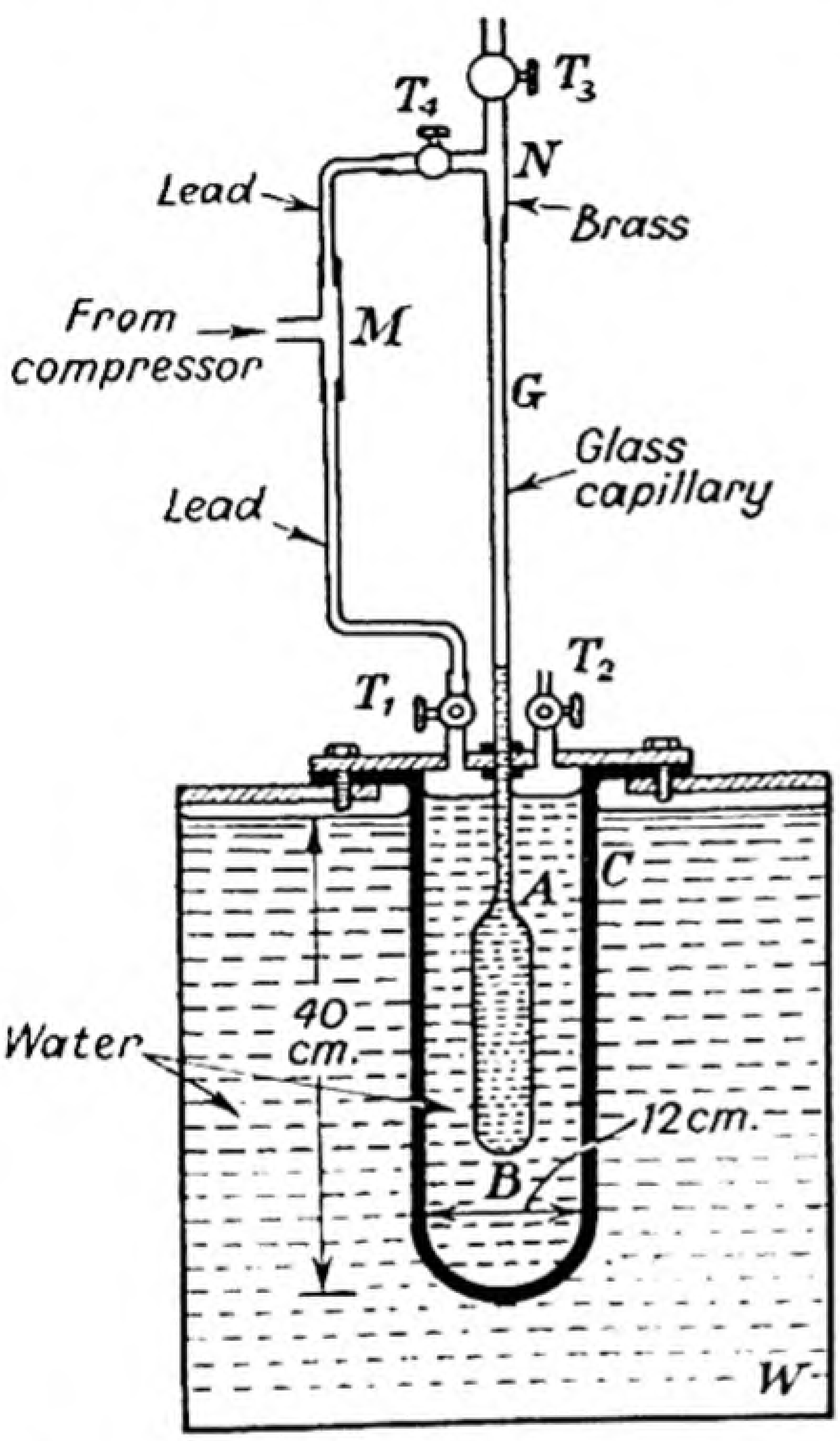


FIG. 9·04.—Regnault's apparatus for measuring the compressibility of a liquid.

Regnault's experiments on the compressibility of liquids.—The apparatus used by REGNAULT† to investigate the compressibilities of different liquids is shown diagrammatically in Fig. 9·04. The liquid under investigation was contained in a glass vessel AB, a cylindrical bulb with hemispherical ends. To this there was attached a graduated glass capillary tube AG, the volume of one division on this tube being equal to that of 0·010271 gm. of mercury at room temperature. AB was about 23 cm. long and 2·4 cm. in diameter. It was immersed in water contained in a copper vessel C, 40 cm. long, and 12 cm. in diameter, and had walls 2 mm. thick. This vessel was provided with a flange to which a lid was fixed by solder. The whole was supported by the lid of a large outer vessel, W, containing water; it was made large in order to reduce the rate at which the temperature of the whole changed the course of an experiment. The capillary tube of the piezometer was fixed with wax into the lid of the copper vessel; its upper end passed into one arm of a three-way brass tube N provided with stop-cocks T_3 and T_4. M was another such tube. T_1 and T_2 were

† *Mem. de l'Acad. Fran.*, **21**, 429, 1847.

stop-cocks fitted to the lid of the copper vessel. These were connected to the rest of the apparatus by means of lead tubes, as indicated, and M was placed in communication with an air compressor.

The following operations were then carried out, in each instance the position of the liquid surface in the graduated stem of the piezometer being observed.

(*a*) The stop-cocks T_1 and T_4 were closed, while T_2 and T_3 were opened. The pressure outside and inside the piezometer was then atmospheric—say p.

(*b*) T_2 was then closed and T_1 opened so that the pressure was increased to $(P + p)$, say, outside, while inside it was still p. The liquid rose in the calibrated capillary tube AG; the change in volume, deduced from this rise, is the change in the internal volume of the piezometer due to the change in the external pressure.

(*c*) T_2 was kept closed and T_1 open; then T_3 was shut and T_4 opened. The pressure was then $(P + p)$ both inside and outside the piezometer.

(*d*) T_1 was shut and T_2 opened so that the pressure external to the piezometer was p, while inside it was $(P + p)$. The liquid descended in the capillary tube.

(*e*) T_4 was then shut and T_3 opened so that the pressure was again atmospheric both inside and outside the instrument. The liquid surface should therefore be at the same position in the stem, if the temperature of the apparatus had not changed appreciably.

For simplicity, it will now be assumed that the piezometer was cylindrical in form, so that we may use the results of the analysis given on pp. 416–8. [Regnault used a still more complicated formula which took account of the fact that the ends of the piezometer were hemispherical.] Let ω_1, ω_2 and ω_3 be the apparent increases in internal volume of the cylinder corresponding to the operations (*b*), (*c*) and (*d*). These changes in volume may be written down from the general formula by putting $p = 0$. Then

$$\omega_1 = \pi a^2 l\left[-\frac{Pb^2}{b^2 - a^2}\cdot\frac{1}{\beta_1} - \frac{Pb^2}{b^2 - a^2}\cdot\frac{1}{n_1}\right],$$

$$= -\frac{\pi a^2 b^2 l P}{b^2 - a^2}\left[\frac{1}{\beta_1} + \frac{1}{n_1}\right].$$

Regnault used this result to calculate β_1, the bulk modulus of the material (glass) of the piezometer. He assumed that σ, Poisson's ratio, was a universal constant equal to 0·25, and used this value and the equations

$$(1 + \sigma) = \frac{E_1}{2n_1}, \qquad \text{and} \qquad (1 - 2\sigma) = \frac{E_1}{3\beta_1},$$

where E_1 is Young's modulus for glass, in order to calculate n_1, the modulus of rigidity for glass. Thus $n_1 = 0{\cdot}6\beta_1$. It is now known that σ is not a universal constant but, for soda-glass, varies from 0·20 to 0·27, so that in this respect Regnault's work is open to criticism.

Also

$$\omega_2 = \pi a^2 l\mathrm{P}\left[\frac{1}{\beta} - \frac{1}{\beta_1}\right],$$

so that β, and hence κ, may be derived.

Again

$$\omega_3 = \pi b^2 l\mathrm{P}\left[\frac{a^2}{b^2 - a^2}.\frac{1}{\beta_1} + \frac{a^2}{b^2 - a^2}.\frac{1}{n_1}\right] + \pi a^2 l\mathrm{P}.\frac{1}{\beta},$$

$$= \frac{\pi a^2 l}{b^2 - a^2}.\mathrm{P}\left[\frac{b^2 - a^2}{\beta} + \frac{a^2}{\beta_1} + \frac{b^2}{n_1}\right].$$

$$\therefore\ \omega_1 + \omega_3 = \pi a^2 l\mathrm{P}.\left[\frac{1}{\beta} - \frac{1}{\beta_1}\right].$$

$$= \omega_2.$$

Regnault used this relation to test the validity of the theoretical investigation of which he had made use. Such a check was very desirable, since in developing the theory it had been assumed that the material used in making the container for the liquid was isotropic and that the walls were uniform in thickness. It is very difficult to realize such conditions in practice.

Regnault also used spherical containers when the necessary formulae were simplified. One of these containers was made of copper, another of brass, the two halves in each instance being silver soldered. The choice of spherical containers was most appropriate for it was more easy to realize with them the conditions laid down in the analysis by Lamé. The same capillary tube was used in all the experiments made by Regnault.

Once the compressibility of a liquid has been determined accurately, an excellent method for finding that of any other liquid which does not react with the standard liquid, is to fill the piezometer with the standard liquid, say mercury, and determine the apparent change in the volume of the mercury when the inside and the outside of the instrument are exposed to the same increase in pressure. The instrument is then filled with the liquid under investigation and the apparent change in volume for the same increase in pressure both internally and externally found. Then

$$\Omega_1 = \text{observed change in volume in the first instance}$$
$$= \pi a^2 l\mathrm{P}[\kappa_{\mathrm{Hg}} - \kappa_{\mathrm{glass}}],$$

and $$\Omega_2 = \pi a^2 l P[\kappa_{\text{liq.}} - \kappa_{\text{glass}}].$$

$$\therefore\ \Omega_2 - \Omega_1 = \pi a^2 l P[\kappa_{\text{liq.}} - \kappa_{\text{Hg}}].$$

Amagat and the compressibility of liquids.—In 1877 Amagat† gave an account of an extensive study he had made concerning the compressibility of several liquids. The apparatus was an improved form of that due to Oersted.‡ The liquid under investigation was contained in a glass piezometer, A, Fig. 9·05, consisting of a cylindrical bulb and a graduated capillary tube which passed through an opening into an iron trough B. The piezometer was surrounded by a water bath, the temperature of which was controlled from underneath by a small flame. The bath consisted of a metal tank with glass plates both front and back. The pressure was given by an air manometer, M, maintained at a constant temperature by a stream of cold water flowing through a glass jacket, C, which surrounded it. This manometer was fitted into the iron trough as indicated. It was provided with a bulb at its lower end, of such volume relative to that of the upper part of the manometer, that when the pressure was nine atmospheres, the mercury was just about to enter the capillary tube, i.e. the manometer was sensitive even at the higher pressures. Pressure was applied from a Cailletet compressor. The necessary joints between different parts of the apparatus were covered with wax which was kept cold.

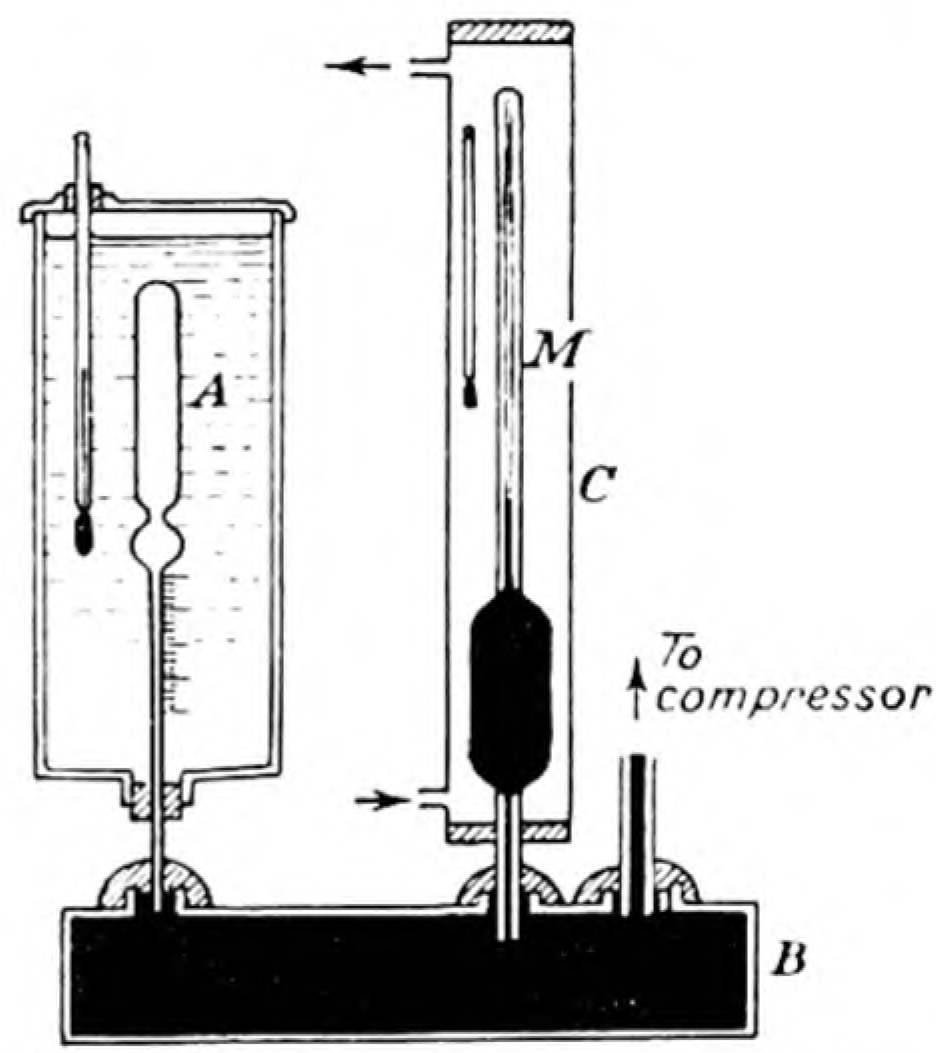


FIG. 9·05.—Amagat's apparatus for comparing the compressibilities of liquids (1877).

With this apparatus Amagat was able to investigate how the compressibility of a liquid varied over a range of temperature from 0° to 50° C. The following remarks will indicate why such an apparatus is particularly suitable for such an investigation.

Amagat was well acquainted with Regnault's work and realized the difficulty of obtaining an accurate measure of the compressibility of the material of the piezometer. He therefore used a liquid of known compressibility in order to find that of glass, and then used

† *Ann. Chim. Phys.*, **11**, 520, 1877.
‡ *I.P.*, p. 153.

this in determining the compressibility of another liquid and how it varied with temperature. [It does not appear that any correction was made for the variation with temperature of the compressibility of glass.] In all instances the liquid was freed from dissolved gases by prolonged boiling and the air manometer was cleaned and dried.

In later years [1882–9] Amagat made further investigations and made use of Jamin's method but, unlike Jamin, interpreted his observations correctly. He used, however, metal cylinders with flat ends, a glass capillary tube being attached by means of wax. In each experiment two cylinders of the same material, length and internal diameter, were used. The external diameters were different. In this way he was able to confirm several important deductions from Lamé's theory, one of which was that the change in internal volume for a given increase in the external pressure was independent of the thickness of the wall for cylinders having the same internal diameter and made from the same (isotropic) material. It is also interesting to note that these experiments were carried out at 4° C., the temperature at which water has a maximum density—this was the liquid used outside the piezometer. The reason for this choice of temperature was that the coefficient of increase in volume with temperature for water is a minimum at 4° C.

In 1893 Amagat† gave an account of further experiments on the compressibilities of liquids in which he increased the pressure to 3000 atmospheres. At that time it was technically impossible to construct an apparatus entirely of glass capable of withstanding high pressures and, simultaneously, permitting the mercury surface to be observed. Amagat therefore used a glass piezometer consisting of a cylindrical bulb A, Fig. 9·06, and a capillary tube B, below; these were housed in a steel container. The end of B dipped below some mercury, and as the pressure increased so the mercury rose in the capillary tube. To locate the position of the mercury at any stage of an experiment he coated the inside of the capillary tube with a substance which dissolved in mercury and in this way, after having taken the apparatus to pieces, he was able to obtain the desired information. This was a tedious procedure. Tait recommended the use of electrical contacts so that the position of the mercury could be determined during, and not after, the experiment. The piezometer was therefore provided with a series of platinum wires as shown, an electrical resistance joining each pair consisting of a coil of wire well insulated and wrapped round the capillary tube B. An insulated platinum wire passed through the steel jacket, through the bulb A, and made contact with the uppermost platinum sealed-in wire. As the mercury rose in B, so did the current in the

† *Ann. Chim. Phys.*, **29**, 68, 1893.

main circuit change by a finite amount as the mercury passed each contact in turn.

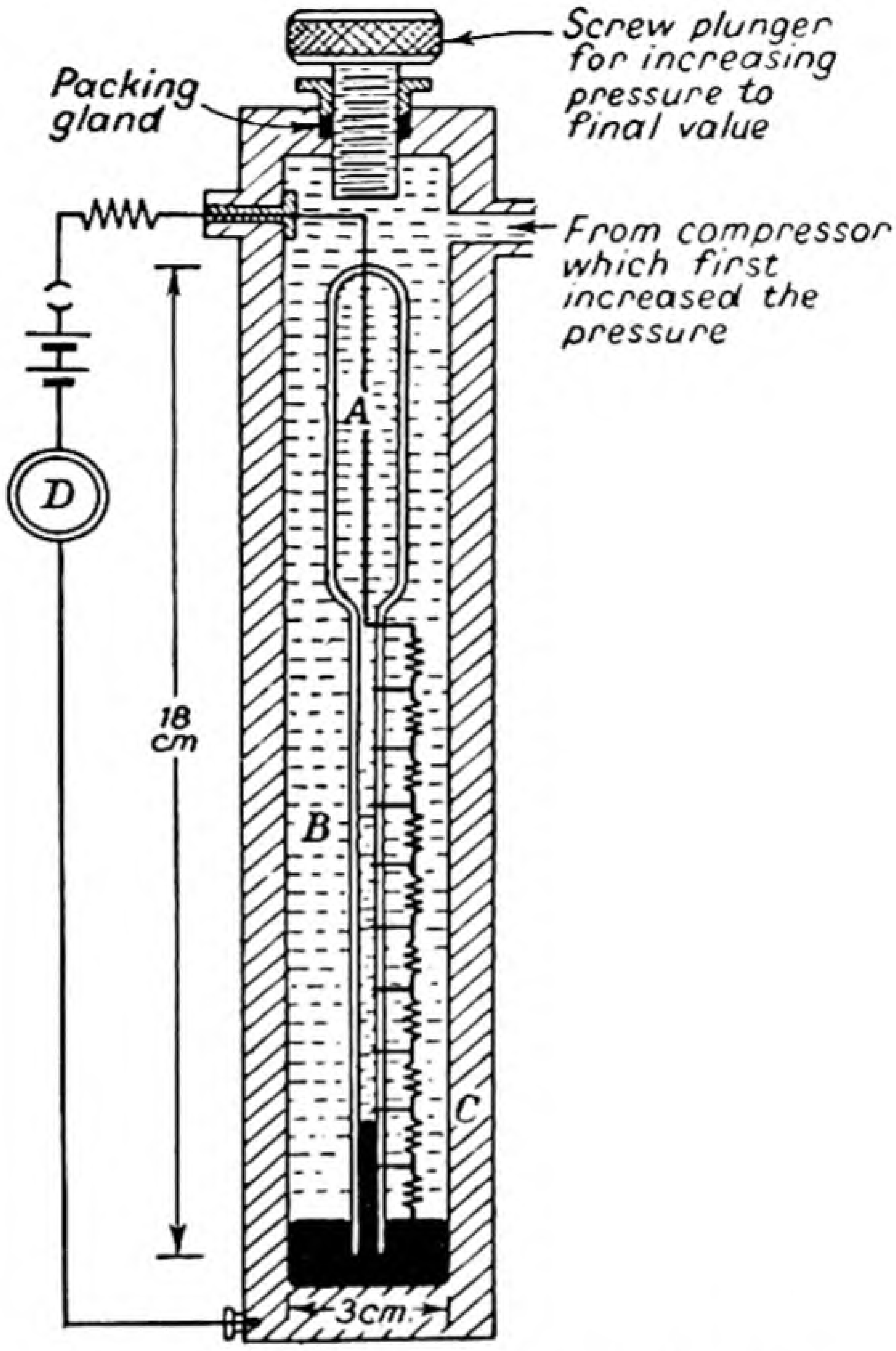


FIG. 9·06.—Amagat's second apparatus for investigating the compressibility of liquids at high pressures.

The work of Richards and Stull.†—The method developed by these investigators was such that the compressibility of solids, even if they existed only in irregular pieces, and of many liquids may be determined with accuracy. The essential feature of the method is the comparison of the compressibility of the substance to be tested with that of a standard liquid by noting the weighed quantities of mercury which must be added to the mercury in the apparatus in order to supply the volume lost under compression at successive pressures. Electrical contacts were used to indicate when the desired extent of compression had been obtained. Many errors were avoided by conducting a series of experiments.

† *Jour. Amer. Chem. Soc.*, **26**, 399, 1904. *Carnegie Inst., Washington, Publication*, No. 7, 1903; No. 76, 1907.

For solids, the apparatus shown in Fig. 9·07(*a*) was used. It consisted of a cylindrical bulb A into which could be fitted a well-ground stopper B provided with a capillary tube D, 1·5 mm. in diameter, and a small funnel E. The optimum diameter for the capillary tube was found to be 1·5 mm.; if larger, the apparatus was insensitive, and if smaller, drops of mercury tended to stick

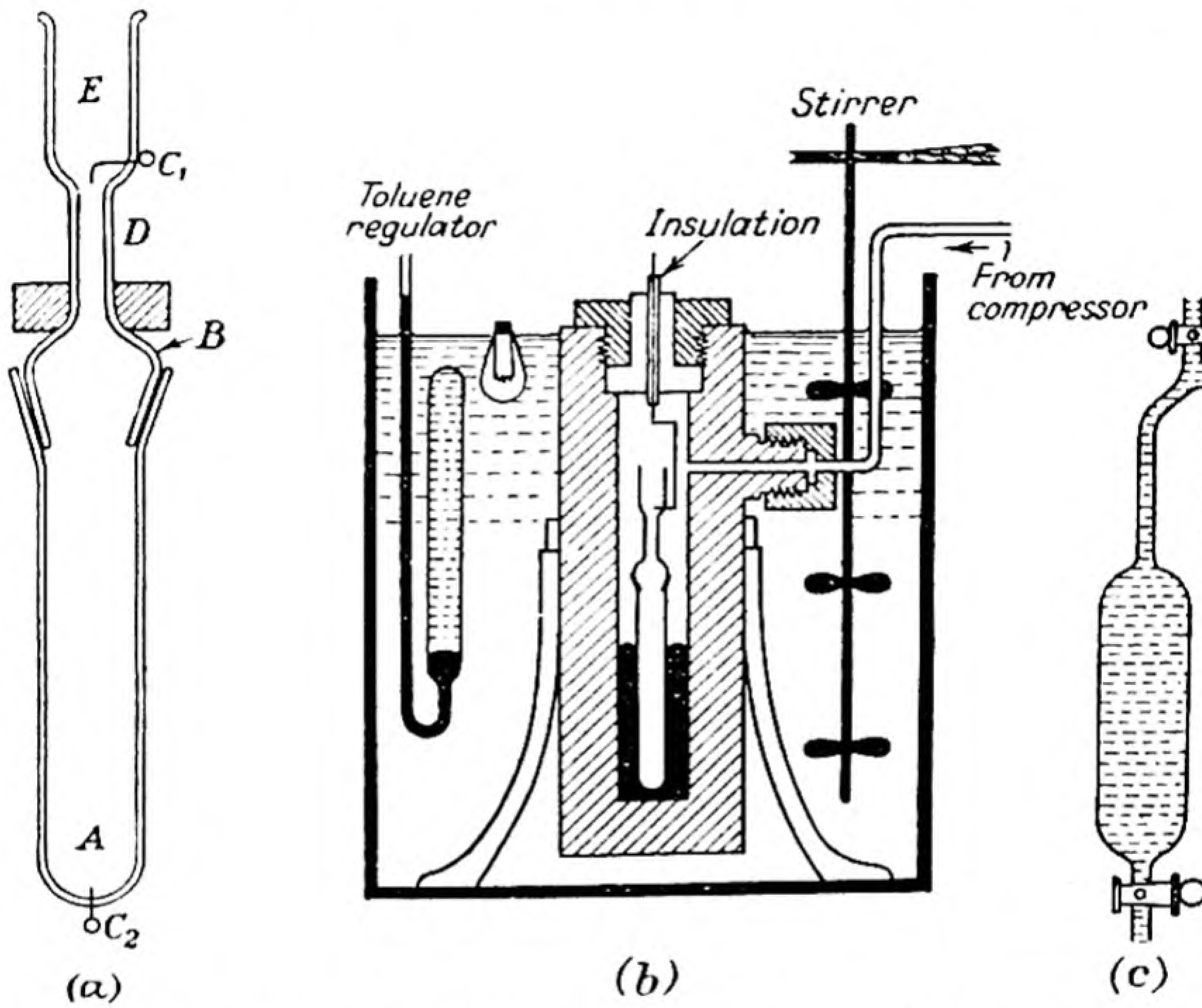


Fig. 9·07.—Richards and Stull: Apparatus for determining the compressibility of solids and liquids.

between the platinum wire and the walls of the tube. C_1 and C_2 were platinum contacts. The whole apparatus was first filled with mercury, a liquid whose compressibility was taken as known, and the pressure was increased to P_1 until contact was just made with C_1. To do this the piezometer was placed in the barrel of a Cailletet compressor, the pressure being read on a carefully calibrated hydraulic-dial gauge. A small weighed quantity of mercury was then added to that in the instrument and the pressure increased until contact between the platinum point C_1 and the mercury was just broken. Actually the pressure was increased beyond this stage and then reduced until contact C_1 was just established. In this way trouble due to adhesion between the two metals was avoided; also, to diminish the effects of sparking at C_1, the mercury was covered with pure water.

The procedure suggested above was repeated until the highest pressures attainable with the apparatus were reached. Observations were made:

(*a*) with the apparatus filled with mercury,

(*b*) with the solid inside, the space above the solid being filled with mercury.

To facilitate the insertion of the solid the apparatus was made in two parts which fitted together. The ground joint between the two portions was the source of serious trouble until the following procedure was adopted. The trouble arose from the fact that mercury leaked, so that the compressibility appeared correspondingly high and the values obtained were not consistent. The difficulty was obviated by wetting the surfaces with a minute drop of liquid (water) thus displacing all the air and preventing the ingress of mercury. The infinitesimal variations due to the compressibility of this practically constant drop of lubricating liquid were much too small to have any appreciable effect. The stopper was firmly tied in place by means of a piece of stout string which passed over a rubber shoulder.

In order to conduct away the heat of compression and to make electrical contact with the lower platinum wire, mercury was poured round the lower two-thirds of the piezometer which was situated in an iron vessel immersed in a thermostat whose temperature was kept constant to within 0·01 deg. C. The general arrangement is shown in Fig. 9·07(*b*).

For liquids which do not attack mercury, the piezometer took the form shown in Fig. 9·07(*c*). The two stop-cocks were introduced in order that the liquid under investigation might be introduced easily, but they are not really necessary, for the piezometer could be filled by a method of alternate heating and cooling.

Richards and Stull also investigated the compressibility of alkali and alkaline earth metals with a slight modification of the latter type of piezometer. Observations were made as follows:—

(*a*) with the piezometer filled with mercury,

(*b*) with mercury in the bend and oil (fractionated paraffin from kerosene) elsewhere,

(*c*) with mercury in the bend, solid in the vessel, and oil filling the remainder of the available space.

Theory of the above piezometer for solids.—Let m_1 be the mass of the mercury required to fill the instrument up to the contact C_1 when no solid is present, the pressure being P_1, and the temperature known and constant. Then, if ρ is the density of mercury at this temperature and pressure, the volume of the mercury is $\frac{m_1}{\rho}$. Let μ be the mass of the solid whose material has a density D at the

temperature of the experiment and at pressure P_1, and let M_1 be the mass of the mercury required to fill the instrument to the same fiducial mark. Then

$$\frac{m_1}{\rho} = \frac{M_1}{\rho} + \frac{\mu}{D}.$$

When the pressure is increased to P_2, let m_2 be the mass of mercury required to fill the piezometer to C_1 when no solid is present; let M_2 be the corresponding mass when the solid is present. Then

$$\frac{m_2}{(\rho + \delta\rho)} = \frac{M_2}{(\rho + \delta\rho)} + \frac{\mu}{(D + \delta D)},$$

since the densities of the mercury and the solid will have been increased by $\delta\rho$ and δD respectively. Let κ be the compressibility of the material of the solid. Then, by definition,

$$\begin{aligned}\kappa &= \frac{D}{\mu}\left[\frac{\dfrac{\mu}{D} - \dfrac{\mu}{D + \delta D}}{P_2 - P_1}\right],\\ &= \frac{D}{\mu(P_2 - P_1)}\left[\frac{m_1 - M_1}{\rho} - \frac{m_2 - M_2}{\rho + \delta\rho}\right],\\ &= \frac{D}{\mu(P_2 - P_1)}\left[\frac{m_1 - M_1}{\rho} - \frac{m_2 - M_2}{\rho\left(1 + \dfrac{\delta\rho}{\rho}\right)}\right].\end{aligned}$$

Let κ_m be the (known) compressibility of mercury. Then

$$\kappa_m = -\frac{1}{v}.\frac{\partial v}{\partial p} = \frac{1}{\rho}.\frac{\partial \rho}{\partial p}.$$

$$\therefore\ \kappa_m(P_2 - P_1) = \frac{\delta\rho}{\rho}.$$

This gives

$$\begin{aligned}\kappa &= \frac{D}{\mu(P_2 - P_1)}\left[\frac{m_1 - M_1}{\rho} - \frac{(m_2 - M_2)}{\rho\{1 + \kappa_m(P_2 - P_1)\}}\right],\\ &= \frac{D}{\mu(P_2 - P_1)}\left[\frac{m_1 - M_1}{\rho} - \frac{m_2 - M_2}{\rho}\{1 - \kappa_m(P_2 - P_1)\}\right],\end{aligned}$$

since $(1 + a)^{-1} = 1 - a$, if a is small.

$$\begin{aligned}\therefore\ \kappa &= \frac{D}{\mu(P_2 - P_1)}\left[\frac{(M_2 - M_1) - (m_2 - m_1)}{\rho}\right] + \frac{D}{\mu}.\kappa_m.\frac{m_2 - M_2}{\rho}.\\ &= \frac{D}{\mu(P_2 - P_1)}\left[\frac{(M_2 - M_1) - (m_2 - m_1)}{\rho}\right]\\ &\qquad + \frac{D}{\mu}\kappa_m\left(\frac{m_1 - M_1}{\rho}\right)\left(1 + \frac{\delta\rho}{\rho} - \frac{\delta D}{D}\right),\end{aligned}$$

since $$m_1 - M_1 = \frac{\mu\rho}{D},$$

and $$m_2 - M_2 = \mu . \frac{\rho + \delta\rho}{D + \delta D} = \mu . \frac{\rho}{D}\left[1 + \frac{\delta\rho}{\rho} - \frac{\delta D}{D}\right].$$

Now the terms $\frac{\delta\rho}{\rho}$ and $\frac{\delta D}{D}$ are negligible with respect to unity, so that we have

$$\kappa = \frac{(M_2 - M_1) - (m_2 - m_1)}{\rho} . \frac{D}{\mu(P_2 - P_1)} + \kappa_m,$$

where $(M_2 - M_1)$ and $(m_2 - m_1)$ are the masses of mercury added in the two instances, viz. with the solid in the piezometer and without it. These masses were determined directly. Calling them μ_2 and μ_1 respectively, we have

$$\kappa = \frac{\mu_2 - \mu_1}{\rho} . \frac{D}{\mu(P_2 - P_1)} + \kappa_m.$$

The mechanical stretching of liquids.—Three methods are known by which, with care, a mercury column of many times the barometric height may be supported by its adhesion to the top of the tube in which it is contained.

(*a*) The first is the method of the inverted barometer. The mercury above the level of the barometric height is in a state of tension. This tension (increases with the height and its effect is propagated in all directions to the walls of the tube. When the upper part of the tube is made elliptical in cross-section and of thin glass, its yielding to the inward pull is easily observed.

(*b*) The second is the centrifugal method, devised by OSBORNE REYNOLDS in which a glass U-tube, ABCD, Fig. 9·08(*a*), closed at both ends, contained air-free liquid in the portion ABC and only the saturated vapour of the liquid in the portion CD. This tube was fixed to a suitable board and whirled about an axis through O, a point a little beyond the end A, and perpendicular to the plane of the board. Let CE be the arc of a circle whose centre is O and radius OC. While the rotation continued the liquid between A and E was in a state of tension, increasing from zero at E (if we neglect the saturation vapour pressure of the liquid) to a maximum at A. In this way Reynolds subjected water to a tensile stress of 5 atmospheres, i.e. 72·5 lb.-wt.in.$^{-2}$. WORTHINGTON, using a similar apparatus, reached 7·9 atmospheres with alcohol, and 11·8 atmospheres with concentrated sulphuric acid.

(*c*) The third method is the method of cooling. It was devised by BERTHELOT.† In these experiments the liquid (water, alcohol,

† *Ann. de Chim.*, **30**, 232, 1850.

ether), freed from air by prolonged boiling, nearly filled a strong thick-walled glass tube, the small residual space being occupied by its saturated vapour. When slightly heated, the liquid expanded and filled the whole tube, but on cooling again it remained extended, still filling the whole tube, of which it at last let go its hold with

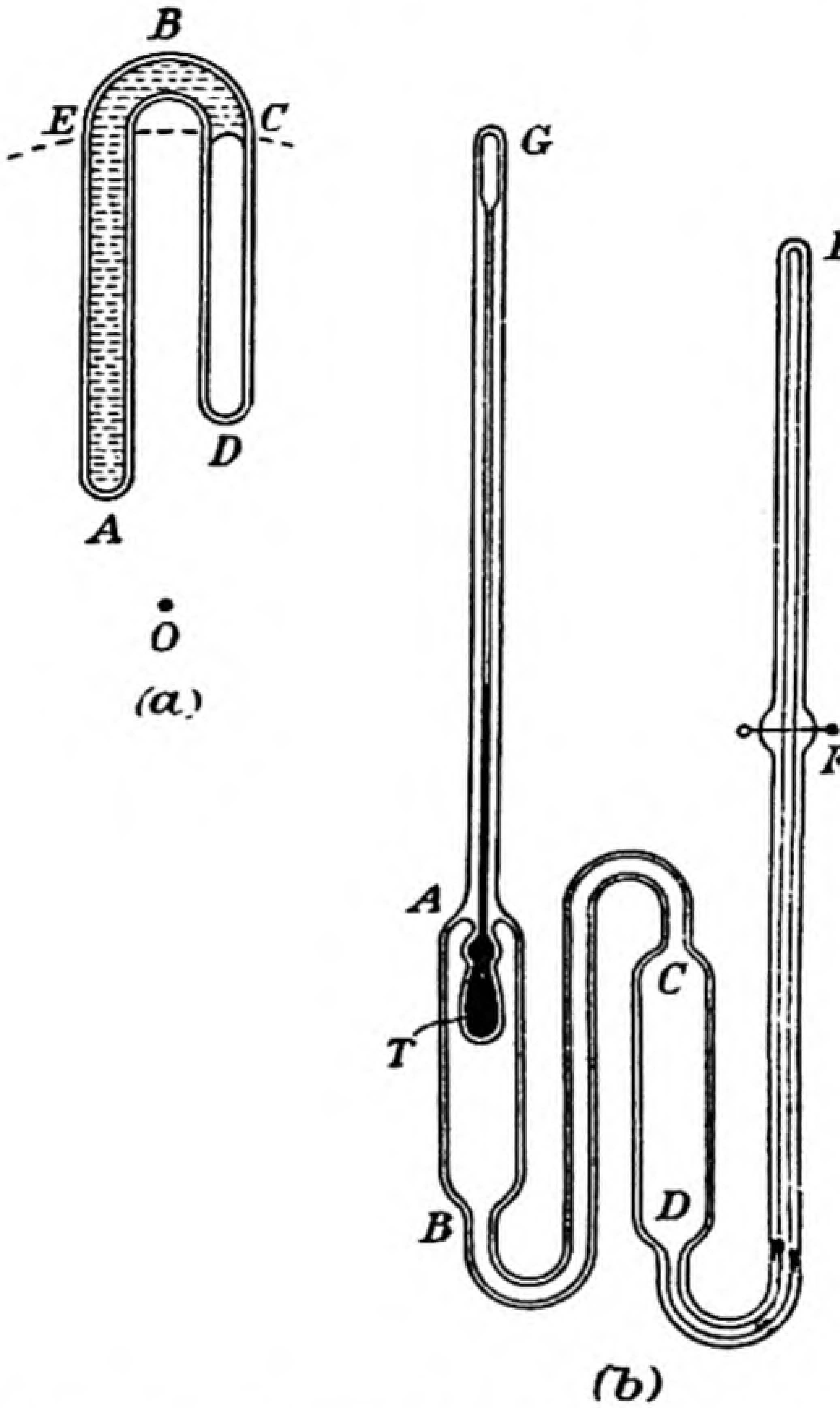


FIG. 9·08.—(*a*) The tensile strength of a liquid (Osborne Reynolds). (*b*) Worthington's tonometer.

a loud metallic click, when the bubble of vapour reappeared. From the length of this bubble the extension (volume strain) was calculated.

Methods (*a*) and (*b*) give a measure of the tensile stress, while method (*c*) gives the strain. WORTHINGTON† devised an apparatus by means of which he was enabled to measure the stress and the strain simultaneously. The liquid was contained in a strong glass vessel which it nearly filled at ordinary temperatures. Saturated vapour of the liquid filled the remainder. Dissolved air and

† *Phil. Trans.*, A, **183**, 355, 1892.

especially the film of air which first lay between the liquid and the walls of the vessel were got rid of, as far as possible, by prolonged boiling before sealing up the vessel. The temperature of part of the vessel and its contents was then raised so that the liquid expanded and filled the whole of the tube. Before the pressure exerted by the expanded liquid had become great enough to rupture the vessel, it was cooled in ice-cold water. It then remained so firmly adhered to the walls of the vessel that it could not contract and remained extended, tugging at the walls, until at last, as the cooling proceeded, the tensile stress became so great that the liquid let go its hold on the walls and a loud metallic click was heard, the volume of the liquid becoming that appropriate to its mean temperature and to the pressure of its saturated vapour. Worthington thinks that it is the adhesion between the glass and the liquid which is first overcome and not the cohesion of the liquid itself.

To measure the tensile stress in the liquid an ellipsoidal bulb T, Fig. 9·08(*b*), filled with mercury, and provided with a graduated stem AG, was sealed into the bulb AB. The bulb was then subjected to pressures up to 60 atmospheres by means of a hydraulic press, and for each pressure the corresponding rise of the mercury in the stem noted. This was found to be directly proportional to the pressure applied. This rise was due to the fact that the bulb had been made less spherical.

On the other hand, when the surrounding liquid was in a state of tension it tugged at the walls of the bulb and made it more spherical and therefore of greater capacity. It was assumed that the enlargement caused by a given tension was equal to the diminution in volume produced by an equal pressure.

To measure the strain, the liquid was caused to let go its hold on the walls of the tube and thus spring back to its unstretched volume, the volume occupied by the vapour being measured. To do this the straight tube attached to CD was traversed at F by a fine platinum wire. This was heated suddenly to redness while the liquid was in a state of tension. The liquid which was tugging at it immediately gave way and the length of the tube occupied by the bubble of vapour which had its upper end at the wire and extended below it was measured. Errors due to parallax were avoided by holding a plane mirror behind the tube. The extension so determined is only the apparent extension—it is necessary to subtract from it the amount by which the volume of the containing vessel had been diminished by the inward pull. To determine directly the yielding of the vessel the liquid under investigation was expelled by boiling and replaced by mercury. The compressibility of mercury is fifty times less than that of alcohol. The mercury was subjected to pressure and the retreat of the mercury in CD noted. If the

mercury were quite incompressible this retreat would be due to the yielding of the glass—it amounted to 7·2 divisions per 10 atmospheres of 1033 gm.-wt.cm.$^{-2}$. When a correction was applied for the finite compressibility of the mercury the above number became 6·6.

For further details concerning the method of filling the apparatus —called a ***tonometer***—and of getting rid of the last traces of air the original paper must be consulted.

In this way Worthington subjected alcohol to a tensile stress of 17 atmospheres when the corresponding apparent strain was 25×10^{-4}. The absolute strain of the alcohol was about 22×10^{-4}.

On the relation between the pressure and volume of a constant mass of gas at a given temperature.—The relation between the pressure and volume of a constant mass of air at room temperature was first given by the Hon. Robert Boyle in a book† which he published in 1662. It appears that he presented a copy of this book to the Royal Society, but an account of this famous experiment was not published by that society. On p. 58 of that treatise the following description of the experiment is given:—

'We took then a long Glaſs-Tube, which by dextrous hand and the help of Lamp was in ſuch a manner crooked at the bottom, that the part turned up was almoſt parallel to the reſt of the Tube, and the Orifice of this ſhorter leg of the Siphon (if I may ſo call the whole Inſtrument) being Hermetically ſeal'd, the length of it was divided into Inches, (each of which was ſubdivided into eight parts) by a ſtraight liſt of paper, which containing thoſe Diviſons was carefully paſted all along it: then putting in as much Quickſilver as ſerved to fill the Arch or bended part of the Siphon, that the *mercury* ſtanding in the level might reach in the one leg to the bottom of the divided paper, and juſt to the ſame height or Horizontal line in the other; we took care, by frequently inclining the Tube, ſo that the Air might freely paſs from one leg into the other by the ſides of the *Mercury* (we took (I ſay) care) that the Air at laſt included in the ſhorter Cylinder ſhould be of the ſame laxity with the reſt of the Air about it. This done, we began to pour Quickſilver into the longer leg of the Siphon, which by its weight preſſing up that in the ſhorter leg did by degrees ſtrengthen the included Air: and continuing this pouring in of the Quickſilver till the Air in the ſhorter leg was by condenſation reduced to take up but half the ſpace it poſſeſs'd (I say; *poſſeſs'd* not *fill'd*) before; we cast our eyes upon the longer leg of the Glaſs, on which was likewise paſted a liſt of paper carefully

† *New Experiments Physico-Mechanical, Touching the Spring of Air,* 1662. The second part of this treatise contains 'A Defence of the Doctrine touching the Spring and Weight of the Air, Propos'd by Mr. R. Boyle in his New Physico-Mechanical Experiments.' It is in this that the above account appears.

divided into Inches and parts, and we observed, not without delight and ſatisfaction, that the Quickſilver in that longer part of the Tube was 29 inches higher than the other.'

Boyle also made a series of measurements of the volume of the air as the pressure was increased.

In another series of experiments, 'On the Debilitated Force of Expanded Air', a slender glass tube was immersed in mercury and closed at the upper end with sealing wax. It contained a quantity of air which occupied a portion of the tube one inch long. This tube was then gradually raised until the air had expanded to 32 times its original volume, the position of the mercury being noted on nineteen different occasions.

Two tables given by Boyle are reproduced on p. 434.

The length of tube occupied by the air was gradually reduced and the pressures were read on twenty-five occasions. These pressures agreed very closely with 'what that pressure should be according to the HYPOTHESIS that supposes the pressures and expansions to be in reciprocal proportions.'

In 1679 E. MARIOTTE published his experimental results on the relation between the pressure and the volume of a given mass of gas at constant temperature, and no doubt his work was independent of that of Boyle.

The isothermal compressibility of an ideal gas.—If p is the pressure of the gas at constant temperature and v its specific volume, i.e. the volume of unit mass, Boyle's law may be expressed by the equation

$$pv = \text{constant}.$$

No actual gas obeys this law over a wide range of pressures, so, for theoretical discussion, we contemplate the existence of an ideal gas whose behaviour when subjected at constant temperature to different pressures is given by the above equation.

The isothermal compressibility of an ideal gas is readily calculated as follows. Since pv is constant under isothermal conditions, differentiating we have,

$$v\,\delta p + p\,\delta v = 0.$$

Hence κ, the isothermal compressibility, is given by

$$\kappa = \left(-\frac{1}{v}.\frac{\partial v}{\partial p}\right)_{\mathrm{T}} = \frac{1}{p},$$

i.e. the isothermal compressibility of an ideal gas is measured by the reciprocal of the pressure to which it is subjected.

A Table of the Condensation of the Air

A	B	C	D	E
12	00	Added to $29\frac{1}{8}$ makes	$29\frac{2}{16}$	$29\frac{2}{16}$
$11\frac{1}{2}$	$01\frac{7}{16}$		$30\frac{9}{16}$	$30\frac{6}{16}$
11	$02\frac{13}{16}$		$31\frac{15}{16}$	$31\frac{12}{16}$
$10\frac{1}{2}$	$04\frac{6}{16}$		$33\frac{8}{16}$	$33\frac{1}{7}$
10	$06\frac{3}{16}$		$35\frac{5}{16}$	35 --
$9\frac{1}{2}$	$07\frac{14}{16}$		37 --	$36\frac{15}{19}$
9	$10\frac{2}{16}$		$39\frac{4}{16}$	$38\frac{7}{8}$
$8\frac{1}{2}$	$12\frac{8}{16}$		$41\frac{10}{16}$	$41\frac{2}{17}$
8	$15\frac{1}{16}$		$44\frac{3}{16}$	$43\frac{11}{16}$
$7\frac{1}{2}$	$17\frac{15}{16}$		$47\frac{1}{16}$	$46\frac{3}{5}$
7	$21\frac{3}{16}$		$50\frac{5}{16}$	50--
$6\frac{1}{2}$	$25\frac{3}{16}$		$54\frac{5}{16}$	$53\frac{11}{13}$
6	$29\frac{11}{16}$		$58\frac{13}{16}$	$58\frac{2}{8}$
$5\frac{3}{4}$	$32\frac{3}{16}$		$61\frac{5}{16}$	$60\frac{18}{23}$
$5\frac{1}{2}$	$34\frac{15}{16}$		$64\frac{1}{16}$	$63\frac{6}{11}$
$5\frac{1}{4}$	$37\frac{15}{16}$		$67\frac{1}{16}$	$66\frac{4}{7}$
5	$41\frac{9}{16}$		$70\frac{11}{16}$	70 --
$4\frac{3}{4}$	45 --		$74\frac{2}{16}$	$73\frac{11}{19}$
$4\frac{1}{2}$	$48\frac{12}{16}$		$77\frac{14}{16}$	$77\frac{2}{3}$
$4\frac{1}{4}$	$53\frac{11}{16}$		$82\frac{12}{16}$	$82\frac{4}{17}$
4	$58\frac{2}{16}$		$87\frac{14}{16}$	$87\frac{3}{8}$
$3\frac{3}{4}$	$63\frac{15}{16}$		$93\frac{1}{16}$	$93\frac{1}{5}$
$3\frac{1}{2}$	$71\frac{3}{16}$		$100\frac{7}{16}$	$99\frac{6}{7}$
$3\frac{1}{4}$	$78\frac{11}{16}$		$107\frac{13}{16}$	$107\frac{7}{13}$
3	$88\frac{7}{16}$		$117\frac{9}{16}$	$117\frac{4}{8}$

A Table of the Rarefaction of the Air

A	B	C	D	E
1	$00\frac{0}{0}$	Subtracted from $29\frac{3}{4}$ leaves	$29\frac{3}{4}$	$29\frac{3}{4}$
$1\frac{1}{2}$	$10\frac{5}{8}$		$19\frac{1}{8}$	$19\frac{5}{6}$
2	$15\frac{3}{8}$		$14\frac{3}{8}$	$14\frac{7}{8}$
3	$20\frac{2}{8}$		$9\frac{4}{8}$	$9\frac{11}{12}$
4	$22\frac{5}{8}$		$7\frac{1}{8}$	$7\frac{7}{16}$
5	$24\frac{1}{8}$		$5\frac{5}{8}$	$5\frac{19}{20}$
6	$24\frac{7}{8}$		$4\frac{7}{8}$	$4\frac{23}{24}$
7	$25\frac{4}{8}$		$4\frac{2}{8}$	$4\frac{1}{4}$
8	$22\frac{0}{0}$		$3\frac{6}{8}$	$3\frac{23}{32}$
9	$26\frac{3}{8}$		$3\frac{3}{8}$	$3\frac{11}{16}$
10	$26\frac{6}{8}$		$3\frac{0}{0}$	$2\frac{39}{40}$
12	$27\frac{1}{8}$		$2\frac{5}{8}$	$2\frac{23}{48}$
14	$27\frac{4}{8}$		$2\frac{2}{8}$	$2\frac{1}{8}$
16	$27\frac{6}{8}$		$2\frac{0}{0}$	$1\frac{55}{64}$
18	$27\frac{7}{8}$		$1\frac{7}{8}$	$1\frac{47}{72}$
20	28--		$1\frac{6}{8}$	$1\frac{9}{80}$
24	$28\frac{2}{8}$		$1\frac{4}{8}$	$1\frac{23}{96}$
28	$28\frac{3}{8}$		$1\frac{3}{8}$	$1\frac{1}{16}$
32	$38\frac{4}{8}$		$1\frac{2}{8}$	$0\frac{119}{128}$

A. The number of equal spaces in the shorter leg, that contained the same parcel of Air diversely extended.

B. The height of the Mercurial Cylinder in the longer leg, that compress'd the Air into those dimensions.

C. The height of a Mercurial Cylinder that counterbalanced the pressure of the Atmosphere.

D. The aggregate of the two last columns B and C, exhibiting the pressure sustained by the included Air.

E. What the pressure should be according to the HYPOTHESIS, that supposes the pressures and expansions to be in reciprocal proportion.

A. The number of equal spaces at the top of the Tube, that contained the same parcel of Air.

B. The height of the Mercurial Cylinder, that together with the spring of the included Air counterbalanced the pressure of the Atmosphere.

C. The pressure of the Atmosphere.

D. The complement of B to C, exhibited the pressure sustained by the included Air.

E. What the pressure should be according to the HYPOTHESIS.

Deviations from Boyle's law.—According to Regnault,† Boyle himself did not consider the law which bears his name to possess the generality later scientists have associated with it. At pressures greater than four atmospheres, this writer continues, Boyle thought that the compression of a given mass of air was less than that calculated from his hypothesis. Towards the end of the eighteenth century MUSSCHENBROEK,‡ ROBISON and others investigated the strict applicability of Boyle's law to actual gases but their final decisions were not conclusive. OERSTED § (1826), and DESPRETZ¶ (1827), showed that for pressures even as low as two atmospheres there were slight departures in the behaviour of the following dry gases—ammonia, hydrogen sulphide and cyanogen—compared with that of an ideal gas. In each instance the volume decreased more rapidly with increase in pressure than would be expected if Boyle's law were accurately true, i.e. the product pv, instead of remaining constant, tended to become smaller as the pressure was increased. In these experiments the gases were enclosed in a number of different barometer tubes, of the same diameter, and standing in the same cistern of mercury. The whole was surrounded by a glass tube so that the pressure could be increased. Initially, equal lengths of each tube were occupied by different gases. When the pressure was increased it was found that the lengths of the tubes occupied by the gases were no longer equal, so that some gases at least did not behave according to Boyle's law.

In 1847 REGNAULT,‖ with his characteristic experimental skill, completed his experiments on the behaviour of certain gases under pressure, from 1 atmosphere to 30 atmospheres. His apparatus is shown diagrammatically in Fig. 9·09. The gas under investigation was enclosed in a glass tube A, 3 metres long, 1·1 cm. wide, and graduated in mm. along its whole length. It was fitted at its upper end with a carefully ground metal stop-cock B, by means of which it could be put into connexion with a compression pump or with a vacuum pump. Its lower end fitted into an iron trough C containing mercury. D was an open mercury manometer, also fitted into C, and by means of this the pressure of the gas in A was measured directly. It was 36 metres long and at intervals of 1·5 metres along it mercury-in-glass thermometers were placed. The mean of the temperatures indicated by these thermometers was taken to be that

* *Mem. de l'Acad. Fran.*, **21**, 330, 1847. 'Boyle avait déjà cru remarquer que pour des pressions supérieures à 4 atmosphères, l'air se comprimait moins qu'il ne devrait le faire d'après la loi énoncée.'

† *Cours de Physique*, traduit par Signand de Lafond, Paris, 1759, Vol. **3**, p. 142.

‡ *Edin. Jour. of Science*, **4**, 224, 1826.

¶ *Ann. Chim. Phys.*, 605, 1827.

‖ *Mem. de l'Acad. Fran.*, **21**, 329, 1847.

of the mercury column, whose height was then corrected to 0° C. A correction (very small) was applied for the compressibility of mercury. The tube A was surrounded by a glass tube G, 8 cm. wide, through which a stream of water at about 4° C. passed. Its temperature was given by a sensitive mercury-in-glass thermometer.

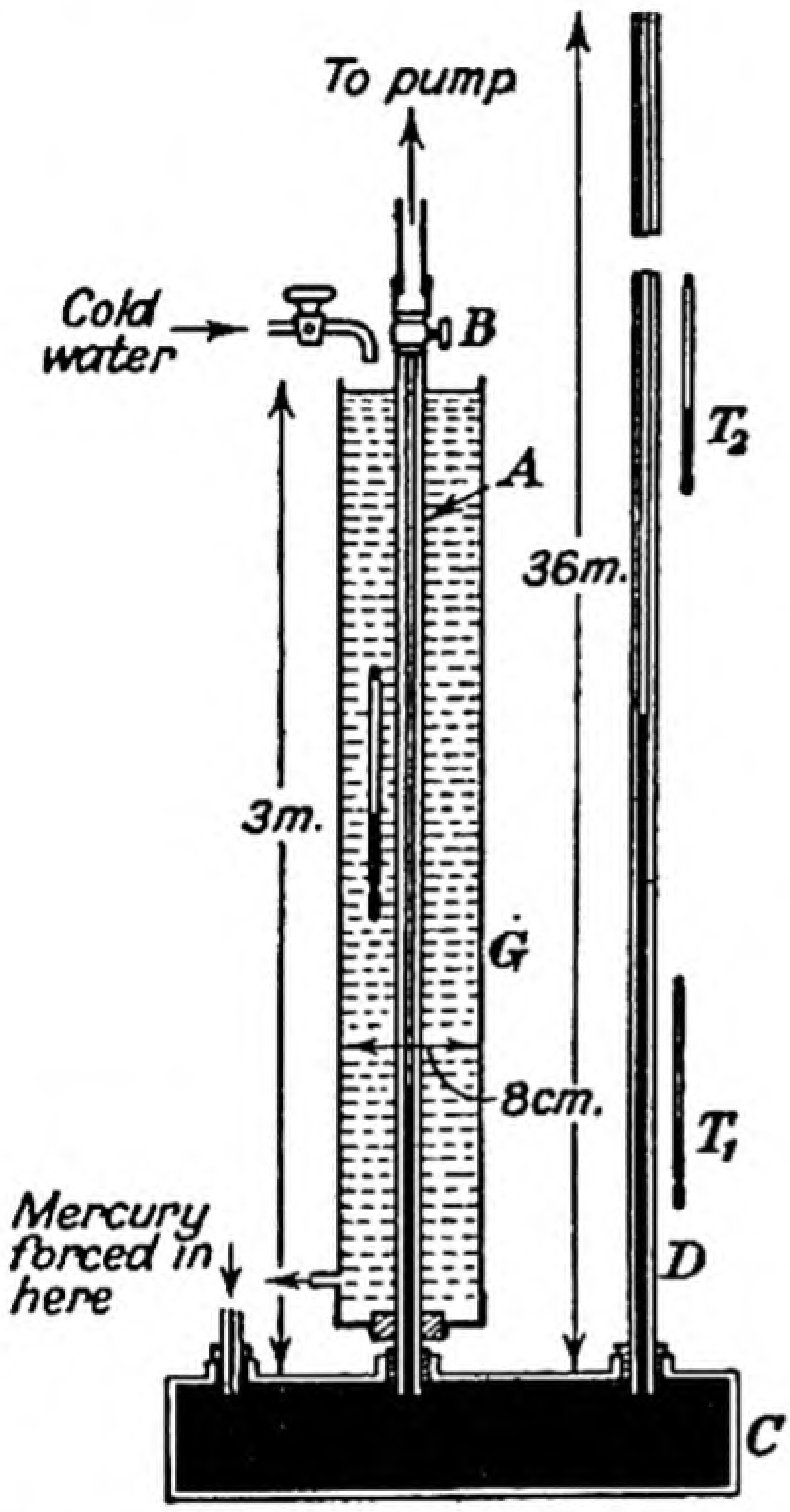


FIG. 9·09.—Regnault's apparatus for investigating the validity of Boyle's law.

In these experiments Regnault sought to determine that pressure which must be applied to a volume of gas at known temperature and pressure in order to reduce its volume at the same temperature to one-half its initial volume. The volume of the tube A up to the 300th mm. division was taken as unity. To determine the position of a mark on the tube which would divide the volume of the tube from the top to the 300th mm. mark exactly into two equal portions, the tube was filled with mercury, an approximate position for the mark being estimated. The masses of mercury filling the tube from the zero mark to the 'half-way' mark, and from there to the 300th mm. mark were determined. If the position of the 'half-way' mark had been correctly chosen, the above masses should be equal. In this way, after many trials, the correct position of the 'half-way' mark was rigorously determined. The volumes between successive divisions in the neighbourhood of the above mark and the 300th mm. mark were also found.

The graduated tube A was dried by exhausting it and then filling it with air, a process which was repeated several times. During this process, to facilitate the drying operations, the water in G was kept at a temperature between 40° and 50° C. The tube was then filled with a dry gas.

The amount of mercury in the apparatus was adjusted until the level of the mercury in A was at the 300th division mark. The barometer was read and the volume of the gas reduced to 0·5, care being taken not to make any observations until the heat generated by compressing the gas had been dissipated. Parallax errors in

estimating the position of the mercury surface were avoided by using a small lens. On each occasion the experiment was repeated and if consistent results were obtained, it was concluded that the stop-cock B had not leaked.

Regnault found that atmospheric air did not obey Boyle's law; the actual compression was a little more than that of an ideal gas subjected to the same increase in pressure.

Let $\left[\dfrac{(pv)_{\text{init.}}}{(pv)_{\text{fin.}}}\right]$ denote the ratio of the initial value of the product 'pv' to the final value of that product in the course of any one of Regnault's experiments, i.e. the ratio of the products 'pv' when the volume of the gas, at a given pressure, is reduced to half that volume under a higher pressure. Let the suffixes 1, 2, 3, . . . denote the values of the above ratio when the initial pressure is approximately, 1, 2, 3, . . . atmospheres.

Regnault found for air, nitrogen, and carbon dioxide, that

$$\left[\frac{(pv)_{\text{init.}}}{(pv)_{\text{fin.}}}\right]_1 = (1 + \alpha_1),$$

where α_1 is a small positive quantity. This fact showed that the product 'pv' decreased a little more than it should have done according to Boyle's hypothesis. Similarly

$$\left[\frac{(pv)_{\text{init.}}}{(pv)_{\text{fin.}}}\right]_2 = (1 + \alpha_2).$$

Now suppose that the pressure acting on a constant mass of gas had been increased in turn from P_0, P_1, P_2, P_3, . . . so that the volume of the gas was reduced to $\frac{1}{2^1}$, $\frac{1}{2^2}$, $\frac{1}{2^3}$, . . . times its original volume, V. Then P_1, P_2, P_3, etc. may be calculated. For we have at once

$$\left[\frac{(pv)_{\text{init.}}}{(pv)_{\text{fin.}}}\right]_1 = \frac{P_0 V}{P_1\left(\dfrac{V}{2^1}\right)} = (1 + \alpha_1),$$

so that P_1 is known.

Similarly $\dfrac{P_1\left(\dfrac{V}{2}\right)}{P_2\left(\dfrac{V}{2^2}\right)} = (1 + \alpha_2)$, so that P_2 is known.

Proceeding in this way it became possible to calculate values for 'pv' when the pressure on a given mass of gas was increased to P_1, P_2, etc.

The following table is taken from Regnault's paper. It contains some of his results for air, nitrogen, carbon dioxide and hydrogen.

GAS	Temp.	V	V_0	P	P_0	$\frac{V}{V_0}$		
Air	4·44	1939·69	969·26	738·72	1476·25	2·001215	1·998389	1·001414
	4·70	1939·91	970·13	9336·41	18551·09	1·999610	1·986962	1·006366
	4·86	1939·86	969·86	6770·15	13483·48	2·000143	1·991607	1·004286
	4·95	1939·48	970·36	4219·05	8404·70	1·998726	1·992084	1·003335
Nitrogen	5·09	1939·10	969·82	753·96	1505·06	1·999434	1·996206	1·001617
	5·18	1938·89	969·72	2159·36	4313·33	1·999433	1·997537	1·000966
	5·07	1941·81	966·70	8628·54	17249·95	2·000706	1·999174	1·004768
Carbon dioxide	3·26	1939·68	970·56	765·77	1517·63	1·998515	1·981835	1·008416
	3·24	1939·10	969·75	4879·77	9331·72	1·999577	1·912328	1·0045625
	2·66	1935·25	970·31	9620·06	17445·23	1·994456	1·813421	1·099830
Hydrogen	10·02	1939·91	968·83	5555·32	11168·86	2·002314	2·010480	0·995938
	9·62	1939·98	969·79	7074·96	14228·28	2·000412	2·011075	0·994697
	9·73	1939·34	969·89	10361·88	20879·18	1·000540	2·015000	0·992327

Regnault found that the product 'pv' was not constant; i.e. the ratio

$$\left[\frac{(pv)_{\text{init.}}}{(pv)_{\text{fin.}}}\right]$$

exceeded unity by a small amount for all the gases he examined, except hydrogen for which the ratio was slightly less than unity. Regnault inferred, following the lines of the argument given above, that within the range of pressure 1–30 atmospheres, the product 'pv' diminished with increasing pressure for all the above gases except hydrogen. The fact that these experiments showed that hydrogen had a compressibility less than that of an ideal gas caused Regnault to make the ironical remark that hydrogen was 'un gaz plus que parfait'. It is now known, from the work of later experimentalists, that the above behaviour is not characteristic of hydrogen alone, but that it is exhibited by all gases under high pressures, provided there is no change of state. The discovery of this fact is due to NATTERER† and it was made during the course of some experiments in which an attempt was made to liquefy the gases under investigation. In this he did not succeed although he reached a pressure of 3000 atmospheres.

In 1870 and the following years, CAILLETET‡ examined the relation between the pressure and volume of several gases. The pressure was measured by a Desgoffe's manometer and the position of the mercury in the piezometer determined by observing how far a coat of gold inside the experimental tube had been dissolved by the mercury. He confirmed the results of Regnault and Natterer. The

† *Wien. Ber.*, **5**, 351, 1850, and later papers. *Pogg. Ann.*, **62**, 139.
‡ *Compt. Rend.*, **70**, 1131, 1870; **88**, 61, 1879.

most complete study of the behaviour of gases under high pressures made during the last century was carried out by AMAGAT.†

Amagat's experiments on the compressibility of gases.—A description of Amagat's work on the compressibility of liquids has already been given. With a modified piezometer, i.e. one in which the bulb was nearer the end which dipped into the mercury, he investigated the compressibilities of gases at room temperature. Such an apparatus, however, was not suitable for work at high pressures, partly on account of the fragile nature of the glass tube at those points where the platinum contacts were sealed and the difficulty attending the effective insulation of the wires, and partly on account of the massive nature of the apparatus and its many joints. Amagat therefore developed the following method which he termed 'une mèthode des regards', i.e. a visual method. The gas under investigation was enclosed in a glass piezometer, P, Fig. 9·10. This was housed in an iron vessel A, the lower part of which contained mercury; the rest was filled with water so that pressure could be transmitted to the gas in the usual way. The upper part of the iron vessel was surrounded by a double-jacketed vapour bath, B, and this surrounded the upper portion of the piezometer, i.e. the only part occupied by the gas when the pressure was high. G_1 and G_2 were glass windows fixed in position with marine glue and screw-on caps to the ends of the side arms attached, as shown, to the iron vessel. The glue was kept from melting by the cooling devices indicated. The uppermost part of the piezometer was attached by means of a specially designed clamp to a plunger whose position could be altered by means of the screw S. In this way the piezometer could always be brought into such a position that one of the marks on its stem could be seen through G_2, a gas lamp near to G_1 providing the necessary illumination. The pressure was then adjusted until the mercury was level with the mark seen through G_2. The visual examination was not always easy, at first, on account of the fact that the water in the side arms became heated in such a way that convection currents were produced so that the mark could not be clearly seen. Amagat finally overcame this difficulty by using quartz 'rods', with polished flat ends, to displace the water from the side arms of the apparatus. These rods were cut from quartz crystals, the optic axis being along the axis of the rod. Glass rods were also tried for this purpose but at high temperatures there was a chemical action between them and the water, so that the hot ends became covered with a layer of material which was not transparent, i.e. they became 'frosted'. The upper portion of the piezometer having been previously calibrated, and the pressures measured by means of a 'free-piston' gauge which Amagat had

† *Ann. Chim. Phys.*, **23**, 353; **29**, 68, 1893.

perfected, the compressibility of the gas under investigation could be calculated.

Some of the results obtained with the above apparatus by Amagat are given in Fig. 9·11, where the product PV is plotted against

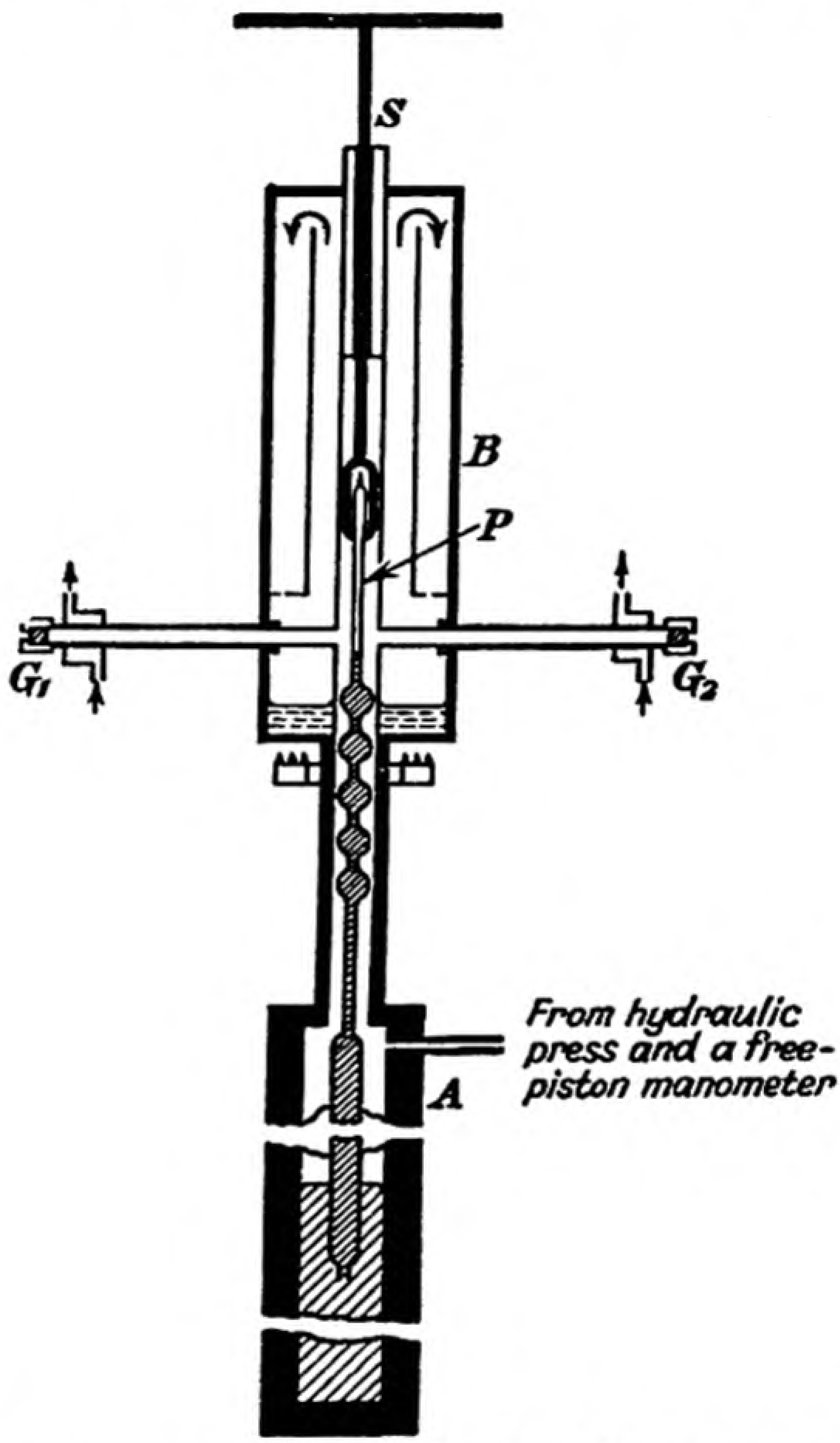


FIG. 9·10.—Amagat's apparatus for investigating the compressibilities of gases at high temperatures and pressures. [360°C.; 1000 atmos.]

corresponding values of P as abscissae. If Boyle's law were rigorously true the curve connecting PV and P at any given constant temperature would in each instance be a straight line parallel to the axis of pressure. The curves actually show that the behaviour of gases with respect to pressure depends upon the nature of the gas and even for the same gas varies as the temperature is changed. For hydrogen cf. Fig. 9·11(a) the curves are practically straight lines parallel to one another, but inclined to the pressure axis in such a way that they show that PV increases with P at constant temperature. Similar remarks may be made with reference to

nitrogen, the curves for which are shown along with those for hydrogen. In this instance, however, it will be noted that each curve shows a minimum. The curves for carbon dioxide are shown in Fig. 9·11(*b*). Each curve for this gas shows a distinct minimum, after which the gas behaves like hydrogen except that the effect is more marked. With respect to the minima for carbon dioxide it

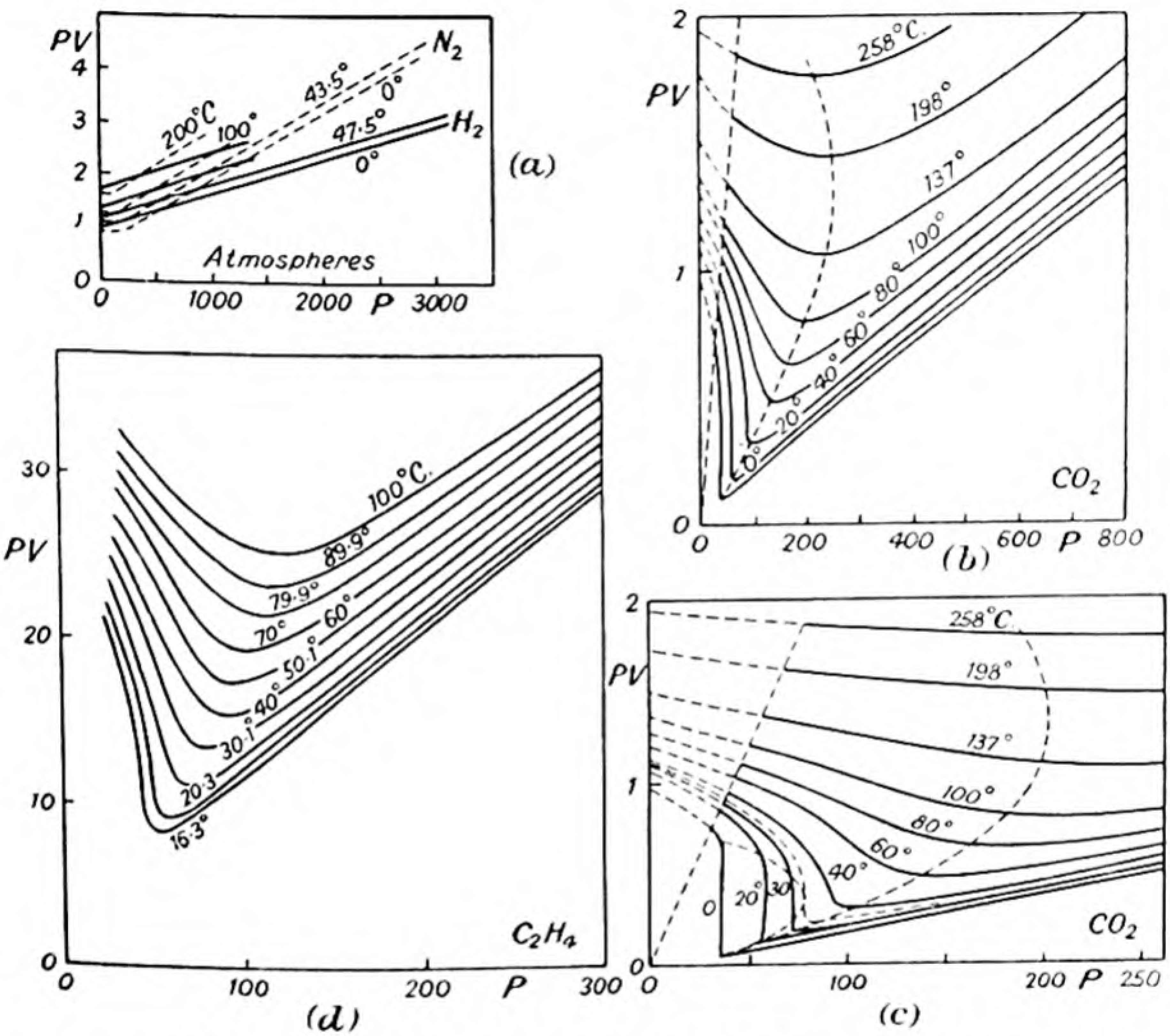


FIG. 9·11.—The variation of PV with P for several gases.

will be noticed that, as the temperature is increased, at first the minimum moves towards the right of the diagram but this displacement does not continue for at still higher temperatures a retrograde motion occurs. The curve passing through these minima is a parabola. These facts are shown more clearly in Fig. 9·11(*c*) which is an enlarged diagram of the left-hand portion of (*b*). In addition to this parabolic curve which now shows itself very strikingly, there is a second parabola in connexion with the curves below the critical point. It passes through those points at which the formation of liquid begins and also through those where it is complete. The dotted isothermal corresponds to 32° C., a temperature just above

the critical temperature for carbon dioxide. The curves for ethylene are shown in Fig. 9·11(*d*). Here, too, it will be observed that as the temperature rises the marked drop in the curve tends to disappear, from which we may rightly infer that at higher temperature the curves would resemble those for hydrogen. From these facts it is concluded that the particular behaviour of hydrogen in that no minima are exhibited is merely a consequence of the fact that at ordinary temperatures this gas is so far removed from its critical temperature that the initial downward slope of the curves has disappeared.

Bridgman's Work on the Properties of Matter at High Pressures

General technique.—Bridgman has made a very comprehensive investigation of the properties of matter at high pressures, and he has been successful because he was able to design a method of packing the piston used to produce the high pressures, and the necessary joints between different parts of the apparatus, so that the leak was negligible. Until Bridgman began his work the favourite method of packing, a pipe connexion for example, was due to Amagat. The pipe A, Fig. 9·12(*a*), was provided with a

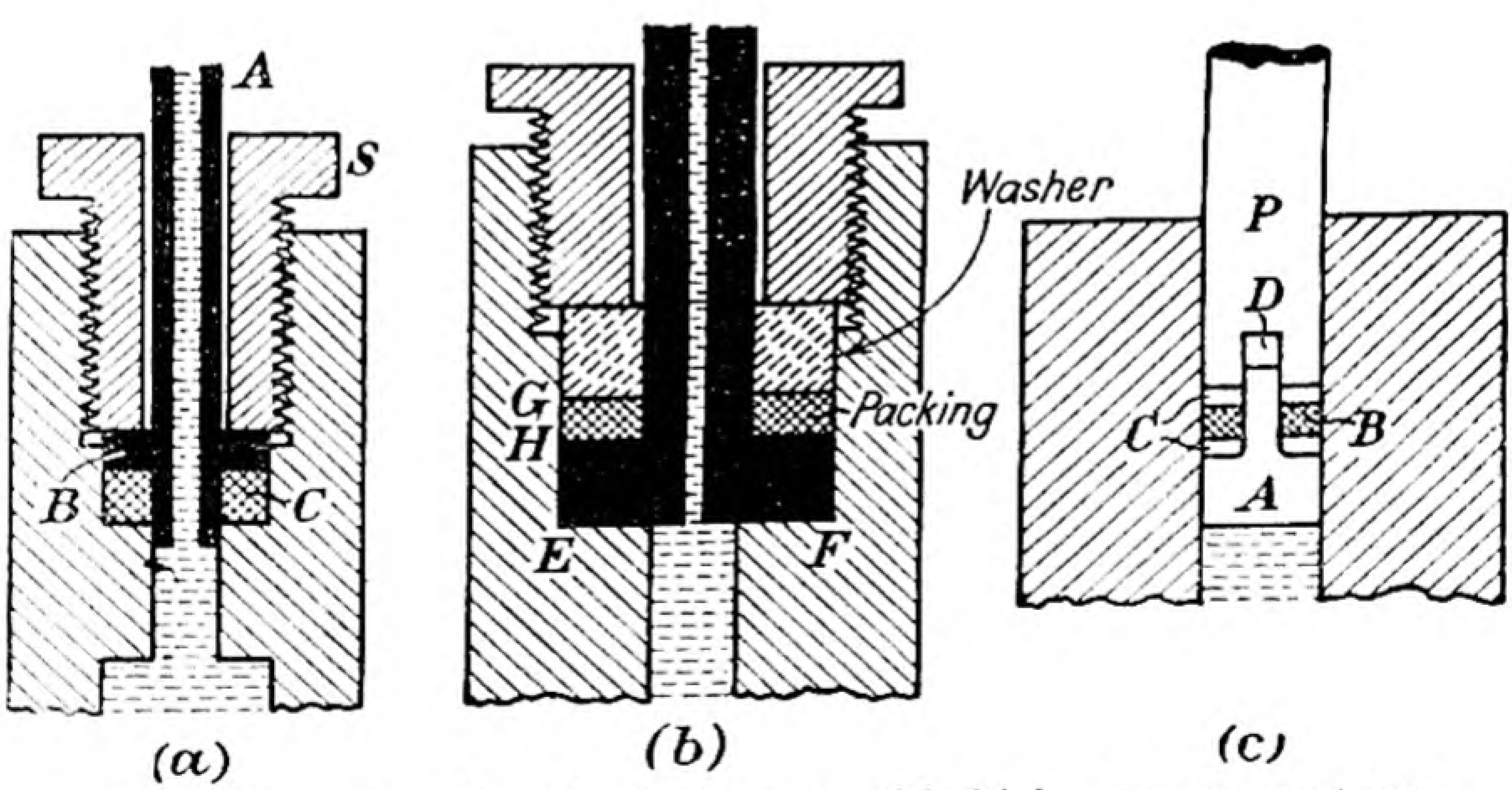


Fig. 9·12.—Packing glands for use with high pressure systems.

flange B near to its end and this flange entered a recess in the side of the vessel, where the high pressure was to be maintained, in such a way that the soft packing material C was completely enclosed. In this way Amagat prevented the packing from being extruded under the initial pressure exerted by the screw S, when this was placed in position, or being blown out when the pressure was

increased inside the apparatus. Amagat reached a pressure of 3000 atmospheres. An upper limit of pressure beyond which this type of joint failed was reached when the pressure inside the apparatus became equal to that exerted by the screw on the packing. When this stage was reached the packing just filled the surrounding space, but at higher pressures it shrank away from the enclosing walls and a leak occurred. If there are irregularities due to faulty machining, leaks will occur below the above theoretical limit. Now we shall be in a better position to appreciate the excellence of the packing device designed by Bridgman.

An example of such a device is shown in Fig. 9·12(*b*). At first it appears only to be a slight modification of that due to Amagat, for the packing has only been removed from the front to the back side of the flange. However, let us consider the forces acting on the packing. The force exerted from below on the packing is equal to the pressure in the liquid filling the apparatus, multiplied by the area EF. For the packing to be in equilibrium this force must be balanced by the force exerted by the screw; it is distributed over the annular area of the packing. This force per unit area, which is the pressure in the packing as long as it remains soft, is therefore automatically maintained at a value exceeding the pressure in the liquid, so that the liquid cannot leak—provided, of course, that the walls of the retaining vessel remain intact. Bridgman calls this method of packing 'that based on the principle of unsupported areas'.

There was one further difficulty which had to be overcome, and that was to prevent the packing material from leaking beyond the points G and H. Eventually this was accomplished by means of steel washers suitably placed.

Fig. 9·12(*c*) depicts Bridgman's method of packing a moving piston. A was a mushroom-shaped plug which is pushed into the high-pressure vessel by a plunger P, constructed of hard steel. The stem of the plug freely projects into a hollow in P and it is the cross-sectional area of this stem which is the 'unsupported area'. For equilibrium the pressure in the packing must be greater than it is in the liquid in the ratio

$$\frac{\text{area of cross-section of plug}}{\text{annular area of packing}}.$$

The packing B (soft rubber) was prevented from leaking by the presence of the copper and steel rings C. The friction between the packing and the walls of the tube in which the piston moved is considerable; the retarding force due to it was kept low by making the thickness of the packing small. Sometimes the packing was made in two layers with a mixture of graphite and vaseline smeared between them to reduce friction.

Such a packing is never used for more than two or three strokes of the piston, but it is readily renewed and consistent results are obtained.

The connecting tubes used in this work were made of copper. ($P < 10^6$ gm.-wt.cm.$^{-2}$), hard-drawn steel ($P < 4 \times 10^6$ gm.-wt. cm.$^{-2}$), and for higher pressures from short pieces of a hard steel alloy, drilled from the solid rod and heat treated. Heavy glass capillary tubes may be used at the lower pressures but they are unreliable.

The measurement of high pressures.—Pressure gauges may be classified conveniently as follows:

(*a*) As ***primary gauges***, i.e. those so constructed that the absolute pressure may be found at once, at least approximately, from the construction and reading of the instrument itself. We say approximately since it is seldom, if ever, possible to design a gauge of this type in which no correction at all has to be applied.

(*b*) As ***secondary gauges***, the readings of which can only be interpreted as pressures when the instruments have been calibrated.

The simplest and earliest form of primary pressure gauge was the open mercury manometer. The upper limit of such a gauge is set by the height of the column, and in practice these gauges have not been used for pressures greater than a few hundred kgm.-wt.cm.$^{-2}$. They have to be used in the shaft of a deep mine or in a tall tower. Among the corrections to be applied are those for temperature, the compressibility of mercury, and for the variation of the intensity of gravity with latitude and with altitude. Practically the only type of primary gauge successfully applied to pressures exceeding 10^6 gm.-wt.cm.$^{-2}$ is some form of the so-called 'free piston gauge', used and greatly improved by Amagat although it may be that Descoffe has the claim to priority. Such a gauge consists essentially of a piston of small cross-sectional area exposed directly at one end to the pressure to be measured and so well machined that it fits the cylinder in which it works so well that leak is unimportant. The force required to keep the piston in place and against the expelling force due to the pressure in the liquid is measured in any convenient way. Generally the small piston is in communication with a much larger one in direct communication with a mercury column. When the system is in equilibrium the pressures involved are inversely proportional to the areas of the pistons. In use, the effect of friction between the pistons and the walls of the cylinders in which they move is reduced by giving each piston a rotary motion before taking an observation.

In almost all high pressure investigations direct measurement of the pressure by means of a free-piston gauge would be inconveniently clumsy; in addition there is invariably a small leak. A secondary

gauge is therefore very desirable and as a basis for such a gauge, any conveniently measurable high-pressure effect may be taken. In all probability the simplest effect on which to base the construction of a secondary gauge is the change with pressure of the electrical resistance of a metallic wire. In ordinary metals, which are never quite homogeneous or free from strain, the effect is not sufficiently reproducible for use in this connexion. Bridgman, using a free-piston gauge, investigated the effect of pressure on the resistance of liquid mercury, at constant temperature, and up to pressures of 7×10^6 gm.-wt.cm.$^{-2}$. The measurements were so accurate that they have become recognized standards. In using such resistances as pressure gauges a number of precautions is necessary, the most important of which concerns the glass capillary in which the mercury must be contained. The compressibility of glass is so great compared with the pressure coefficient of resistance of mercury that the compressibility of the glass must be measured, and its variation with pressure determined. Since glass is not homogeneous it is impossible to get results reproducible to the necessary degree of accuracy, so that the desired advantage of reproducibility, which mercury apparently offered, is lost. Moreover, it was found essential carefully to anneal the glass and to use the glass and its contained mercury at a definite temperature.

Later on Bridgman found that wires made from such soft metals as lead could be used; for two samples of lead the pressure coefficient of resistance was identical to within 0·1 per cent.

These, and other considerations, led Bridgman to return to the manganin gauge introduced by Liselle, and it became an essential part of all his work. A typical gauge is made from well annealed manganin wire having a diameter 0·005 in., and being about 5 metres long, so that its resistance under ordinary circumstances is about 120 ohms. A gauge constructed from this material is calibrated by determining its change in resistance when it is subjected to such a pressure that mercury will freeze at 0° C. This is determined by plotting the position of the piston by which the pressure is generated against the resistance of the coil. Freezing is indicated by a discontinuity in the motion of the piston, the total amount of the discontinuity corresponding to the difference in volume between the liquid and the solid mercury. The freezing pressure of mercury at 0° C. is known by previous measurement with a free-piston gauge to be $7{\cdot}640 \times 10^6$ gm.-wt.cm.$^{-2}$. Bridgman had previously established, using a primary pressure gauge, that the change of resistance with pressure for manganin up to 13×10^6 gm.-wt.cm.$^{-2}$ was a linear function of the pressure—an extension of the result found by Liselle.

In carrying out a calibration by the above method the freezing

point of mercury should be approached in two directions, viz., liquid → solid, and solid → liquid, and consistent results should be obtained if the apparatus is working properly.

Bridgman has used such a gauge up to pressures of 12×10^6 gm.-wt.cm.$^{-2}$, the results at the higher pressures being obtained by extrapolation. Such a procedure, as Bridgman admits, is not perfectly safe, but in this instance the error is not considered to be large. A possible check on the method would be to measure simultaneously the changes of resistance of a coil of manganin and one of some other metal or alloy, and then compare the values of the pressure deduced by extrapolation in the two instances.

The behaviour of gases at high pressures.—Bridgman's experiments on gases were made with hydrogen, helium, nitrogen, argon and ammonia. The gas under an initial pressure of 2×10^6 gm.-wt.cm.$^{-2}$ was passed into a cylindrical vessel A, Fig. 9·13, placed inside a larger cylinder B; A was closed at its lower end with a specially designed valve C opening inwards. The lower end of B was closed by means of a plug D which carried a manganin gauge M—the 'lead-in' was specially insulated. A known mass of gas-free kerosene was then placed inside so that the space between B and A was completely filled with this liquid. P was a plug of the conventional type, moved inwards by a piston Q operated from a hydraulic press. In this way the pressure inside the apparatus could be increased. As the plug was gradually pushed in, the pressure increased until it reached that of the gas in A; the valve C then opened and from the advance of the piston Q the change in volume due to the fact that both the kerosene and the gas were compressible deduced. Until this change was reached the motion was determined solely by the apparent compressibility of the kerosene. The apparent compressibility of the kerosene being known, the compressibility of the gas could be calculated. A small correction for the change in internal volume of A with pressure did not exceed one per cent at the highest pressure applied (16×10^9 gm.-wt.cm.$^{-2}$).

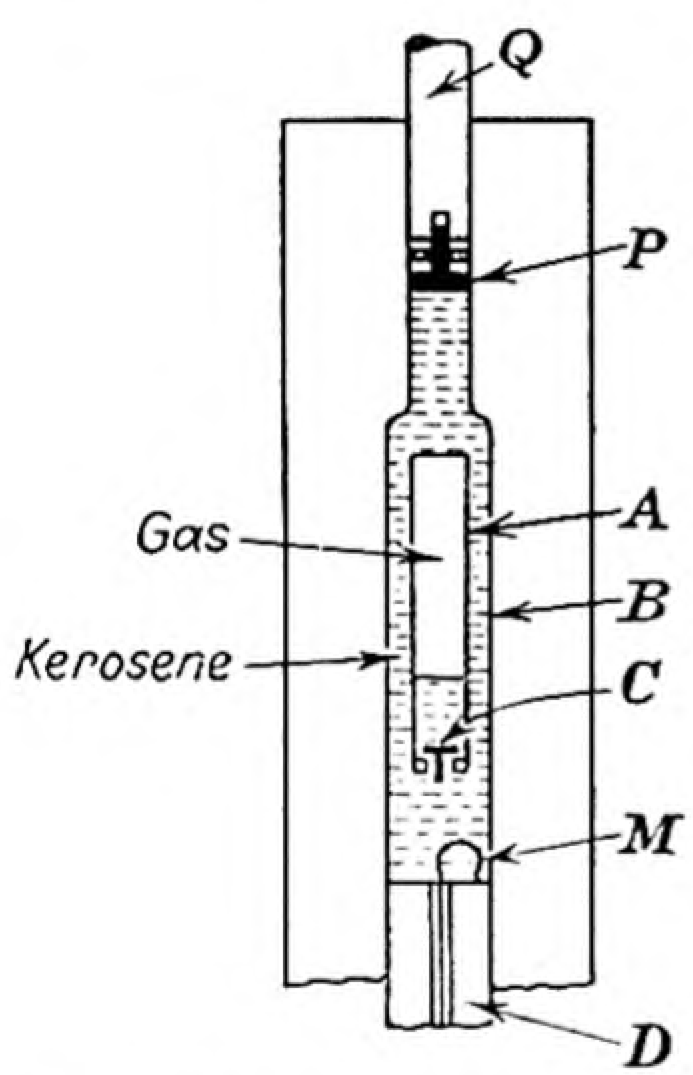


Fig. 9·13.—Principle of Bridgman's piezometer for use with gases at high pressures.

The fact that the gas was in contact with kerosene raised the possibility of the results being affected by the solution of the gas

in this liquid, but it was found that the volume of gas, at a given pressure, was independent of the time during which the gas and liquid remained in contact. The mass of gas was determined by weighing the bomb A before and after filling.

The results obtained for hydrogen were least accurate for at high pressures hydrogen passed freely through the pores of the steel cylinders. Air and oxygen may behave in a similar manner, but to a less extent.

Bridgman found that for nitrogen, hydrogen and helium, the product 'pv' was almost a linear function of the pressure—cf. Fig. 9·14(a). At $p = 15 \times 10^9$ gm.-wt.cm.$^{-2}$, the volume of nitrogen

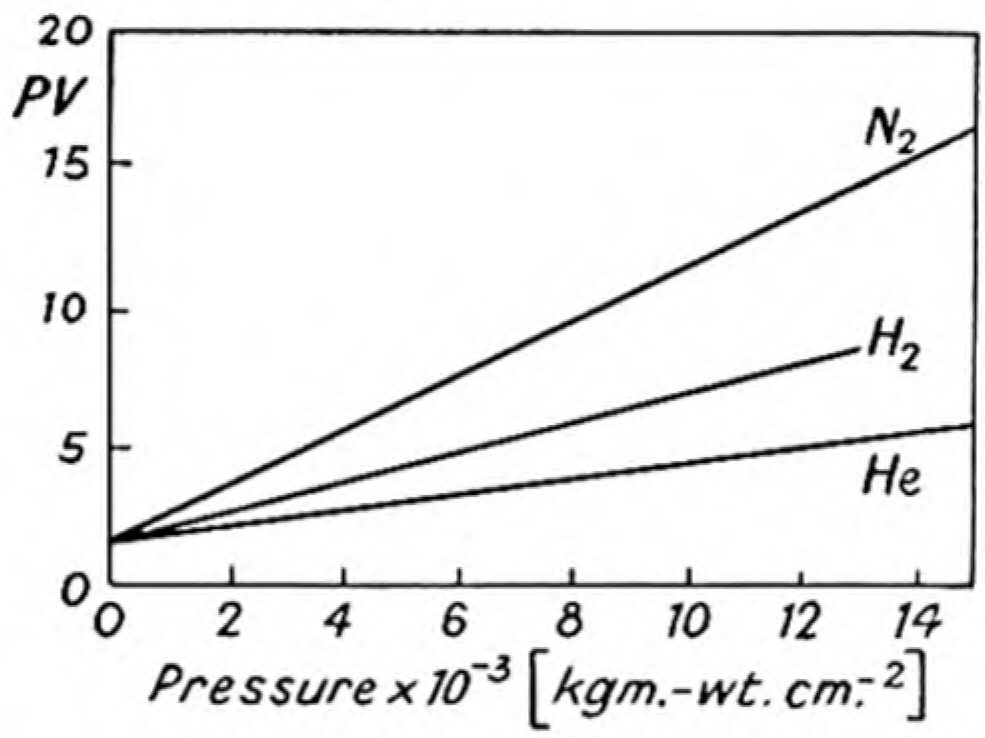


FIG. 9·14(a).—The product PV as a function of pressure for some gases.

was more than sixteen times what it would have been if it had behaved as an ideal gas. Hydrogen and helium behaved similarly. At these high pressures the above gases have compressibilities which are comparable with those of liquids at ordinary pressures.

Fig. 9·14(b) shows an effect which all the gases examined exhibit and which could not be inferred from their behaviour at relatively low pressures. In the diagram the change in volume in cm.^{3} per mole is plotted against the pressure. The initial volume was that of one gram-molecule at a pressure of 3×10^6 gm.-wt.cm.$^{-2}$. A significant feature of this diagram is that the curves for nitrogen and helium intersect at a point other than at the origin, and from the general trend of the curves for hydrogen and argon it appears that they would intersect at a point corresponding to a pressure somewhat above the highest pressure reached by Bridgman. The intersection in the former instance is due to a change in the relative compressibility of the gases with increasing pressure.

To account for this behaviour in the case of helium and nitrogen, it must be remembered that at the comparatively low pressures to which the initial parts of the curves correspond, the decrease in the volume of the gas with pressure has its origin in the diminution

of the 'free or empty space' between the atoms or molecules. At higher pressures this 'empty space' has been so much reduced that further diminution in it is very slow, but it is most probable that the actual space occupied by a molecule or atom is still capable of being reduced by pressure. Now the complicated nitrogen molecule is much larger than an atom of helium so that at the lower pressures

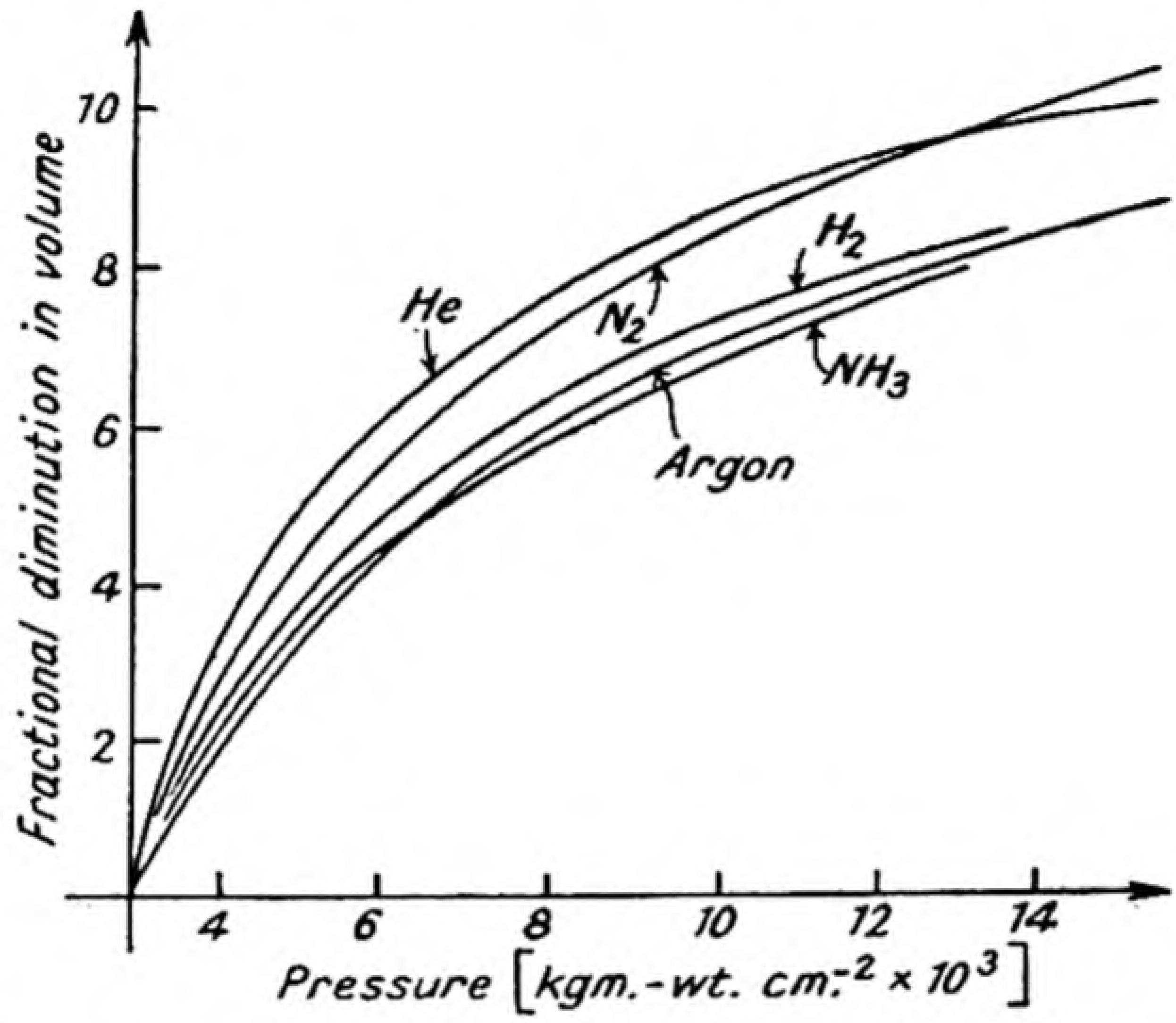


FIG. 9·14(*b*).—The change in volume in cm.^{3} per mole as a function of the pressure.

the main reduction in volume, brought about by a reduction in the 'empty space', is greater in the case of helium, i.e. its compressibility is greater than that of nitrogen. At the higher pressures, however, where it is the electronic configuration of the fundamental unit which largely determines the compressibility, it is the nitrogen molecule which offers the greater possibility in this direction.

The expected crossing of the argon and helium curves is explained in a similar way; the crossing of the ammonia and argon curves is also readily explained when one remembers that a molecule of ammonia has ten extra-nuclear electrons, while argon has eighteen.

The compressibility of liquids at high pressures.—One of Bridgman's latest forms of piezometer for use with liquids is shown diagrammatically in Fig. 9·15. It is a development of a method initiated by PERKINS.† Bridgman had previously improved the method but in its latest form the piston is fitted with extreme accuracy and continuous readings are possible, i.e. it is no longer necessary to open the cylinder after every change of pressure to see the effect produced. The piston is the same diameter as the

† *Trans. Roy. Soc.*, **72**, 324, 1819–20.

body of the steel cylinder, the clearance between them being 10^{-5} in.: recent improvements in the technique of grinding have made such a working limit possible. An electrical device of great sensitivity is used to measure the motion of the cylinder. A high-resistance wire W_1 is attached to the plunger, P, and is in contact with a horizontal wire W_2 rigidly fixed to the cylinder, but insulated from it with mica. C_1 and C_2 are the current leads; P_1 and P_2 the potential leads. W_1 moves across W_2 when the position of the piston changes. A potentiometer is used to measure the potential difference between the sliding contact A and a given point B in the wire W_1, the error from contact resistance being eliminated by this method. The current through W_1 is also measured, and so the resistance of the portion AB of the wire W_1 may be calculated. From the change in this resistance, the displacement of the piston P is determined. The method is of the same order of sensitivity as that in which use is made of the phenomenon of optical interference. On account of its great sensitivity the apparatus can be made small and therefore be mounted in a separate cylinder from that containing the manganin gauge. In this way the temperature of the liquid under examination can be varied while that of the gauge is kept constant. Errors due to a change in the temperature of the gauge are therefore avoided.

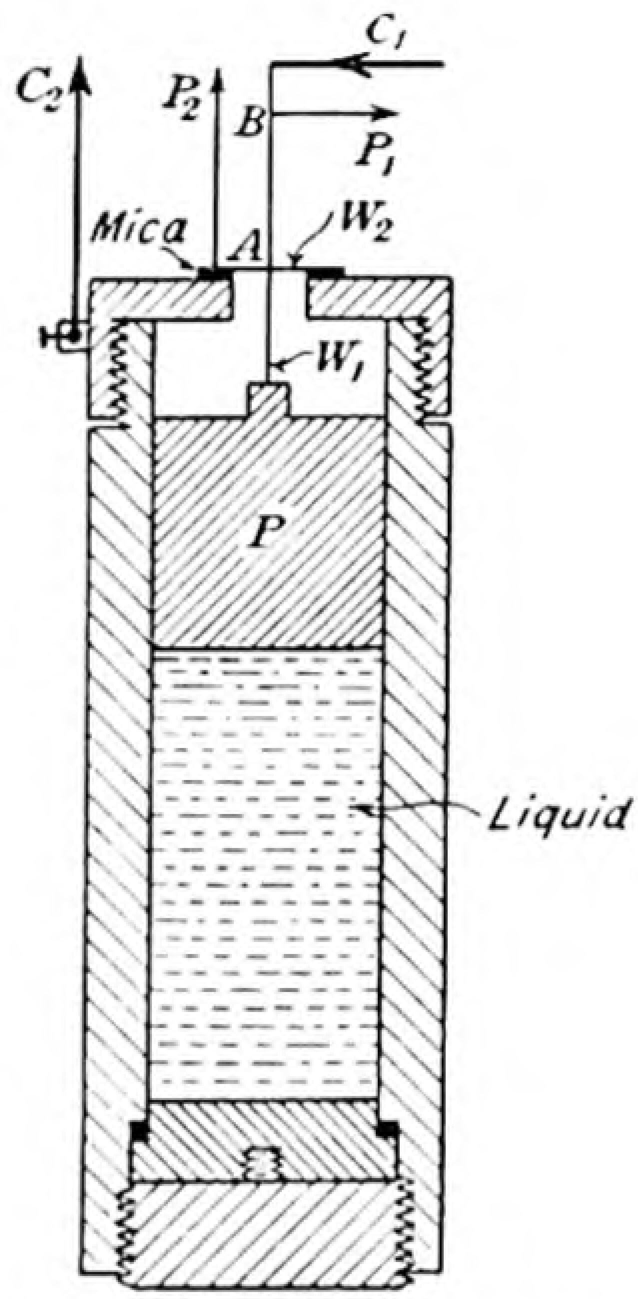


FIG. 9·15.—Bridgman's piezometer for measuring the compressibilities of liquids at high pressures.

Water, mercury, glycerol, several alcohols, ether, acetone, kerosene and three ethyl halides, were among the liquids investigated. With increasing pressure it was found that the compressibility diminished considerably, an absolute necessity if the volume is not ultimately to become zero. A natural explanation of this is that at the relatively low pressures the molecules fit loosely together with a considerable amount of 'free space'. Until this has been considerably diminished the compressibility will be relatively large, but when it has been made more or less nonexistent, the changes in volume which do occur with increasing pressure must then be caused by the shrinkage of the molecules themselves.

The compressibility of matter in the solid state.—No general

method which is direct has yet been devised for measuring the compressibility of a solid; the reason is that the volume of a solid is only obtainable in general by a method which involves immersing it in a liquid contained in some vessel. When a piezometer for solids is used corrections must be applied for the compressibility of the liquid and for the change in volume of the containing vessel under pressure. The method is therefore indirect. We have already discussed the method due to RICHARDS and STULL [cf. p. 425]. In an earlier method due to MALLOCK, the change in length of the major portion of a long cylindrical tube when exposed to internal pressure is measured; according to the theory of elasticity this change is a function of the compressibility of the material of the tube and is independent of the other elastic constants. In this manner GRÜNEISEN has determined the compressibility of several metals at low temperatures.†

Among the methods adopted by Bridgman for measuring the compressibility of a solid, only one will be described. In it the change in length of an iron rod subjected to hydrostatic pressure is measured relative to the change in length of the container, this being made from the material under investigation. It is first necessary, however, to know the compressibility of iron. It is determined in the following way.

An iron bar, 30 cm. long, is placed inside a heavy cylinder and exposed to pressure, this being transmitted to the apparatus by a fluid with which it is filled. The relative change in length of the rod and the cylinder is measured, and at the same time the change in length of the cylinder is measured at outside points. The absolute change in length of the iron rod can then be computed. In making this, it is assumed that the change in length of the cylinder externally, where it is measured, is the same as that internally where it is in contact with the rod. This assumption is legitimate since the cylinder extended a considerable distance beyond the ends of the rod so that end effects are negligible. In any case, however, the correction for the change in length of the cylinder is small, not exceeding a few per cent.

The method of determining the relative change in length between the iron and the cylinder is the potentiometer method cited above. The constant current is passed through the wire independently of the potential terminals. Readings of the potential difference between the sliding contact and a fixed point in the wire are never made until after the lapse of a few minutes from the time of changing the pressure. This is essential in order to allow the heat of compression to be dissipated. In connexion with this electrical method Bridgman emphasizes that the wire must be exceedingly uniform

† *Ann. Phys.*, **33**, 1269, 1910; **39**, 257, 1912.

and should be of such a material that its resistance is not appreciably affected by pressure changes. Moreover its resistivity must be large. For these reasons the ultimate choice of material was nichrome.

For pure iron Bridgman† found, at temperatures indicated by suffixes,

$$\kappa_{30} = [5{\cdot}87 - (2{\cdot}1 \times 10^{-5}P)](\text{kgm.-wt.})^{-1}\text{cm.}^2,$$

and

$$\kappa_{75} = [5{\cdot}93 - (2{\cdot}1 \times 10^{-5}P)](\text{kgm.-wt.})^{-1}\text{cm.}^2,$$

where P is the pressure in kgm.-wt. cm.$^{-2}$.

When the absolute compressibility for one metal had been obtained accurately, those of other metals could be investigated by a differential method. At relatively low pressures Richards had made extensive use of such a method. He realized fully, however, that his results would have to be corrected when more accurate values for the compressibility of mercury and that of the material (iron) of the containing vessel were known.‡ In this he was justified for he used a value for the compressibility of iron which was about 30 per cent too low.

When the changes in length of the iron and the material under investigation were relatively large, Bridgman used the apparatus shown in Fig. 9·16. S is the specimen in the form of a long rod; it is kept pressed against the bottom of the iron vessel in which it is contained by a spring, represented very diagrammatically by the two 'force vectors,' FF, acting downwards. A high resistance wire (nichrome) W_1 is attached to a holder at the upper end of the specimen; it is capable of movement across a contact A, attached to the containing vessel but insulated from it with mica washers. A spring N keeps W_1 in contact with A. C_1 and C_2 are current leads, while P_1 and P_2 are

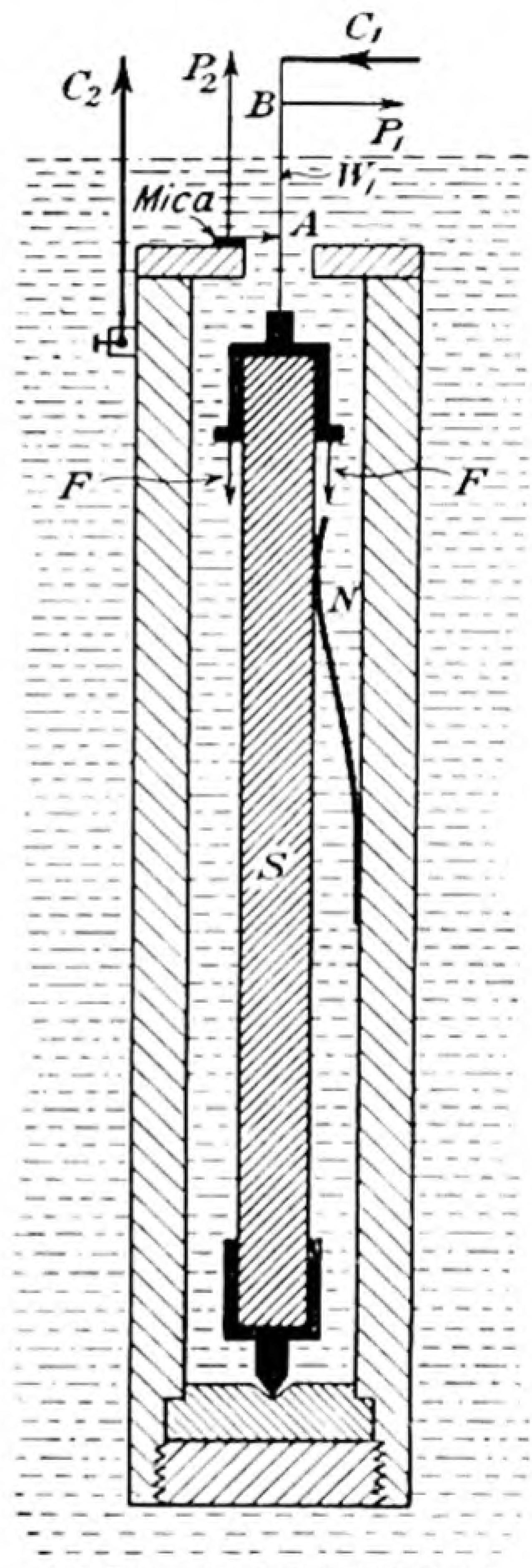


FIG. 9·16.—Bridgman's piezometer for determining the relative linear compressibility with respect to iron of materials available in the form of rods.

† cf. Appendix.

‡ *Jour. Amer. Chem. Soc.*, **37**, 1643, 1915.

potential leads. From the observed changes in the difference of potential between the sliding contact A and a fixed point B in W_1, the current being kept constant, the relative motion of the rod and the iron cylinder is found. To bring about these changes in length the whole apparatus is placed in a high pressure apparatus and the pressure applied hydraulically.

To eliminate the effects of various disturbing factors, such as the change in length of the holder whereby W_1 is fixed to the specimen, and for the changes in the thickness of the mica washers, a 'blank experiment' is performed in which S is replaced by a cylinder of pure iron, i.e. by a material of known compressibility.

When only short specimens are available the relative motion of one end of the specimen is magnified by an arrangement of levers, and then a potentiometer method is used to determine the change in position of the end of the last lever.

The following results were obtained:—

For tungsten,

$$\kappa_{30} = [3{\cdot}15 - (1{\cdot}6 \times 10^{-5}P)] \times (10^{-7}\ \text{kgm.-wt.})^{-1}\text{cm.}^2.$$

$$\kappa_{75} = [3{\cdot}16 - (1{\cdot}5 \times 10^{-5}P)] \times (10^{-7}\ \text{kgm.-wt.})^{-1}\text{cm.}^2.$$

For gold,

$$\kappa_{30} = [5{\cdot}77 - (3{\cdot}1 \times 10^{-5}P)] \times (10^{-7}\ \text{kgm.-wt.})^{-1}\text{cm.}^2.$$

$$\kappa_{75} = [5{\cdot}70 - (2{\cdot}1 \times 10^{-5}P)] \times (10^{-7}\ \text{kgm.-wt.})^{-1}\text{cm.}^2.$$

General remarks on the compressibility of solids.— Richards first pointed out, and Bridgman substantiated his remarks, that the compressibility of the elements in the solid state is a periodic function of the atomic number. The alkali elements are the most compressible of all the elements in the solid state which have been investigated. It is probable that the compressibility of the rare gases in the solid state would be greater, but a direct determination, of the severest experimental difficulty, has yet to be made.

EXAMPLES IX

9·01. Explain what is meant by the elasticity of a solid. How is the volume elasticity defined? How may its value be determined in the case of a given solid?

9·02. Obtain a value for the density of sea water at a depth of 4 miles below the surface assuming that the density at the surface is 1·025 gm.ml.$^{-1}$ and that the compressibility of sea water is $4{\cdot}3 \times 10^{-5}$ dyne.bar^{-1}. [1 bar = 10^6 dyne.cm.$^{-2}$.]

CHAPTER X

SURFACE TENSION

A fundamental property of all liquid surfaces and a molecular explanation thereof.—All liquids not in contact with solids tend to alter the shape of their surfaces so that these have a minimum area. This is a fundamental property of every liquid and exhibits itself in many ways. For example, small quantities of liquids tend to become spherical in shape, and soap films tend to become less extended. PLATEAU (1873) made an exhaustive study of the shapes assumed by the surfaces of liquid drops when they were not affected by gravitational forces. He used (*a*) soap films, which are so thin that their weight is negligible, and (*b*) drops of olive oil suspended in a mixture of water and alcohol of practically the same density. Spheres several centimetres in diameter were obtained in this way. He came to the conclusion that if R_1 and R_2 are the principal radii of curvature at any point on the surface of a liquid drop in a zero gravitational field, then

$$\frac{1}{R_1} + \frac{1}{R_2} = \text{constant}.$$

Geometrically, it can be shown that, for a given volume of liquid, the surface area is a minimum when the principal curvatures at any point on the surface satisfy the above relation.

To account, on a molecular hypothesis, for this inherent property of all liquids, it must be remembered that in fluids (i.e. liquids and gases) the constituent molecules, which are always particles of definite size and shape, are free to move relatively to each other; in liquids the motions are much restricted by the cohesional forces between the molecules. This freedom of the molecules to move in liquids and gases distinguishes them at once from solids. They are said to possess ***fluidity***.

Now molecules in the interior of a liquid are subjected to molecular forces in all directions, i.e. to forces due to the attraction between neighbouring molecules. The resultant of these forces on any given such molecule is, on the average, very small and the molecule does not rapidly move away from a given position. For molecules near to the surface of the liquid, however, conditions are essentially different. Considering any one such molecule, it appears that it

can only be attracted by forces which have a component directed towards the interior of the liquid, if we neglect the effect of the relatively few molecules existing as vapour near to the surface. Every such molecule will therefore experience a relatively large resultant force tending to make it move inwards, and while there will be a large number of molecules experiencing such forces there will be relatively few molecules moving outwards to take their vacant places. Thus the inherent tendency for a liquid surface to contract is readily explained.

Surface energy and surface tension.—In virtue of the resultant forces on surface molecules whereby the surface of a liquid is caused to contract until it contains a minimum number of molecules, i.e. the surface is the smallest possible for a given volume under the action of any external forces which may be present, it follows that every surface molecule must possess potential energy, i.e. energy in virtue of its position. The amount of this energy per unit area of the surface is termed the ***intrinsic surface energy*** or the ***surface energy density*** of the substance. The surfaces of both liquids and solids possess surface energy but it is only when the surface is mobile that its effects become apparent. The fact that a liquid surface is the seat of potential energy manifests itself very vividly when a soap film is ruptured (with the aid of a pointed piece of filter paper, for example), for the liquid is projected in all directions with a considerable velocity, i.e. the potential energy has been converted into kinetic energy.

Let a liquid film be formed between two limbs of a bent wire, BAC, Fig. 10·01, and a horizontal straight wire, XY, placed across them. Suppose that a force, F, acting normally to XY is necessary to maintain equilibrium when the film is vertical. Then F must be balanced by a force on the wire due to the film. Suppose γ is the magnitude of this force per unit length of the wire. If the length of XY is l, the total force on the wire from the above cause is $2\gamma l$, the factor 2 being introduced since the film has two sides. Hence

$$F = 2\gamma l.$$

γ is termed the ***surface tension*** of the liquid.

[It should be noted that if parallel wires are used for the purpose of forming a film between them, the system is unstable. For example, if F is too large, the force $2\gamma l$ never becomes sufficient to balance F for l remains constant. The instability does not matter as far as theory is concerned, but with the stable arrangement here adopted a rough estimate of γ may be made. If the weight of the wire is not sufficient it may be loaded. Then $F = mg$, where m is the total mass of the wire and its load.]

On the relation between intrinsic surface energy and surface tension.—Let XY, Fig. 10·01, move through a small distance δx to a parallel position X_1Y_1, the external force on the wire being F. Now when a film is stretched in this way its temperature falls unless heat (thermal energy) is communicated to it. We shall suppose that the heat necessary to restore the film to its original temperature has been supplied.

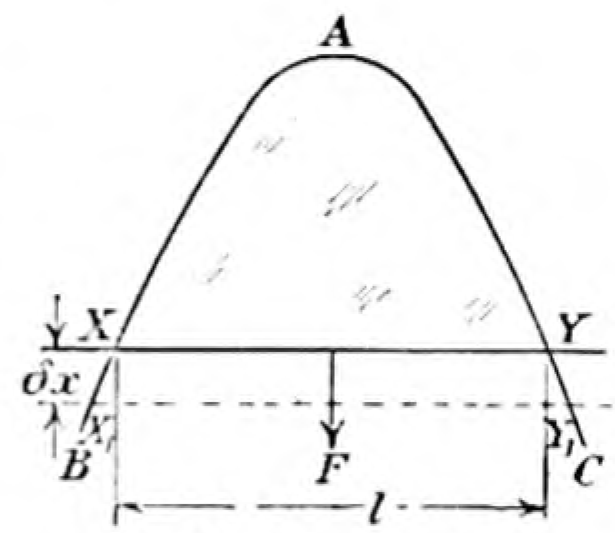


Fig. 10·01.—Surface tension.

If ε is the intrinsic surface energy of the film, i.e. the potential energy per unit area of the surface, the increase in potential energy of the 'surface' molecules is $(2l\ \delta x)\varepsilon$, the factor 2 being introduced since the film has two surfaces. The work done by the stretching force is F δx. These two quantities cannot be equated, however, for heat has been communicated to it from external bodies. If δQ is the heat (thermal energy) supplied to restore the temperature of the film to its original value, we have

$$(2l\ \delta x)\epsilon = \mathrm{F}\ \delta x + \delta \mathrm{Q}.$$

Now the force F is equal and opposite to the pull of the film on the wire XY, when the film is in equilibrium. If γ is the pull per unit length, then $2l\gamma = \mathrm{F}$, and the above equation becomes

$$(2l\ \delta x)\epsilon = 2\gamma l\ \delta x + \delta \mathrm{Q},$$

or

$$\epsilon = \gamma + \left(\frac{\delta \mathrm{Q}}{2l\ \delta x}\right).$$

This may be written

$$\epsilon = \gamma + \eta,$$

where $\eta = \dfrac{\delta \mathrm{Q}}{2l\ \delta x}$, the thermal energy supplied per unit increase in area of the film.

Now the force γ exerted on each unit length of the wire is called the ***surface tension*** of the liquid and the above shows that the intrinsic surface energy of a liquid is really the sum of the two quantities—a 'thermal' part denoted by η, and a 'mechanical' part γ, or $\epsilon - \eta$; we see, therefore, that the surface tension is equal to the 'mechanical' part of the intrinsic surface energy. Helmholtz called this 'mechanical' part of the surface energy density the ***free surface energy density.*** It will be noted also that the increase in the total free surface energy of a surface is equal to the external work done on that surface, provided heat is supplied to keep its temperature constant.

Free surface energy density or surface tension.—In solving problems concerning the shapes and positions of liquid surfaces under various conditions, it is often very convenient to use the following mathematical device; any necessary virtual change in the shape of the surface is assumed to take place under isothermal conditions and the free surface energy per unit area is replaced by a force in the surface of the liquid. If a short line of length δs is drawn in the surface, the force on it is $\gamma\, \delta s$, and is directed along the normal to δs in the surface of the liquid. γ is called the ***surface tension*** of the liquid. Such a process is always possible since the dimensions of free surface energy and of surface tension are respectively $\frac{ML^2T^{-2}}{L^2}$ and $\frac{MLT^{-2}}{L}$, both of which reduce to MT^{-2}.

The sharpness of a liquid surface in contact with its own vapour (or with air).—Experimental evidence supports the view that the transitional layers between a liquid and its own vapour (or the superincumbent gas—generally air) are only one or two molecules thick, i.e. the rate of change of density in a direction normal to the surface is very rapid in the neighbourhood of the surface. The strongest evidence in support of the above comes from the theory of optics. If the transition from a liquid of refractive index μ to a gas or vapour above it is absolutely abrupt, light reflected from such a surface will be completely plane polarized if the angle of incidence is $\tan^{-1} \mu$, but if there is a transitional layer of gradually changing refractive index from that of the liquid below to that of the gas above, then the reflected light will always be elliptically polarized. RAYLEIGH† working with a water surface from which the accidental layer of grease had been removed—by a method to be described later [cf. p. 465]—showed that there was practically no trace of ellipticity in the light reflected from water when the angle of incidence was $\tan^{-1} 1{\cdot}333$. It is now known that such a grease layer is only one molecule thick so that the transition is very abrupt.

More recently, RAMAN and RAMDAS have detected a slight ellipticity even for very clean water, but this is probably to be attributed to the fact that the molecules at the surface are thermally agitated.

Example.—A soap film is enclosed within a triangular frame which is in a vertical plane with two of its sides equally inclined to the vertical and with the wire forming the base of the triangle horizontal but free to slide up under the action of surface tension forces or down under its own weight. Show that an equilibrium position may be reached and, neglecting damping forces, determine the period of oscillations of the wire base about its position of static equilibrium.

Fig. 10·02(a) shows the static position of the wire, mass m and length

† *Phil. Mag.*, **33**, 1, 1892.

l in contact with the film. If γ is the surface tension of the solution from which the film is formed, for equilibrium, we have

$$mg = 2\gamma l,$$

where g is the intensity of gravity. In Fig. 10·02(b) the wire is shown displaced downwards a distance z from its equilibrium position. If l' is

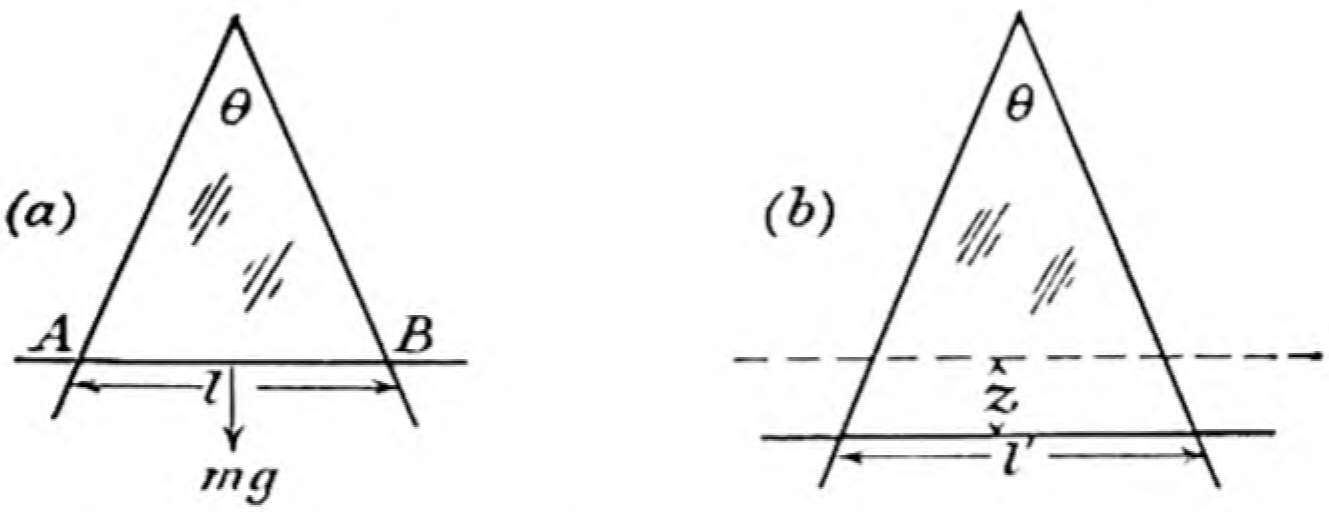


FIG. 10·02.—A soap film on a triangular framework.

the length of wire in contact with the film and θ the angle at the vertex of the triangle,

$$\tan \tfrac{1}{2}\theta = \frac{l'-l}{2z}.$$

The equation of motion is, if it is assumed that isothermal conditions prevail and damping is neglected,

$$m\ddot{z} = mg - 2\gamma l' = 2\gamma(l - l')$$

$$= -\frac{mg}{l}.2z.\tan \tfrac{1}{2}\theta.$$

$$\therefore \ddot{z} + \left(\frac{2g}{l}\tan \tfrac{1}{2}\theta\right) z = 0,$$

so that the motion is simple harmonic with a period T given by

$$T = 2\pi\sqrt{\frac{l}{2g \tan \frac{1}{2}\theta}}.$$

[N.B. If $\theta \rightarrow 0$, $T \rightarrow \infty$ but so many assumptions have been made in deriving the above expression that it cannot be used to discuss the stability of the wire.]

The pressure difference across a spherical surface.—Let r, Fig. 10·03(a), be the radius of a spherical bubble of gas in a liquid. Let P be the pressure outside the bubble. We have to show that the pressure inside is equal to $P + p$, where p is a quantity to be determined. For this purpose let r become $r + \delta r$, where δr is so small that the pressure inside is not altered thereby. Moreover, let heat be supplied to the surface of the bubble so that its original temperature is restored. If $A = 4\pi r^2$, then δA, the increase in area of the surface associated with the change in radius is given by $\delta A = 8\pi r\, \delta r$. If γ is the surface tension of the liquid or, as we have just seen, its free surface energy per unit area, the increase in the

total free surface energy is $8\pi\gamma r\,\delta r$. This is equal to the work done in expanding the bubble. To evaluate this work let δS be an element of the surface of the bubble. The resultant force acting outwards on it is

$$(P + p)\,\delta S - P\,\delta S = p\,\delta S,$$

so that the work done in effecting the expansion is

$$\delta r . \int p\,dS = p . 4\pi r^2 . \delta r.$$

Equating the two expressions obtained for the work done, we have

$$4\pi r^2 p\,\delta r = 8\pi\gamma r\,\delta r,$$

or

$$p = \frac{2\gamma}{r}.$$

If the bubble had been a soap bubble this excess pressure would have been $\frac{4\gamma}{r}$, for a soap film has a double surface.

The fact that the pressure inside a soap bubble diminishes as the radius increases is shown by the following experiment. Two brass cups, X and Y, Fig. 10·03(*b*), about 2 cm. in diameter and 1 cm. long, are connected to stop-cocks A, B and C as shown.

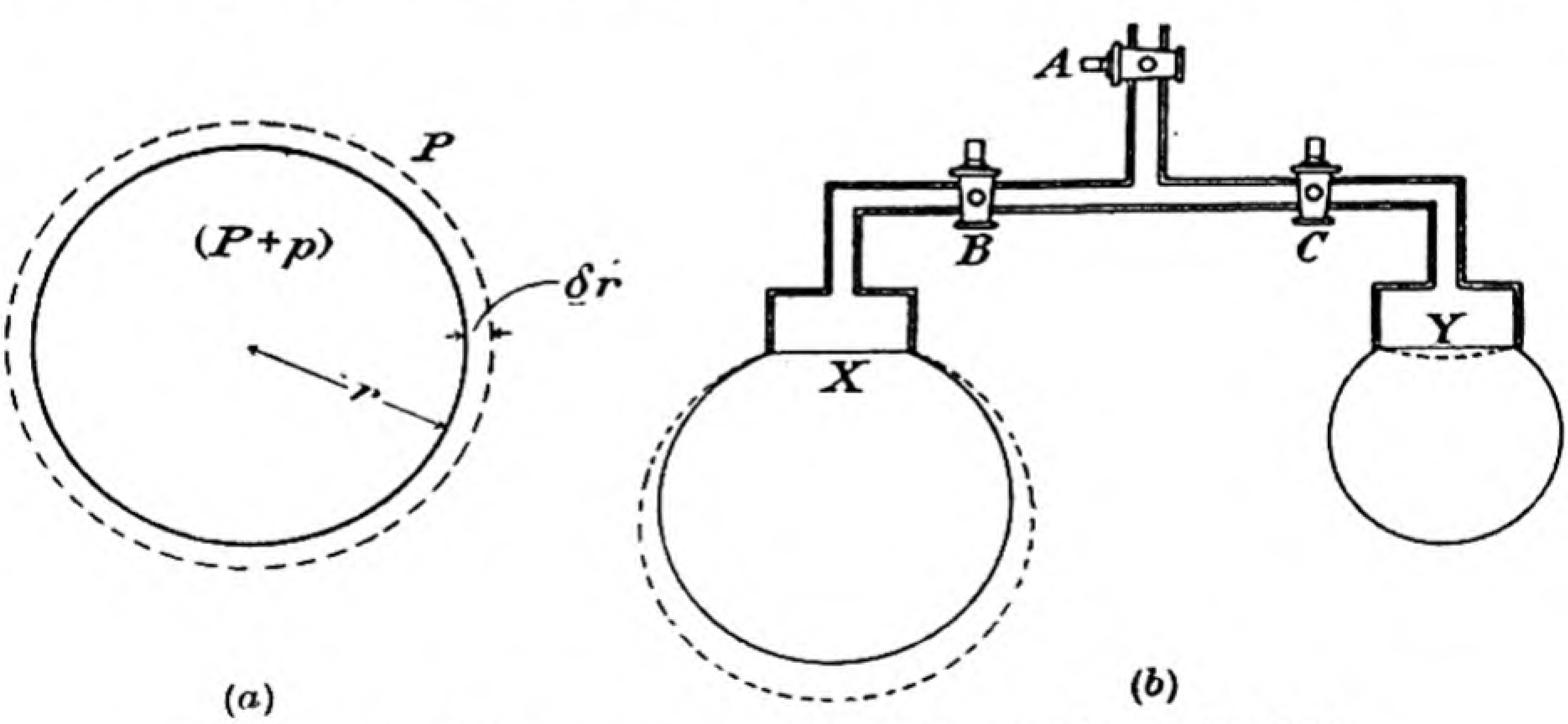


Fig. 10·03.—The pressure excess inside a spherical bubble.

The open ends of X and Y are immersed in a soap solution and soap bubbles differing considerably in diameter blown. B is open and C closed while the larger bubble is being formed, and vice versa. A is then closed and the two bubbles are placed in communication with each other by opening the stop-cocks B and C. Air passes from the smaller bubble into the larger one, causing the latter to expand and the former to shrink. This process continues until the

radius of curvature of the larger bubble is equal to the radius of curvature of the soap film which finally protrudes below the open end of Y and which is a portion of a spherical surface, cf. see the dotted outlines on the diagram. After a time the thickness of the walls of the large bubble become so thin that it bursts: the film remaining on Y at once becomes flat, and after some time very thin; it finally breaks.

Pressure difference across a cylindrical surface.—Let us now assume that Fig. 10·03(a) represents the cross-section of a cylindrical bubble. Since it is difficult to produce such a bubble in a liquid we will assume that it consists of a soap film having ***two*** surfaces. Consider a length l of this cylinder. When r becomes $r + \delta r$, as before, the increase in area is $2[2\pi(r + \delta r - r)l]$. Let thermal energy be supplied to the film so that its temperature assumes its original value. The increase in the free surface energy is $4\pi l\gamma\,\delta r$. Now the work done, calculated by the method used in the previous paragraph, is $2\pi rlp\,.\,\delta r$. Equating these two quantities we have

$$p = \frac{2\gamma}{r}.$$

When there is only one cylindrical surface the excess pressure is $\dfrac{\gamma}{r}$.

The pressure difference across any curved surface free from discontinuities.—The expression for the difference in pressure across a curved surface of the general type just stated which we now proceed to obtain, was known to Young and to Laplace. It is the foundation of most precision methods for determining the surface tension of a liquid. Let ABCD, Fig. 10·04(a), be a small portion of a curved surface liquid, the angles between adjacent sides being right angles so that opposite sides of the element under consideration are equal. Let the surface be convex with reference to a point above it. Let the normals to the surface at A and B intersect at O_1, so that $AO_1 = BO_1 = r_1$ (say), while those at B and C intersect at O_2, where $BO_2 = CO_2 = r_2$ (say). Let P be the pressure on the convex side and $(P + p)$ that on the concave side of the surface. Let γ be the surface tension of the liquid, i.e. its free surface energy density, and suppose the drop of liquid of which ABCD is part of its surface expands so that this element occupies the position $A_1B_1C_1D_1$, the temperature remaining constant. Let δn be the displacement of each point of the element of surface considered. The forces, due to the pressures on the respective sides of the element, are P (area ABCD) and $(P + p)$ (area ABCD). Hence, as far as this portion of the surface is concerned, the external work done is

$$p(\text{ABCD})\,\delta n.$$

The increase in the free surface energy of the element of surface considered is γ times its increase in area, i.e.

$$\gamma(A_1B_1C_1D_1 - ABCD) = \gamma[(A_1B_1 . B_1C_1 - AB . BC)],$$

since each surface element is rectangular.

But

$$\frac{A_1B_1}{r_1 + \delta n} = \frac{AB}{r_1}, \qquad \text{or} \qquad A_1B_1 = AB\left(1 + \frac{\delta n}{r_1}\right),$$

and similarly $$B_1C_1 = BC\left(1 + \frac{\delta n}{r_2}\right).$$

$\therefore$ Increase in free surface energy

$$= \gamma\left[AB . BC\left(1 + \frac{\delta n}{r_1}\right)\left(1 + \frac{\delta n}{r_2}\right) - AB . BC\right]$$

$$= \gamma\ \delta n\left(\frac{1}{r_1} + \frac{1}{r_2}\right)AB . BC,$$

if the term containing $(\delta n)^2$ is neglected. Since this increase in free surface energy is equal to the work done in expanding the surface

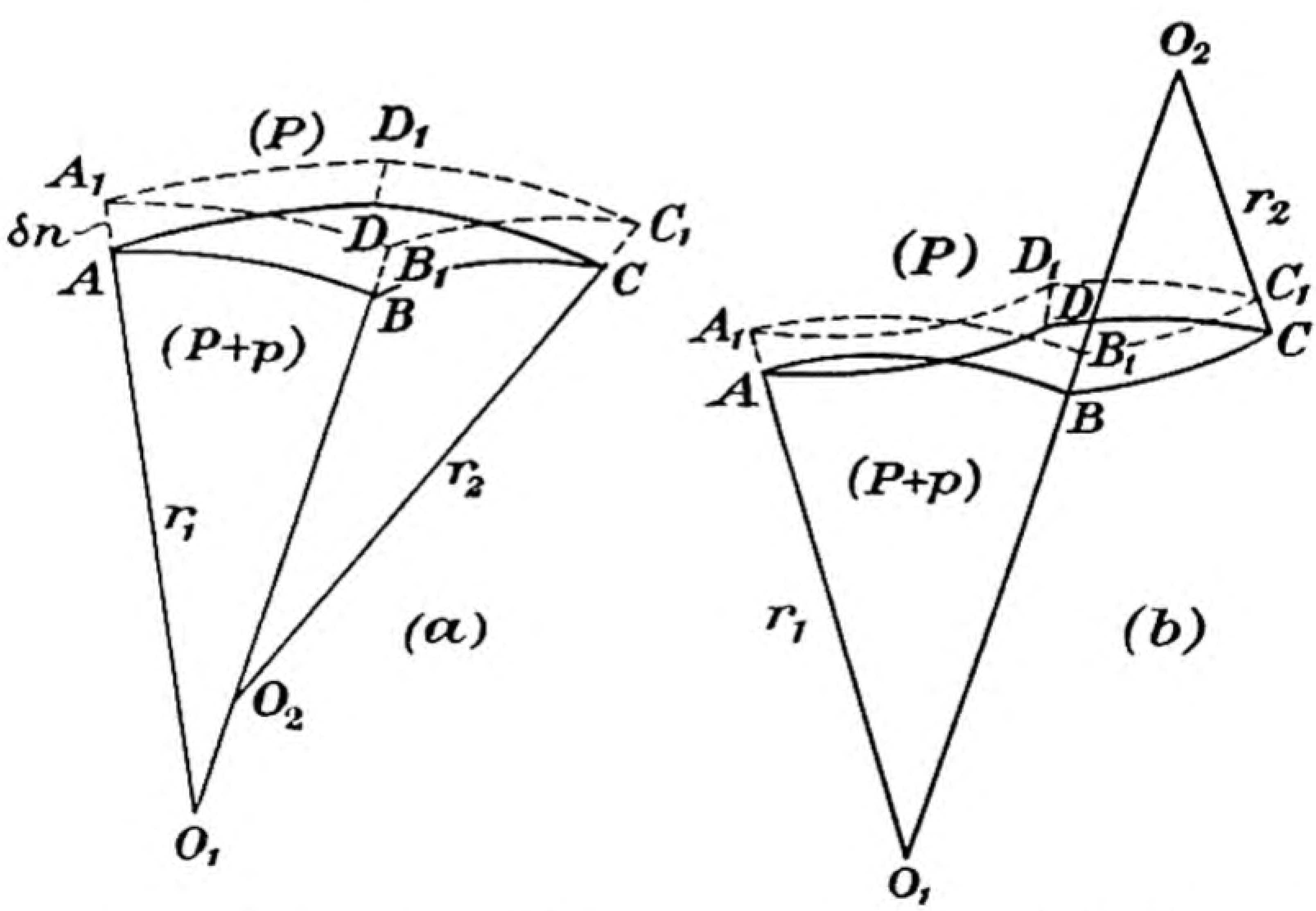


FIG. 10·04.—The pressure difference across any curved liquid surface.

under the condition that heat is supplied to the film so that there is no change in temperature, we have

$$p = \gamma\left(\frac{1}{r_1} + \frac{1}{r_2}\right).$$

So far it has been assumed that the points O_1 and O_2 lie on the same side of the surface. If they do not, but are as in Fig. 10·03(*b*), then

$$\frac{B_1C_1}{BC} = \frac{r_2 - \delta n}{r_2}, \qquad \text{or} \qquad B_1C_1 = BC\left(1 - \frac{\delta n}{r_2}\right).$$

Hence, as before,

$$p.(AB.BC).\delta n = \gamma AB\left(1 + \frac{\delta n}{r_1}\right)BC\left(1 - \frac{\delta n}{r_2}\right),$$

or

$$p = \gamma\left(\frac{1}{r_1} - \frac{1}{r_2}\right).$$

If it is agreed to consider a radius of curvature as positive or negative according as it lies on the side where the pressure is greatest or least, respectively, then we have the general formula

$$p = \gamma\left(\frac{1}{r_1} + \frac{1}{r_2}\right),$$

from which at once follow the values for the pressure difference when the surface is spherical or cylindrical.

If the surface is that of a film, the pressure difference will in each instance be doubled since there are two surfaces; of course the film is so light that its weight is negligible compared with the other forces involved.

Interfacial tension and the adhesion between two liquids.— When two non-miscible liquids are at rest, i.e. in static equilibrium, there is a definite surface of demarcation between them. Let us suppose that at this surface there is a layer of molecules of one type 'resting' on a layer of molecules of the other type with no inter-mixing and that only molecules of one type are to be found either above or below the liquid interface. First consider those in the upper layer. They will be attracted upwards, on the whole, by the molecules above them, but will also experience an attraction in the opposite direction due to the neighbouring molecules of the other liquid. Thus the molecules we are considering will possess energy in virtue of their position. Similarly for the molecules in the second layer. In virtue of these facts there must exist a definite amount of surface energy at the interface between two liquids, the amount per unit area being termed the ***intrinsic interfacial energy*** of the liquids. It follows, therefore, that when an interface between two liquids expands work will have to be done. Consider two non-miscible liquids, A and B, in a column of ***unit cross-sectional area***.

Let ϵ_{ab}, ϵ_a and ϵ_b, be the intrinsic surface energies of the interface and of the liquids respectively. Let γ_{ab}, γ_a and γ_b be the corresponding free surface energy densities or surface tensions.

Suppose that by means of a direct pull it were possible just to separate the two liquids completely. Let W_{ab} be the work done. Let η_a and η_b be the amounts of heat (measured in work units) to be supplied per unit area of the new surfaces formed in order to maintain isothermal conditions in the case of the liquids A and B; let η_{ab} be the corresponding quantity for the interface. Then in the present instance unit area of the interface has been destroyed and two new surfaces, one of liquid A and the other of liquid B, each of unit area, have been produced. The heat to be supplied will therefore be $-\eta_{ab} + \eta_a + \eta_b$ for both surfaces. After the change has been brought about the intrinsic surface energy will be $\epsilon_a + \epsilon_b$. Hence, from the principle of the conservation of energy,

$$\epsilon_{ab} + W_{ab} - \eta_{ab} + \eta_a + \eta_b = \epsilon_a + \epsilon_b.$$

Since, in general, $\epsilon = \gamma + \eta$, the above equation may be written

$$\gamma_{ab} + \eta_{ab} + W_{ab} - \eta_{ab} + \eta_a + \eta_b = \gamma_a + \eta_a + \gamma_b + \eta_a,$$

or

$$W_{ab} = \gamma_a + \gamma_b - \gamma_{ab}.$$

This equation is due to DUPRÉ† and is of fundamental importance. In the first place it shows that for given values of the surface tension of two liquids, γ_a and γ_b, the greater the ***work of adhesion***, i.e. W_{ab}, the smaller is the interfacial tension γ_{ab}, where the work of adhesion is defined as the work required per unit area of an interface just to separate two liquids from each other by a pull normal to the common interface and against the forces of adhesion between them, the temperature remaining constant. Secondly, it provides the necessary condition for the complete miscibility of two liquids, viz. that the interfacial tension between the two liquids should be zero or negative. Now when the interfacial tension γ_{ab} is negative it means that any interface will tend to increase in area, for when it is positive the interface tends to decrease in area. The tendency for the interface to increase implies, at once, that the liquids will tend to mix, i.e. they will be miscible. In other words the movement of the molecules across any imaginary boundary which we may have supposed to exist takes place quite freely.

There is, at present, no direct method for measuring the work of adhesion between two given liquids; using Dupré's equation its value must be deduced from those of the surface tension of the liquids and their interfacial tension.

Contact between solids, liquids and gases.—Let Fig. 10·05 represent a vertical section through a solid, liquid and a gas, all in contact and in equilibrium, the line of contact being assumed

† *Théorie Mécanique de la Chaleur*, 1869.

perpendicular to the plane of the paper and of length δl. The forces acting on the molecules in this line are γ_{LG}, γ_{LS} and γ_{SG} per unit length where the meaning of these symbols is at once apparent. Let ϕ be the angle indicated, i.e. the angle between the tangent planes to the liquid and to the solid along their line of contact, or the ***angle of contact*** for the given liquid and solid. Since there is equilibrium, we have,

$$\gamma_{LS} + \gamma_{LG} \cdot \cos \phi - \gamma_{SG} = 0 \quad . \quad \text{(i)}$$

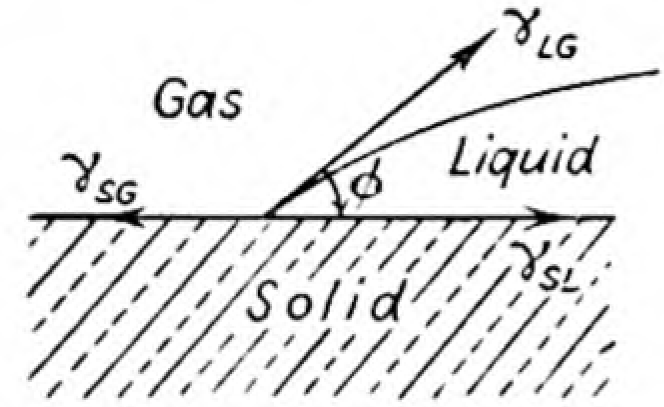


FIG. 10·05.—Liquid resting on a solid surface.

But Dupré's equation for the liquid-solid system is

$$W_{LS} + \gamma_{LS} - \gamma_{LG} - \gamma_{SG} = 0.$$

Subtracting the above two equations in order to eliminate the unknown solid-liquid and solid-gas surface tensions, we have

$$W_{LS} = \gamma_{LG}(1 + \cos \phi) \quad . \quad . \quad \text{(ii)}$$

Now W_{LS} is a measure of the adhesion between the liquid and solid while γ_{LG}, or better $2\gamma_{LG}$, is a measure of the adhesion in the liquid itself, i.e. its cohesion, a fact at once obtained by writing down Dupré's relation for a liquid in contact with itself when the interfacial tension is necessarily zero. Hence the above equation proves that ϕ, the angle of contact between the liquid and solid, in a given instance, is determined by the relative magnitudes of the adhesion between the solid and liquid and the cohesion of the liquid. When ϕ is zero, i.e. the liquid completely 'wets' the solid with which it is in contact, $W_{LS} = 2\gamma_{LG}$, so that in this instance the adhesion between the liquid and solid is equal to the cohesion in the liquid. [If $0 < \phi < \frac{1}{2}\pi$, the solid is only partially wetted by the liquid; if $\frac{1}{2}\pi < \phi < \pi$, there is no wetting.]

Angles of contact and their measurement.—It has just been shown that, in general, there is an angle of contact for every liquid at rest on a solid. Thus, if a piece of clean glass is inserted into water so that it is in a vertical position, it will be found that the liquid near the glass has been drawn some distance beyond the level of the rest of the water. The $\widehat{ABC}$, Fig. 10·06(a), i.e. the angle between the solid surface in the water and the tangent to the water surface where it meets the glass, is called the angle of contact for a water-glass interface. For water in contact with glass this angle is very small, while for benzol in contact with glass it is zero.

When the above experiment is repeated with mercury the liquid near the glass is depressed below the general level of the mercury surface. The angle of contact is again ABC, Fig. 10·06(*b*), but it is now quite large (approximately 135°). It should be noted that, although the surface tensions of two liquids may be equal, they may not exhibit the same capillary phenomena, for their angles of contact with a given material may be different.

The effect of the angle of contact on the shape of a small quantity

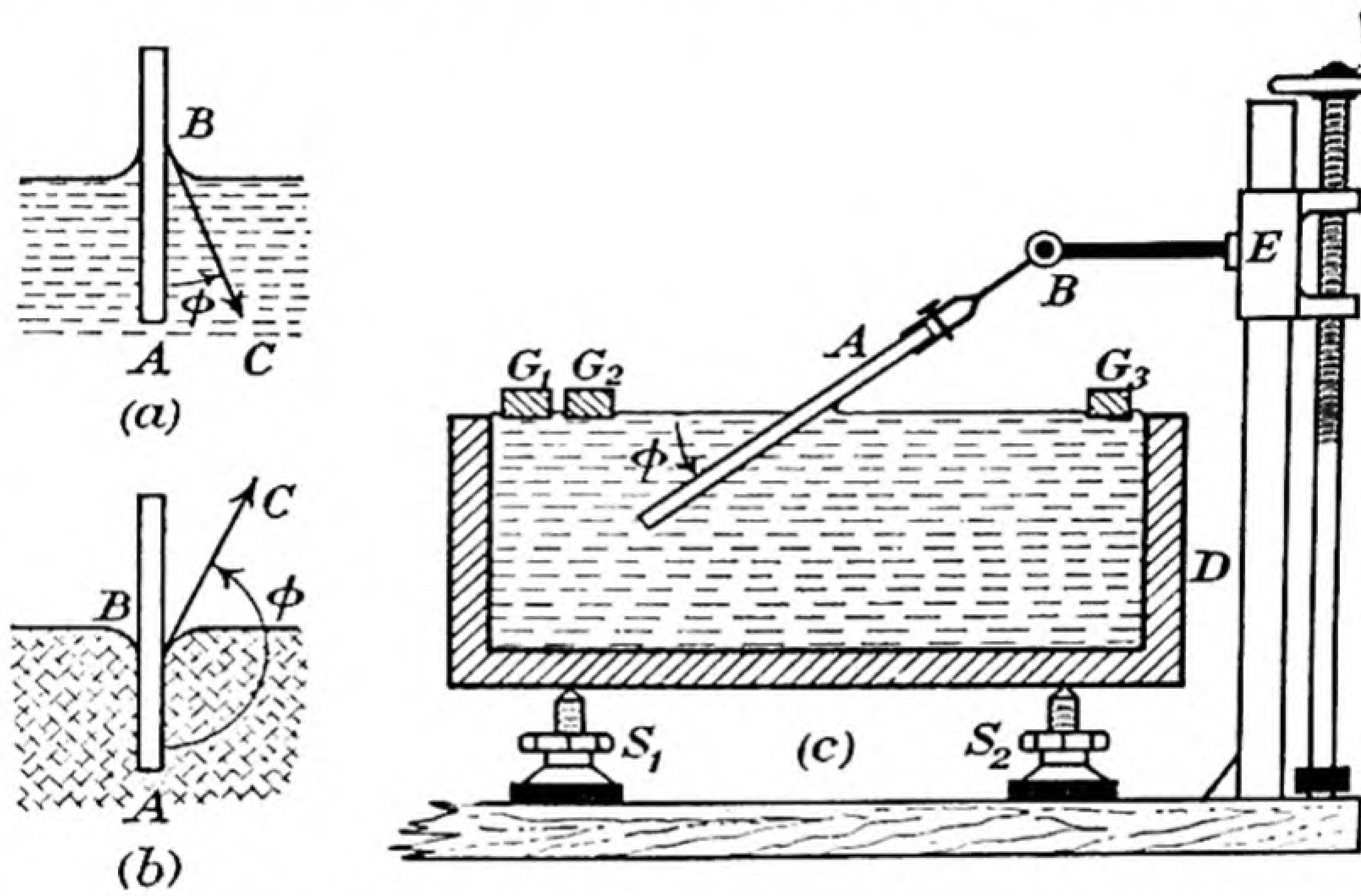


FIG. 10·06.—Angles of contact and their measurement.

of liquid placed on a flat surface is easily shown as follows. Water placed on a clean glass surface spreads itself over the glass, but if water is similarly placed on a greased plate it remains as a 'drop'. Traces of dirt or grease alter the angle of contact very considerably; that is the reason why rainwater persists as a drop when it alights on a window-pane, for such a piece of glass is never chemically clean.

To determine the angle of contact between water and glass coated with paraffin wax, N. K. ADAM used an apparatus similar to that shown in Fig. 10·06(*c*). A is a section of the plate at right angles to its faces. It is held in a clamp which may be rotated about a horizontal axis through B. The clamp may be moved vertically by means of the screw C, and the carriage E which it operates.

D is a glass trough, coated inside with paraffin wax so that it may be filled with water above the level of its sides which have been ground flat on the top. This surface is made horizontal with the aid of screws S_1 and S_2. G_1, G_2 and G_3 are rectangular pieces of glass coated with wax; they rest on the sides of the trough and are

in contact with the liquid. By moving G_1 and then G_2 across from the right-hand side of the trough to the positions indicated, the surface of the liquid is freed from contamination. The plate is set in turn at various angles of inclination until a position is found for which the water-surface on one side of the plate remains undistorted right up to the line of contact with the solid. If ϕ is the angle between the trace of the plate and the undistorted surface of the water (as measured with the aid of a protractor), then ϕ is the angle of contact required.

An interesting method for investigating the angle of contact between mercury and glass is as follows:—The level of some mercury in an inverted spherical flask is adjusted by raising or lowering the reservoir D, Fig. 10·07, until the mercury surface in the flask is plane at points where it meets the glass. The angle BAC $= \phi$ is the required angle of contact. If 2λ is the length AC, and r the radius of the flask, $\phi = \sin^{-1}\dfrac{\lambda}{r}$; it must be remembered that $\dfrac{\pi}{2} < \phi < \pi$.

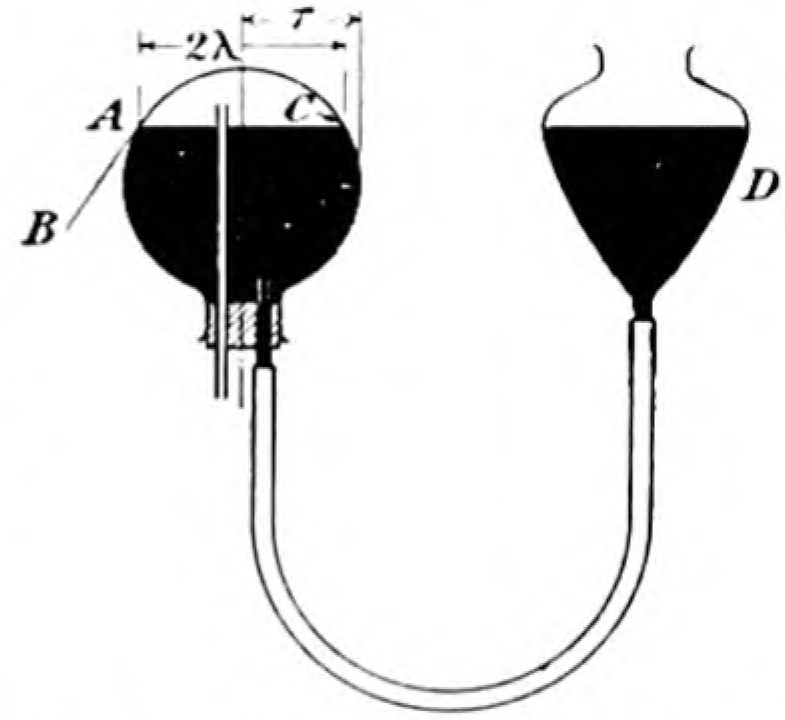


FIG. 10·07.—Angle of contact of mercury with glass.

Careful measurements of the angle of contact between different liquids and solids have shown that for a given liquid–solid combination the angle of contact may vary between two extremes. This variation is easily seen when a rain-drop is travelling slowly down a window-pane, i.e. a piece of glass which is not chemically clean. The angle is greatest at those points where the drop is advancing and least where it is receding. ADAM and JESSOP† accounted for this variation in the angle of contact by assuming the existence of a frictional force between the solid and liquid. Let ϕ_A and ϕ_R be the 'advancing' and 'receding' angles of contact. When the drop is just on the point of advancing the frictional force will tend to oppose the motion so that it will act in the direction of γ_{LS}, cf. Fig. 10·05, p. 463, so that

$$\gamma_{SG} = F + \gamma_{LS} + \gamma_{LG} \cos \phi_A \quad . \quad . \quad \text{(iii)}$$

When the surface is just on the point of receding F acts in the direction of γ_{SG}, and then

$$\gamma_{SG} + F = \gamma_{LG} \cos \phi_R + \gamma_{LS} \quad . \quad . \quad \text{(iv)}$$

† *Jour. Chem. Soc.*, **127**, 1863, 1925.

Eliminating F from equations (iii) and (iv), and combining the result with equation (i), we have

$$\cos\phi_A + \cos\phi_R = 2\cos\phi, \qquad . \quad . \quad \text{(v)}$$

or the cosine of the angle of contact which would be observed in the absence of friction is the arithmetic mean of the cosines of the advancing and receding angles. Equation (v) may be written

$$2\cos\tfrac{1}{2}(\phi_A + \phi_R)\cos\tfrac{1}{2}(\phi_A - \phi_R) = 2\cos\phi.$$

If $(\phi_A - \phi_R)$ is small, a condition nearly always found in practice, then $\cos\frac{1}{2}(\phi_A - \phi_R) \rightarrow 1$, i.e.

$$\phi_A + \phi_R = 2\phi.$$

Another method for measuring the angle of contact between a liquid and a solid, and one in which the difference between the advancing and receding angles is measured, is due to ABLETT.† Essentially the method consists in immersing a cylinder, with its axis horizontal, to such a depth in a liquid that the free liquid surface is horizontal right up to the line of contact with the curved cylindrical surface—cf. Fig. 10·08(a). If O is the centre of a cross-section of the cylinder, BA the liquid surface, then $\widehat{BAT}$ is ϕ, the

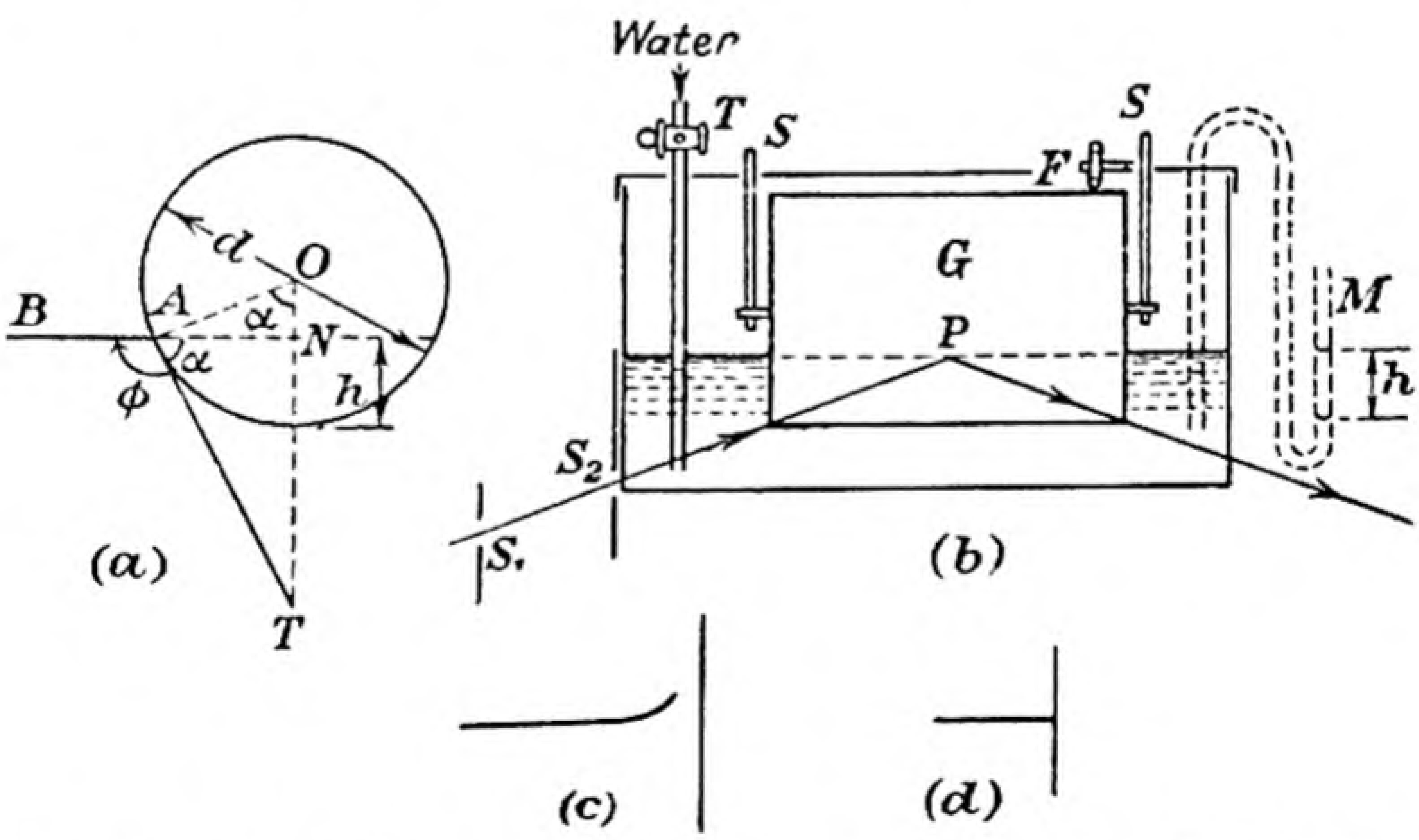


FIG. 10·08.—Ablett's method for determining angles of contact.

angle of contact, where AT is the tangent to the circle at A. Then, if ON is perpendicular to BA produced,

$$\alpha = \widehat{AON} = \pi - \phi.$$

† *Phil. Mag.*, **46**, 244, 1923.

Let d be the diameter of the circle and h the height of N above the lowest generator of the cylinder. Then

$$\cos\alpha = \frac{\mathrm{ON}}{\mathrm{OA}} = \frac{\frac{d}{2} - h}{\frac{d}{2}} = 1 - \frac{2h}{d} = -\cos\phi.$$

$$\therefore\ \cos\phi = \frac{2h}{d} - 1,$$

so that if $h < \frac{d}{2}$, ϕ is obtuse.

Fig. 10·08(b) shows the essential features of the apparatus as finally used. It consisted of a solid glass cylinder G, about 3 in. long and 3 in. in diameter, covered with a layer of paraffin wax several millimetres thick. To do this, the cylinder was placed in a bath of molten wax and slowly spun until a layer of wax 3 to 4 mm. thick had formed on it. It was then allowed to cool in a dust-free atmosphere and when cold was centred accurately in a lathe and turned smooth. The cylinder was then mounted in a rectangular glass tank with its axis horizontal and parallel to the longer sides of the tank. A small electric motor, suitably geared down, drove the friction wheel F, i.e. a small wheel covered with a rubber tyre just bearing on the wax surface, so that a rotatory motion could be imparted to the cylinder when necessary. By using a closely wound helical spring as a driving belt from the motor, the transmission of vibration to the cylinder and hence to the water surface were prevented.

The curvature of the water–air surface was tested by using it as a curved mirror upon which a narrow beam of parallel light was incident. S_1 was a horizontal slit somewhat below the level of another horizontal slit S_2, which consisted of a fine slot in a piece of black paper attached to the left-hand side of the water tank. By adjusting the position of S_1 and the optical apparatus for producing the parallel beam with respect to S_2 the beam of light could be caused to fall at any desired point, say P, of the water surface near to the curved surface of the cylinder. The image formed was observed with the unaided eye through the right-hand side of the tank. When the surface was not flat the image appeared as in Fig. 10·08(c), small irregularities in the wax surface being seen as brightly illuminated spots. When the surface had been made flat by running in more water, the image of the slit was level right up to the surface of the cylinder, as shown in Fig. 10·08(d).

To measure h, the depth of the lowest generator of the surface of the cylinder below the free surface of the water, use was made of the

manometer M as follows. Water from an aspirator was run very slowly into the tank, the rate of flow being controlled by the tap T. When the water just touched the cylinder the position of the water in M was noted, using a low power microscope. It was noted again when the liquid surface had been made flat at P. Thus h was determined directly from the manometer readings. The diameter of the cylinder having been measured in several places and a mean value calculated, the angle of contact could be deduced from the formula already established.

The angle of contact was also measured when the cylinder was rotated slowly, first anticlockwise and then clockwise. Thus the 'advancing' and the 'receding' angles of contact could be obtained, since if P is between the axis of the cylinder and the nearer wall of the tank an anticlockwise motion of the cylinder will establish conditions equivalent to the case of a rising meniscus; clockwise to a falling one. Within the limits of experimental error it was found that

$$\phi = \tfrac{1}{2}(\phi_A + \phi_R),$$

the angles being constant for peripheral speeds in excess of 0·44 mm.sec.$^{-1}$.

The mean results obtained under the above conditions were

$$\phi_A = 113^\circ\, 9',$$

$$\phi_R = 96^\circ\, 20',$$

and $$\phi = 104^\circ\, 34',$$

the mean of the first two angles being $104^\circ\, 45'$.

For peripheral speeds below 0·44 mm.sec.$^{-1}$, ϕ_A and ϕ_R were found to depend upon the speed, but in each instance

$$\phi = \tfrac{1}{2}(\phi_A + \phi_R).$$

Yarnold's method for measuring angles of contact.—This dynamical method† was first used to measure the 'advancing' and 'receding' angles of contact of mercury with steel or glass; later YARNOLD and MASON applied the method to water in contact with paraffin wax.

To understand the principles involved it is necessary to consider a solid sphere of radius a when it is suspended so that it rests partly immersed in a liquid, density ρ; let z be the height of O, Fig. 10·09(a), the centre of the sphere, above the plane surface of the liquid. Let γ be the surface tension of the liquid and g gravity. If ϕ is the angle of contact and θ the semi-vertical angle of the cone AOB, the surface tension acts at an angle $(\phi + \theta - \pi)$ to the horizontal.

† *Proc. Phys, Soc.* **58,** 120, 1946; **B, 62,** 125, 1949.

To calculate the upthrust on the sphere let us consider the pressure due to the liquid at all points on the horizontal ring shown in Fig. 10·09(*b*); its limits M and N are defined by the angles α and $\delta\alpha$.

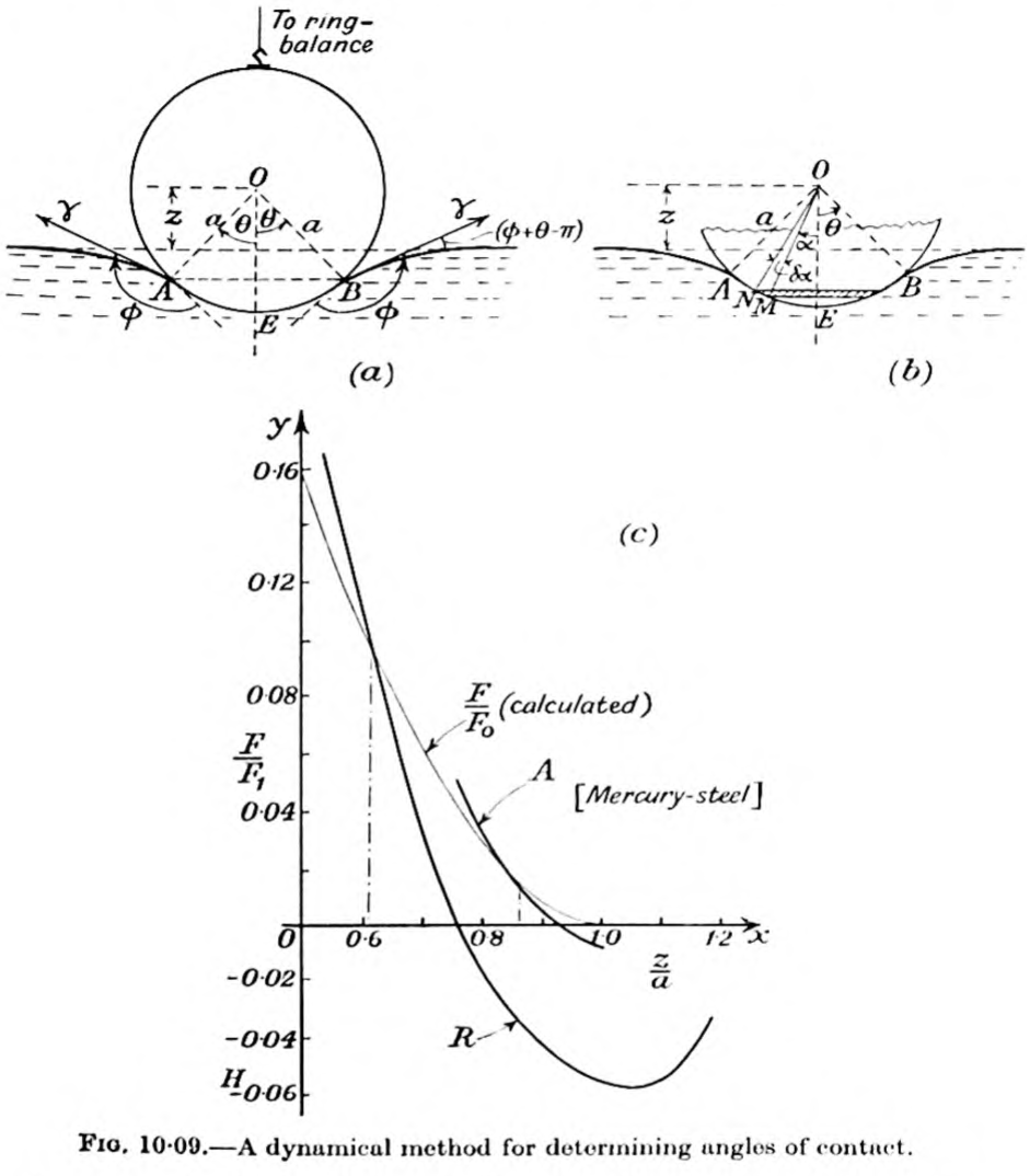


FIG. 10·09.—A dynamical method for determining angles of contact.

Since the depth of M below the horizontal surface of the liquid is $(a \cos \alpha - z)$, the pressure at all points on the ring is

$$g\rho(a \cos \alpha - z).$$

Thus δU, the upthrust on the ring, is given by

$$\delta U = [g\rho(a \cos \alpha - z) . 2\pi\, a \sin \alpha . a\, \delta\alpha] \cos \alpha.$$

$\therefore$ Total upthrust due to this cause is U, where

$$\mathrm{U} = 2\pi g\rho \int_0^\theta (a \cos \alpha - z) a^2 \sin \alpha \cos \alpha \, d\alpha.$$

The total upward force, F, on the sphere and due to the liquid is obtained by evaluating this integral and adding the force due to surface tension. The result is

$$\mathrm{F} = \pi a^3 g\rho[\tfrac{2}{3} - \tfrac{2}{3} \cos^3 \theta] - \pi a^2 g\rho z(1 - \cos^2 \theta)$$
$$+ 2\pi\gamma a \sin \theta . \cos \{\tfrac{1}{2}\pi - (\phi + \theta - \pi)\}$$
$$= \ldots\ldots\ldots\ldots - 2\pi\gamma a \sin \theta . \sin (\theta + \phi).$$

If the depth to which the sphere is immersed is such that the mercury surface is horizontal at all points in the neighbourhood of the sphere, $(\theta + \phi) = \pi$ and $z = a \cos \theta$; if F_0 is the total upward force under such conditions then

$$\mathrm{F}_0 = \pi a^3 g\rho[\tfrac{2}{3} - \cos \theta + \tfrac{1}{3} \cos^3 \theta].$$

If the sphere is completely immersed the upward force becomes F_1 and is given by $\mathrm{F}_1 = \frac{4}{3}\pi a^3 g\rho$. Hence

$$\frac{\mathrm{F}_0}{\mathrm{F}_1} = \frac{1}{2} - \frac{3}{4}\left(\frac{z}{a}\right) + \frac{1}{4}\left(\frac{z}{a}\right)^3.$$

To make use of this formula the sphere is suspended from a Sucksmith balance, cf. p. 407, so that small continuously applied and slowly varying forces may be measured with precision. The liquid is brought into contact with the lowest point on the suspended sphere and then slowly raised, in steps of about 0·01 cm., until about one half of the sphere is immersed and subsequently lowered until contact with the sphere is lost; the time taken for the total displacement through a few millimetres is about an hour.

The variation of $\frac{\mathrm{F}}{\mathrm{F}_1}$ with $\frac{z}{a}$ during the raising and lowering of the liquid (mercury) is shown in Fig. 10·09(c) by the curves A (advancing) and R (receding); the thin curve represents how $\frac{\mathrm{F}_0}{\mathrm{F}_1}$ varies with $\frac{z}{a}$. The points of intersection of the curves give the values of z when the liquid surface is horizontal right up to the surface of the sphere. Then

$$\cos(\pi - \phi) = \cos \theta = \frac{z}{a},$$

where ϕ is the required angle of contact.

The advantages of this method are (a) that it is not necessary to

know the surface tension of the liquid and (*b*) that it does not depend upon a purely visual observation of the angle of contact.

It is found that values for the 'advancing' angle vary over a considerable range, while those for the 'receding' angle all lie within a comparatively narrow range. The difference between the two angles is usually ascribed, cf. p. 465, to a frictional force per unit length acting normally to the line of contact. There it is assumed that this force is independent of the direction of the motion of the liquid; this is equivalent to assuming that the irreversible work expended in the formation of unit area of the solid–air interface is identical with the irreversible work expended in the formation of unit area of the solid–liquid interface. It is difficult to justify such an assumption.

When this apparatus is used with water, in contact with paraffin wax, it is found that if the term 'angle of contact' is to be precise, it is necessary not only to indicate the state of the surfaces but also their relative velocity and the time during which they have been in contact, for it is found that the angles involved depend upon the time of immersion. Yarnold and Mason conclude that the term 'equilibrium angle of contact' is practically meaningless.

Waves on the surface of a liquid.—When waves advance across the surface of a liquid it is usually permissible to neglect the mass of air which is set in motion by the liquid. The condition to be satisfied is that the pressure at the liquid surface must everywhere

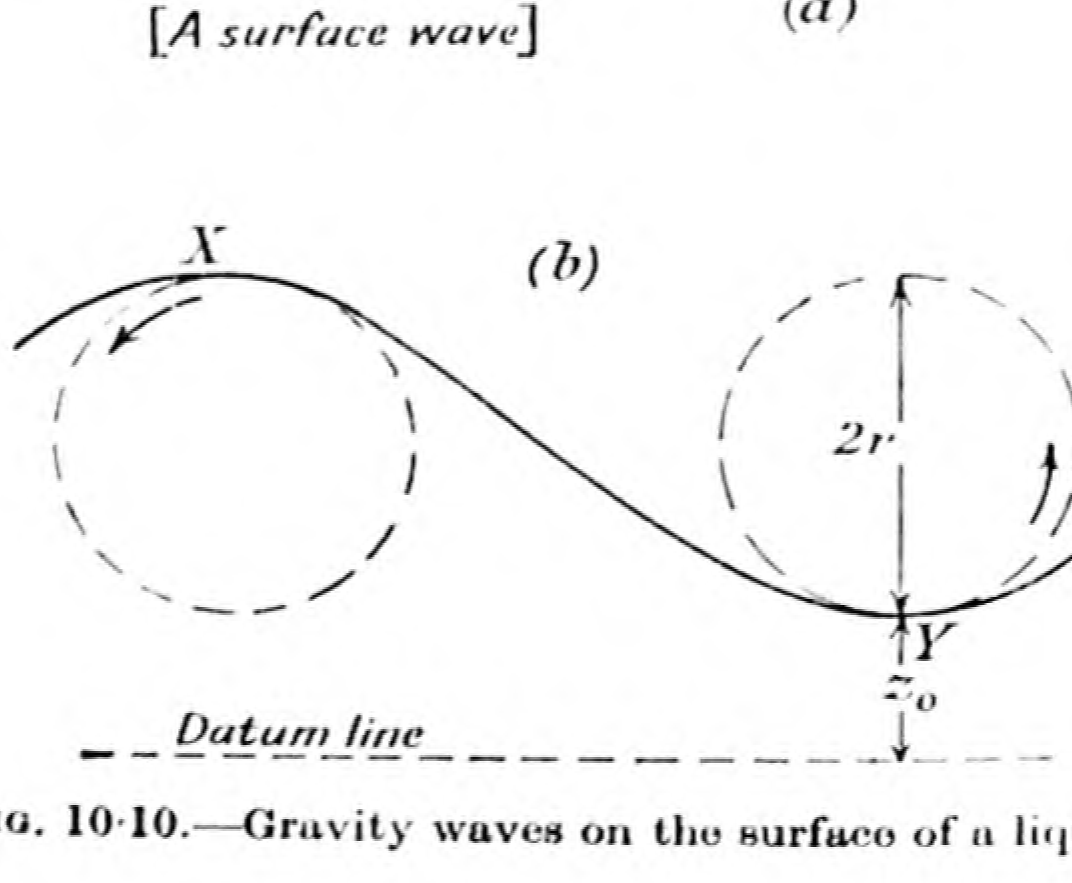


FIG. 10·10.—Gravity waves on the surface of a liquid.

be equal to p_0, the atmospheric pressure. In the simplest cases of wave motion it is known that the individual particles at the surface of the liquid describe paths which may be taken as circles in a vertical plane. To obtain a steady motion to which Bernoulli's theorem, cf. p. 614, may be applied, let us select a system of co-ordinates advancing with the waves and having a velocity c equal to that of the troughs and crests as noted by a stationary observer.

Let r, Fig. 10·10(a), be the radius of the circular path of a particle near the surface of the liquid and T the period of its motion. In this time the particle, as seen by an observer at rest, will advance a distance λ, i.e. one wavelength. Hence $\lambda = c\mathrm{T}$ and the speed of the particle in its circular path is $\dfrac{2\pi r}{\mathrm{T}}$.

The velocity of a particle at X, cf. Fig. 10·10(b), the crest of a wave is $c - \dfrac{2\pi r}{\mathrm{T}}$; at a trough Y it is $c + \dfrac{2\pi r}{\mathrm{T}}$. Let z_0 be the height of Y above a datum level. Then from Bernoulli's theorem,

$$g(2r + z_0) + \frac{1}{2}\left(c - \frac{2\pi r}{\mathrm{T}}\right)^2 + \frac{p_0}{\rho} = gz_0 + \frac{1}{2}\left(c + \frac{2\pi r}{\mathrm{T}}\right)^2 + \frac{p_0}{\rho},$$

where ρ is the density of the liquid and g is gravity.

$$\therefore\ c = \frac{g\mathrm{T}}{2\pi} = \sqrt{\frac{g\lambda}{2\pi}}. \qquad [\because\ \lambda = c\mathrm{T}.]$$

A simple method for calculating the velocity of propagation of capillary waves on a liquid surface.—The velocity c with which long plane waves travel across a liquid surface has just been proved to be given by the equation $c = \sqrt{(g\lambda/2\pi)}$. The profile of a portion of such a wave is shown in Fig. 10.11(a). Let P be a point on this profile and N its projection on the undisturbed surface of the liquid. If gravity were the only force acting, the increase in pressure at N due to the wave would be $g\rho$PN, where ρ is the density of the liquid. Surface tension will give rise to an additional pressure γr^{-1}, where γ is the surface tension of the liquid and r is the radius of curvature at P. If the amplitude is small compared with λ, the profile of the wave may be regarded as the curve traced out by a point near to the centre of a circle rolling along a plane surface. Let a be the radius of this circle and b the distance of P from its centre O, Fig. 10·11(b). Let the position of P be given with reference to a system of fixed rectangular axes P_0x and P_0y, where $\mathrm{P}_0\mathrm{O}$ is parallel to the plane of rolling. Let time be measured from the instant when P is at P_0. Let θ be the angle of roll; then $\widehat{\mathrm{QOC}} = \theta$, where the point Q will be the point of contact which the rolling circle makes with the plane of rolling when

P_0 has moved to P_1, where $P_1\widehat{O_1}P_0 = \theta$ and $OO_1 = a\theta$. The instantaneous coordinates of P_1 are then

$$x = b + a\theta - b\cos\theta = a[\theta + (b/a) - (b/a)\cos\theta]$$
$$= a\theta, \qquad [\text{if } (b/a) \rightarrow 0],$$

and $\quad y = b\sin\theta.$

$$\therefore \frac{dy}{dx} = \frac{b}{a}\cos\theta, \quad \text{and} \quad \frac{d^2y}{dx^2} = -\frac{b\sin\theta}{a^2}.$$

If $\dfrac{dy}{dx}$ is small, a condition appropriate to the present problem, the radius of curvature is $\left(\dfrac{d^2y}{dx^2}\right)^{-1}$. Thus

$$r = \left(\frac{d^2y}{dx^2}\right)^{-1} = -\left(\frac{b\sin\theta}{a^2}\right)^{-1} = -\frac{a^2}{\text{PN}}.$$

Hence, neglecting the negative sign, in the case of a wave the excess pressure due to surface tension is

$$\frac{\gamma}{r} = \gamma\frac{\text{PN}}{a^2}.$$

But $2\pi a = \lambda$, so that the above excess pressure may be put equal

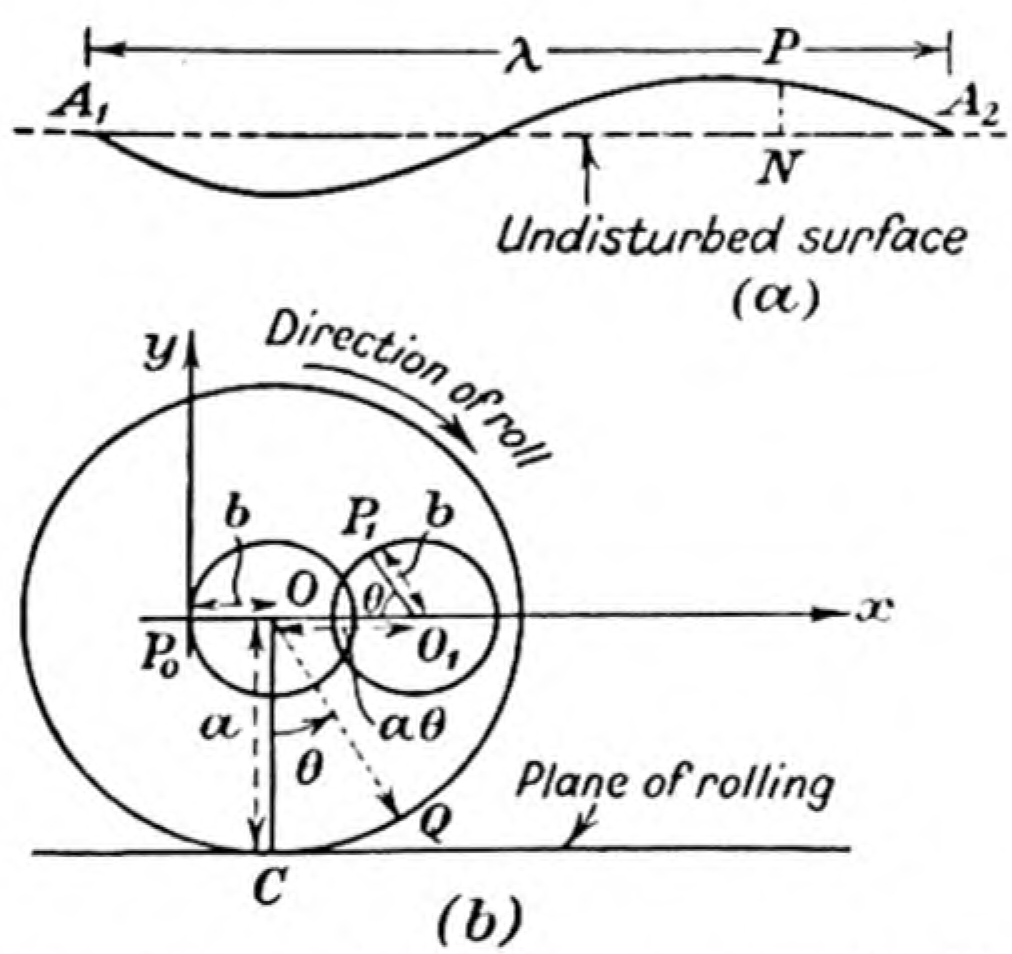


FIG. 10·11.—Velocity of capillary waves on a liquid surface.

to $(4\pi^2.\gamma/\lambda^2)$. Now when the pressure at N is $g\rho$ PN, the velocity is $\sqrt{(g\lambda/2\pi)}$. Therefore, when the pressure at N becomes $[g\rho + (4\pi^2\gamma/\lambda^2)]$PN, the velocity will be given by

$$c = \sqrt{\frac{g\lambda}{2\pi}\left(1 + \frac{4\pi^2\gamma}{\lambda^2 g\rho}\right)} = \sqrt{\frac{g\lambda}{2\pi} + \frac{2\pi\gamma}{\lambda\rho}}.$$

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It seems justifiable to do this since we may regard the effect as due to an increase in $g\rho$ in the ratio $\left[1 + \frac{4\pi^2\gamma}{\lambda^2 g\rho}\right] : 1$.

Since $c^2 = \frac{g\lambda}{2\pi} + \frac{2\pi\gamma}{\rho\lambda}$, we have

$$2c\,\frac{dc}{d\lambda} = \frac{g}{2\pi} + \frac{2\pi\gamma}{\rho}\left(-\frac{1}{\lambda^2}\right).$$

Hence for a maximum or minimum

$$\frac{g}{2\pi}\cdot\frac{\rho}{2\pi\gamma} = \frac{1}{\lambda^2},$$

for $\frac{dc}{d\lambda}$ is then zero, and this may be written

$$\lambda = 2\pi\sqrt{\frac{\gamma}{g\rho}},$$

and since $c \to \infty$ when $\lambda \to 0$ and when $\lambda \to \infty$, this value for λ must be the wavelength of those ripples which travel with minimum velocity across a liquid surface.

METHODS FOR DETERMINING THE SURFACE TENSION OF LIQUIDS AND INTERFACIAL TENSIONS

General remarks.—It has already been shown that there is a difference in pressure on the two sides of a curved liquid surface. When the form of the surface is spherical the expression for the above pressure difference has been proved to be given by

$$p = \frac{2\gamma}{r},$$

where γ is the surface tension of the liquid and r is the radius of the spherical surface. Hence, to measure the surface tension, γ, in any given instance the first essential is to obtain a spherical surface; one very common method of doing this is to place a uniformly narrow tube vertically in the liquid when the liquid surface in the tube is part of a sphere. It then remains to measure the radius of this surface and the pressure difference across it. If the angle of contact between the liquid and the material of the tube is zero, the above radius is equal to that of the cross-section of the tube where the liquid rests, provided the tube is sufficiently narrow. The pressure difference is generally determined from observations on the height of a certain column of liquid. If this liquid is identical with that under investigation its density must be known, but if the

pressure can be measured in terms of the height of a column of some other liquid of known density, then it is not necessary to know the density of the liquid under examination. In actual practice we shall find that both methods of measuring the pressure are used.

Some theorems of practical importance.—(*a*) ***The ascent of a liquid in a capillary tube:*** For the sake of simplicity we shall first assume that the tube is narrow and the angle of contact zero. Let AC, Fig. 10·12(*a*), be the surface of a liquid in a capillary tube

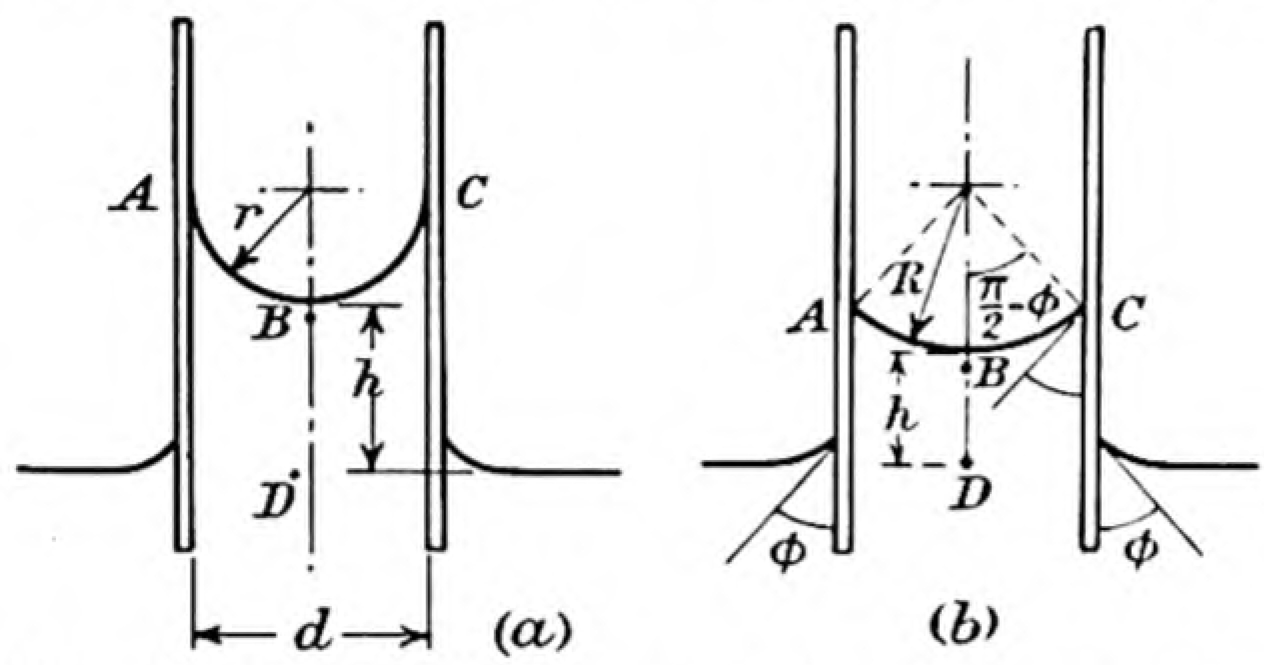


Fig. 10·12.—Ascent of a liquid in a vertical and narrow capillary tube.

of radius r. We assume that AC is part of a sphere of radius r. The pressure over the curved surface is everywhere atmospheric. At B, a point just below the surface and therefore in the liquid, the pressure is less than atmospheric by an amount $\frac{2\gamma}{r}$ [cf. p. 458]. At D, a point below B and lying in the same horizontal plane as the surface of the liquid outside the tube, the pressure is atmospheric. Now the difference in pressure between the two points B and D is equal to the pressure exerted by a column of liquid of height DB $= h$ (say). If ρ is the density of the liquid, this difference is $g\rho h$, where g is gravity. The pressure at B is therefore less than atmospheric by this amount and it has already been shown that this difference in pressure is $\frac{2\gamma}{r}$. We therefore have

$$\frac{2\gamma}{r} = g\rho h.$$

Now suppose that the angle of contact between the liquid and the material of the tube is ϕ, cf. Fig. 10·12(*b*). Let R be the radius of curvature of the liquid surface at its lowest point; if the bore of the capillary is small, R is constant at all points on the liquid surface. Then, as before, if P_0 is the atmospheric pressure,

$$\text{pressure at B} = P_0 - \frac{2\gamma}{R}.$$

But pressure at D $= P_0 =$ pressure at B $+ g\rho h$.

$$\therefore \frac{2\gamma}{R} = g\rho h.$$

But $r = R \cos \phi$; therefore $\dfrac{2\gamma \cos \phi}{r} = g\rho h$.

It should be mentioned, perhaps, that if ϕ is finite, values of the surface tension of a liquid deduced from measurement of its rise in capillary tubes are unreliable, since the magnitude of ϕ is always

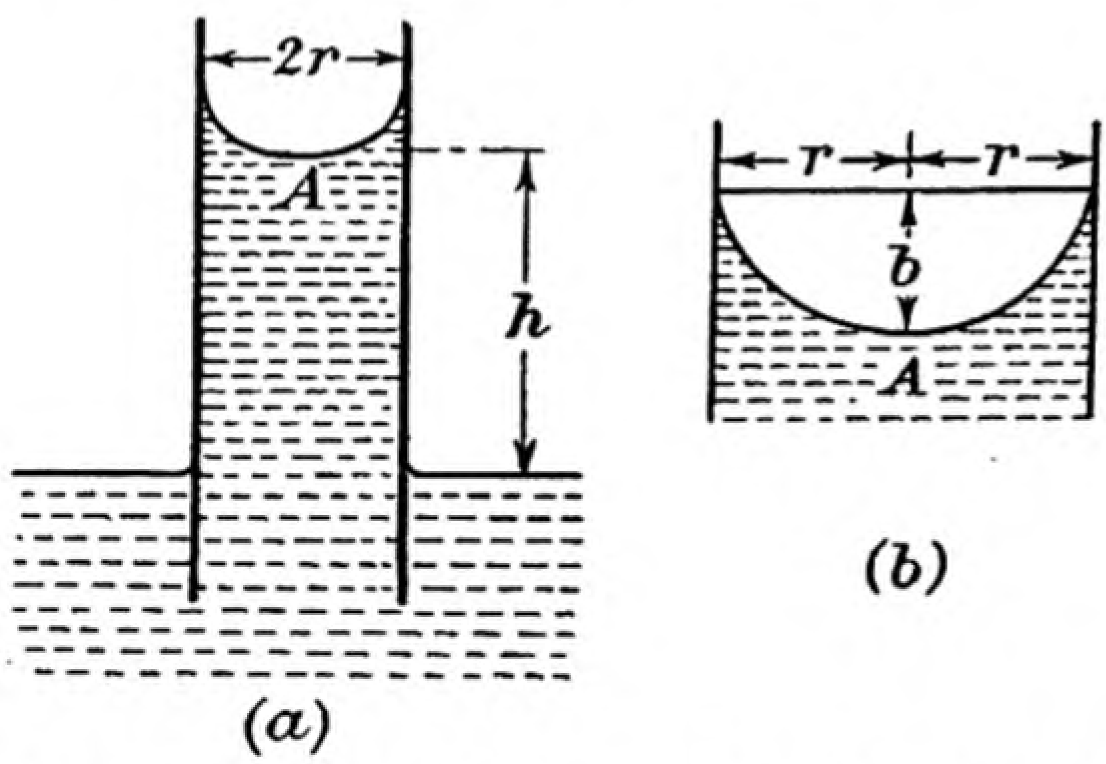


FIG. 10·13.—Rise of a liquid in a capillary tube of wider bore.

uncertain; moreover, ϕ varies considerably with the degree of contamination of the surfaces in contact. The above theory is necessary, however, for academic purposes.

It is now necessary to examine how the formula we have obtained for the ascent of a liquid (zero angle of contact) is modified when the capillary tube is so wide that it is no longer justifiable to assume that the meniscus surface is hemispherical. The following analysis is due to FERGUSON. By the excess-pressure principle

$$g\rho h = \frac{2\gamma}{R},$$

where R is the radius of curvature at A, the lowest point of the meniscus—Fig. 10·13(a). Putting for short $a^2 = \dfrac{\gamma}{g\rho}$, where a^2 is known as the ***capillary constant*** or ***specific cohesion***† of the liquid, then $2a^2 = Rh$.

This is an exact equation, whatever the angle of contact. The difficulties begin when an attempt is made to find a value for R which shall be less approximate than that obtained above.

† In chemistry $2a^2$ is often called the specific cohesion.

It will be assumed that the trace of the meniscus in the plane of the diagram, cf. Fig. 10·13(b), is an ellipse with semi-axes r and b. The radius of curvature at A, cf. p. 10, is $\mathrm{R} = r^2b^{-1}$. Hence the equation

$$2a^2 = \mathrm{R}h,$$

becomes

$$2a^2 = \frac{r^2h}{b}, \qquad \text{or} \qquad b = \frac{r^2h}{2a^2}.$$

If we assume that the tube is perfectly cylindrical we may equate the resultant force $2\pi r\gamma$, which acts upwards on the liquid along its line of contact with the tube, to the total weight of liquid raised. Then

$$2\pi r\gamma = \pi r^2 h\rho g + \tfrac{1}{3}\pi r^2 b\rho g.$$

$$\therefore\ 2a^2 = rh\left(1 + \frac{r^2}{6a^2}\right). \qquad . \quad . \quad . \quad \text{(i)}$$

Now equation (i) may be written

$$12a^4 - 6a^2rh - r^3h = 0,$$

a quadratic in a^2, giving

$$a^2 = \frac{6rh \pm 6rh\left(1 + \frac{4}{3}.\frac{r}{h}\right)^{\frac{1}{2}}}{24},$$

and the positive sign must be taken, since in the limit when $\frac{r}{h} \rightarrow 0$, we must have $2a^2 = rh$.

$$\therefore\ \frac{\gamma}{g\rho} = a^2 = \tfrac{1}{2}rh\left[1 + \frac{1}{3}.\frac{r}{h} - \frac{1}{9}.\frac{r^2}{h^2} + \ \ldots\right].$$

In passing it may be noted that

$$\mathrm{R} = r\left(1 + \frac{r^2}{6a^2}\right).$$

This gives a close approximation to the correct value of R but it is not required here.

(b) ***The rise of a liquid between vertical plates.*** (i) ***Parallel plates:*** To calculate the amount of this rise we may use Fig. 10·12(a). Let the vertical lines in that diagram now represent sections of the two parallel plates at distance d apart. We assume AC to be a section of a cylindrical surface of diameter d so that the pressure at B is less than atmospheric by an amount $\frac{\gamma}{r}$, or $\frac{2\gamma}{d}$, since $d = 2r$. Proceeding as before we obtain, if the contact angle is zero,

$$\frac{2\gamma}{d} = g\rho h.$$

(ii) ***Inclined plates:*** Fig. 10·14 represents two vertical glass plates, AOB and OAD, inclined to one another at a small angle θ. When these are inserted in a liquid the latter rises between the plates. To determine the shape of the curve in which AOC, the vertical plane through OA and bisecting the angle θ, i.e. the plane of co-ordinates, intersects the liquid surface, consider an element PQR of the surface at right angles to the intersection of the liquid surface with the plane AOC. Let (x, y) be the co-ordinates, referred to OC and OA as axes, of Q the middle point of the element PQR. [$P_1Q_1R_1$ is another such element. Notice that the projections p, q and r of the points P, Q and R respectively, on the horizontal plane through Ox do not lie in a straight line.] Then if the liquid wets the glass, the surface at PQR is part of that of a cylinder whose diameter is equal to the distance between the plates at Q. This distance is $x\theta$, since θ is small. The height y to which the liquid rises is therefore given by

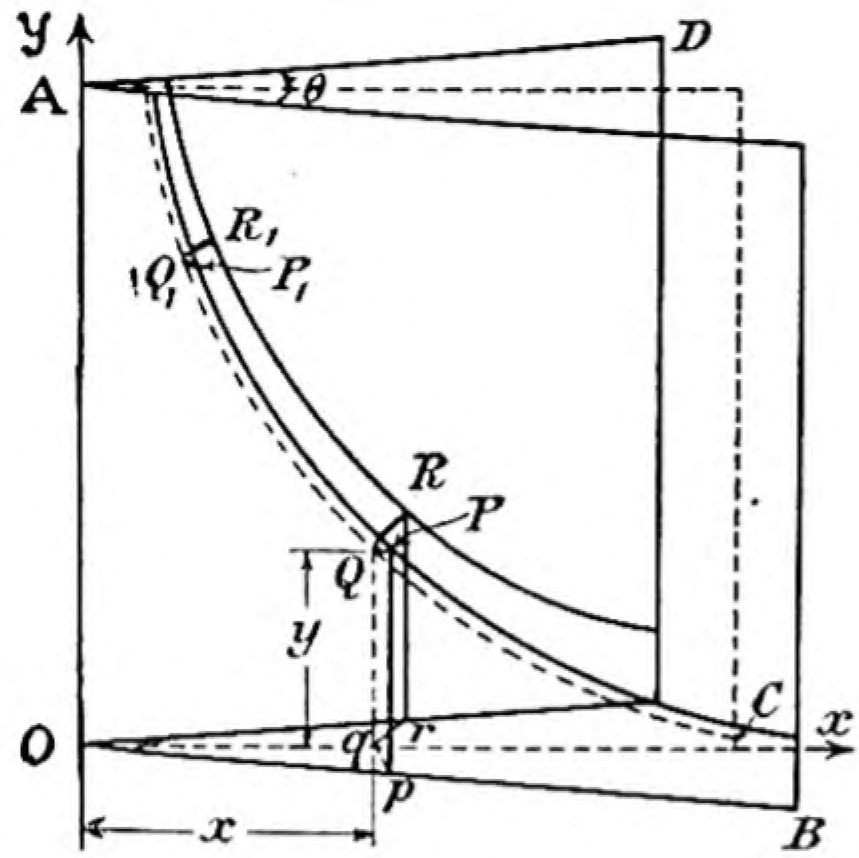


FIG. 10·14.—Rise of a liquid between inclined vertical plates (end-effect neglected).

$$\frac{2\gamma}{x\theta} = g\rho y,$$

i.e. $xy = 2\gamma/g\rho\theta =$ constant. The surface is therefore part of a hyperbola, whose asymptotes are the axes of coordinates.

The energy changes associated with capillary rise.—The energy required to cause a liquid to rise to its position of static equilibrium in a vertical capillary tube is derived from the diminution of surface energy which accompanies the movement of the liquid. R. C. BROWN has shown how use may be made of this fact to establish the well known formula for the rise of a liquid in a capillary tube. For the sake of simplicity Brown assumes that initially no part of the tube is submerged below the general level of the liquid, which is contained in a wide vessel, cf. Fig. 10·15(a). Fig. 10·15(b) shows the liquid in the tube when the length occupied is x and Fig. 10·15(c) shows the equilibrium stage, i.e. when the liquid has reached a height h; $0 < x < h$. It will be assumed that the temperature remains constant.

If r is the radius of cross-section of the tube, the loss of free surface energy which occurs when an area $2\pi rh$ of the wall of the

tube is wetted is $2\pi rh(\gamma_{SG} - \gamma_{SL})$, which, by the relation given on p. 463, is $2\pi rh\gamma_{LG} \cos \phi$; ϕ is the angle of contact and the other symbols have their usual meanings.

Now the most obvious gain of energy is the potential energy of the raised column, viz., $(\pi r^2 h\rho)g \cdot \frac{1}{2}h = \frac{1}{2}\pi r^2 g\rho h^2$, where g is the intensity of gravity. The problem is *not* solved by equating this to the loss of free surface energy for work is done in overcoming the viscous forces which arise whenever a liquid moves slowly. The

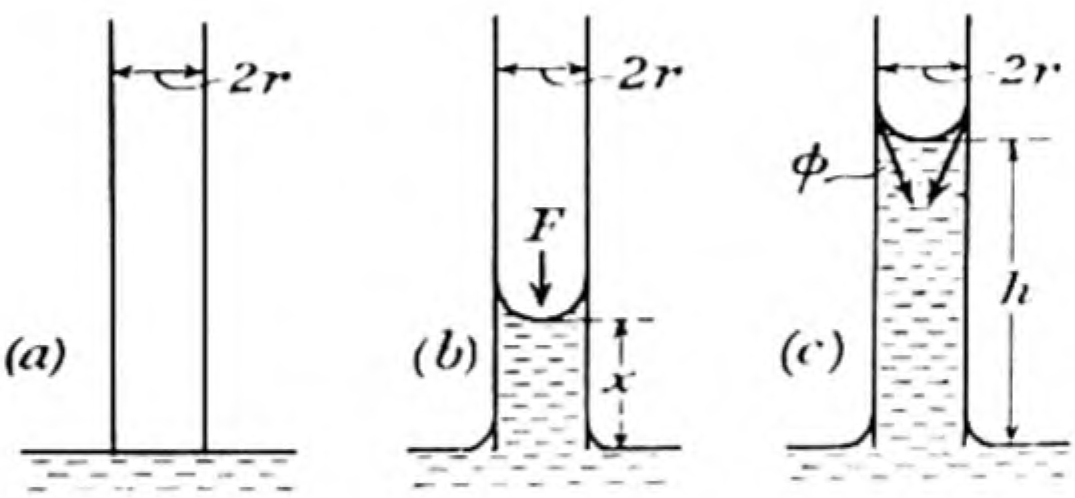


FIG. 10·15.—Energy changes associated with capillary rise.

amount of energy thus dissipated may be calculated by supposing a downward force F to act on the liquid column as it moves from (*a*) to (*c*), the magnitude of F being just sufficient at all stages to prevent acceleration. At the intermediate stage represented in Fig. 10·15(*b*), $F = \pi r^2 g\rho(h - x)$ for this weight of liquid added to that in (*b*) gives us the stage (*c*), when no further rise takes place.

The work done against F is

$$\int_0^h F\,dx = \pi r^2 g\rho \int_0^h (h - x)\,dx = \tfrac{1}{2}\pi r^2 g\rho h^2.$$

Hence $$2\pi rh\gamma_{LG} \cos \phi = \tfrac{1}{2}\pi r^2 g\rho h^2 + \tfrac{1}{2}\pi r^2 g\rho h^2,$$

or $$\gamma_{LG} \cos \phi = \tfrac{1}{2} g\rho hr,$$

which is the relation required.

The stability of a small liquid index in a vertical capillary tube which is slightly conical.—The liquid index, density ρ and length h, is shown in Fig. 10·16(*a*). In this diagram the capillary tube tapers upwards so that if r and R are the radii of curvature of the upper and lower menisci $r < R$. If Π is the atmospheric pressure, γ the surface tension of the liquid, and A, B, C and D are the points indicated, we have

$$p_B = \Pi - \frac{2\gamma}{r}, \qquad \text{and} \qquad p_C = \Pi - \frac{2\gamma}{R}.$$

But
$$p_C = p_B + g\rho h,$$
so that
$$g\rho h = 2\gamma\left[\frac{1}{r} - \frac{1}{R}\right].$$

Since $r < R$, the quantity within the square brackets is positive and the liquid takes up a position so that the above equation is satisfied.

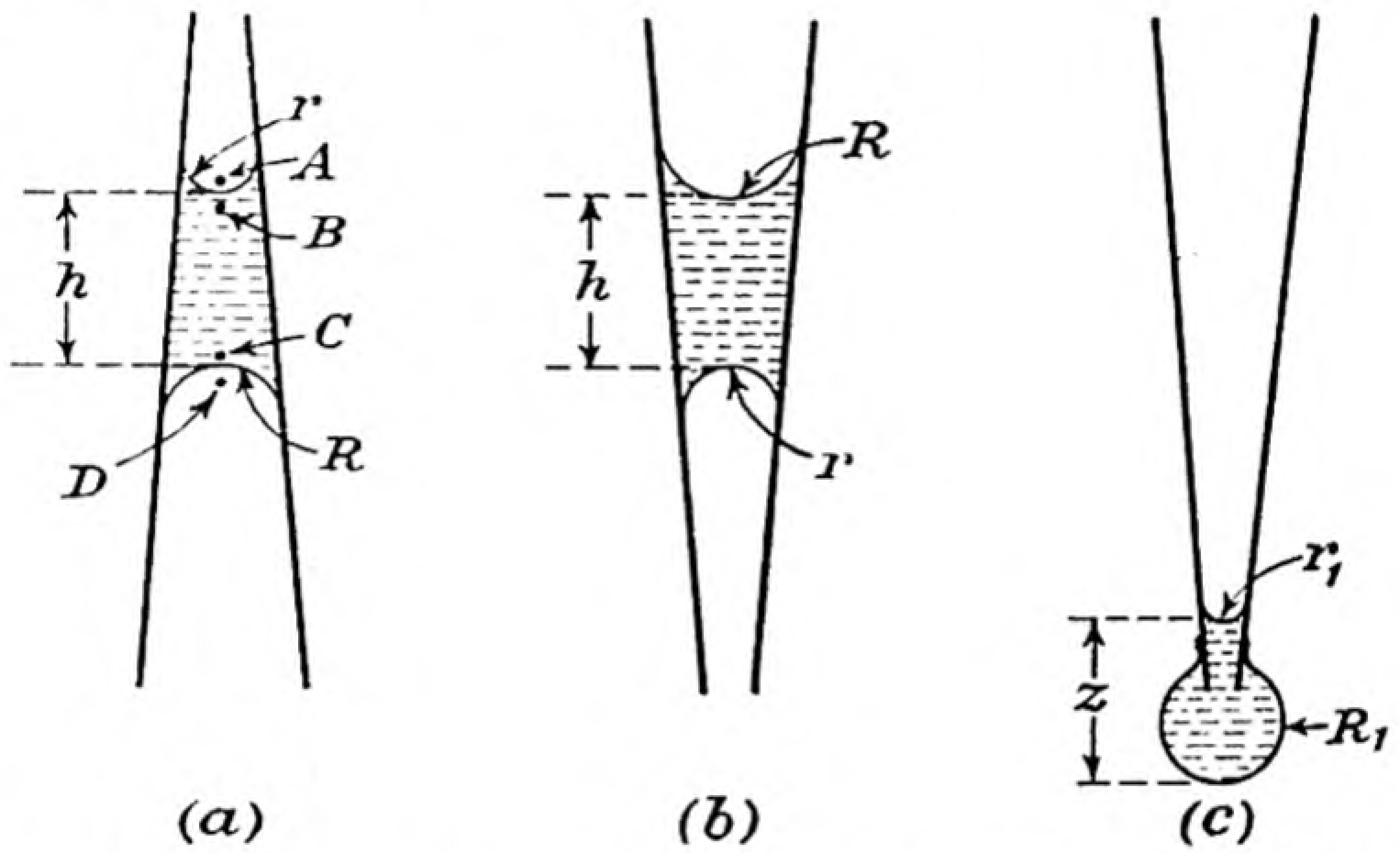


FIG. 10·16.—The stability of a liquid index in a vertical and slightly conical capillary tube.

When the more narrow end of the capillary tube points downwards, as in Fig. 10·16(*b*), we have
$$g\rho h = 2\gamma\left[\frac{1}{R} - \frac{1}{r}\right].$$
which it is impossible to satisfy since $R > r$. The liquid index therefore moves to the lower end of the tube and there forms a small drop as in Fig. 10·16(*c*). If r_1 and R_1 are the radii of curvature of the surfaces indicated, and z the overall length of the drop, we have
$$\Pi - \frac{2\gamma}{r_1} + g\rho z = \Pi + \frac{2\gamma}{R_1}.$$
$$\therefore\ g\rho z = 2\gamma\left[\frac{1}{r_1} + \frac{1}{R_1}\right],$$
which is always positive whatever may be the value of r_1.

The shape of large sessile† drops and bubbles.—(*a*) ***Large drops:*** It will be assumed that the drop under consideration is so large that it is flat at its highest point, and that at any point on its

† Attached at the base.

surface the curvature in planes other than the meridional plane is negligible. A meridional section of such a drop is given in Fig. 10·17(*a*). Let (x, y) be the coordinates of any point Q on the surface, referred to axes Ox, Oy as shown. If P_0 is the atmospheric

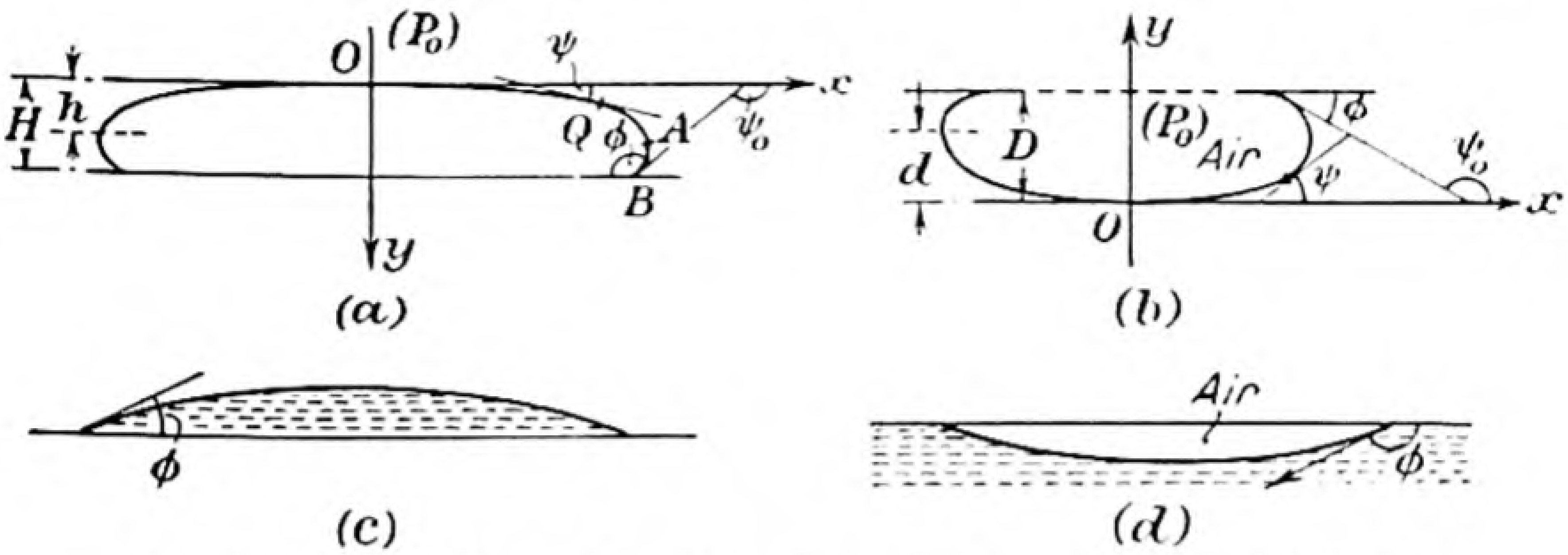


FIG. 10·17.—Meridional sections through large drops and bubbles.

pressure that at Q is either given by $P_0 + g\rho y$ or $P_0 + \dfrac{\gamma}{R}$, where R is the radius of curvature in the meridional plane.

$$\therefore\ g\rho y = \gamma.\frac{1}{R} = \gamma.\frac{d\psi}{ds} = \gamma.\frac{d\psi}{dy}.\frac{dy}{ds},$$

where ψ is the angle which the tangent to the section at Q makes with Ox, and s is the arc OQ.

$$\therefore\ \frac{g\rho}{\gamma}.\ y = \frac{d\psi}{dy}.\sin\psi.$$

If A is the point on the curve at which the tangent is vertical, and h is the depth of A below Ox, then

$$\frac{g\rho}{\gamma}\int_0^h y\,dy = \int_0^{\frac{\pi}{2}} \sin\psi\,d\psi.$$

$$\therefore\ \frac{g\rho}{2\gamma}.h^2 = -\Big[\cos\psi\Big]_0^{\frac{\pi}{2}} = 1.$$

$$\therefore\ h^2 = \frac{2\gamma}{g\rho}.$$

Also, if ψ_0 is the angle which the tangent at B, the point at which the curve cuts the surface on which the drop is formed, makes with Ox, so that $\psi_0 = \phi$, the angle of contact appropriate to the liquid and surface,

$$\frac{g\rho}{\gamma}\int_0^H y\,dy = \int_0^{\psi_0} \sin\psi\,d\psi = \int_0^{\phi} \sin\psi\,d\psi,$$

where H is the maximum height of the bubble.

$$\therefore\ \frac{g\rho}{2\gamma}(\mathrm{H}^2) = -\cos\phi + 1, \quad \text{or} \quad \mathrm{H}^2 = \frac{2\gamma(1-\cos\phi)}{g\rho}.$$

(*b*) ***Large bubbles:*** A section of such a bubble is shown in Fig. 10·17(*b*). For convenience the positive direction of the axis of y is reversed. Then, as above,

$$\mathrm{P}_0 - g\rho y + \gamma.\frac{d\psi}{ds} = \mathrm{P}_0.$$

$$\therefore\ g\rho y\, dy = \gamma \sin\psi\, d\psi.$$

$$\therefore\ \frac{1}{2}.\frac{g\rho}{\gamma}.\mathrm{D}^2 = \Big[-\cos\psi\Big]_0^{\psi_0} = -\cos(\pi-\phi) + 1 = 1 + \cos\phi,$$

and $\frac{1}{2}.\frac{g\rho}{\gamma}.d^2 = 1,$

if D and d are the depths indicated on the diagram.

This discussion on sessile drops and bubbles is incomplete for it has been assumed that for the former $\frac{\pi}{2} < \phi < \pi$, while for the latter that $0 < \phi < \frac{\pi}{2}$. The two other possible instances are shown in Fig. 10·17(*c*) and (*d*). The total height (or depth) is given by the appropriate formula already obtained but it must be noticed that no value for 'h' or 'd' can be found. In these instances a value for ϕ can be obtained only if γ is known.

It must also be pointed out that for such large drops or bubbles as those shown in Fig. 10·17 there cannot exist any point of inflexion, for the existence of such a point would mean that the pressure in the liquid would be the same at points at different levels. With smaller drops, such as those hanging from a thin vertical glass rod etc., points of inflexion do occur, when the pressure change associated with the radius of curvature in a plane normal to that of the diagram prevents a violation of the elementary principles of hydrostatics.

Experimental determination of surface tension.—(*a*) ***The rise in a capillary tube method:*** Select a piece of glass tubing about 0·4 cm. diameter and heat it in a bunsen flame, rotating the tube all the time. When the glass begins to soften, apply a gentle pressure along its length so that the walls of the tube thicken. Then remove the glass from the flame and ***slowly*** pull the ends apart. The capillary tube thus constructed is clean, a condition which is absolutely essential if a reliable value for γ is to be obtained. When the tube is cold select a length from the centre of the drawn-out portion and attach to it a very thin glass rod, R, drawn out to a point and bent twice at right angles as in Fig. 10·18(*a*). Bands B_1

and B_2 cut from a length of rubber tubing enable this rod to be attached to the tube easily.

Now clamp the capillary A in a vertical position and place the liquid whose surface tension is to be measured below the tube so that the latter is immersed to a greater depth than that at which it is to be used and then raise it slightly. If the liquid falls back readily as the tube is raised we may assume that the tube and liquid are not contaminated. Continue to raise the tube until the end of

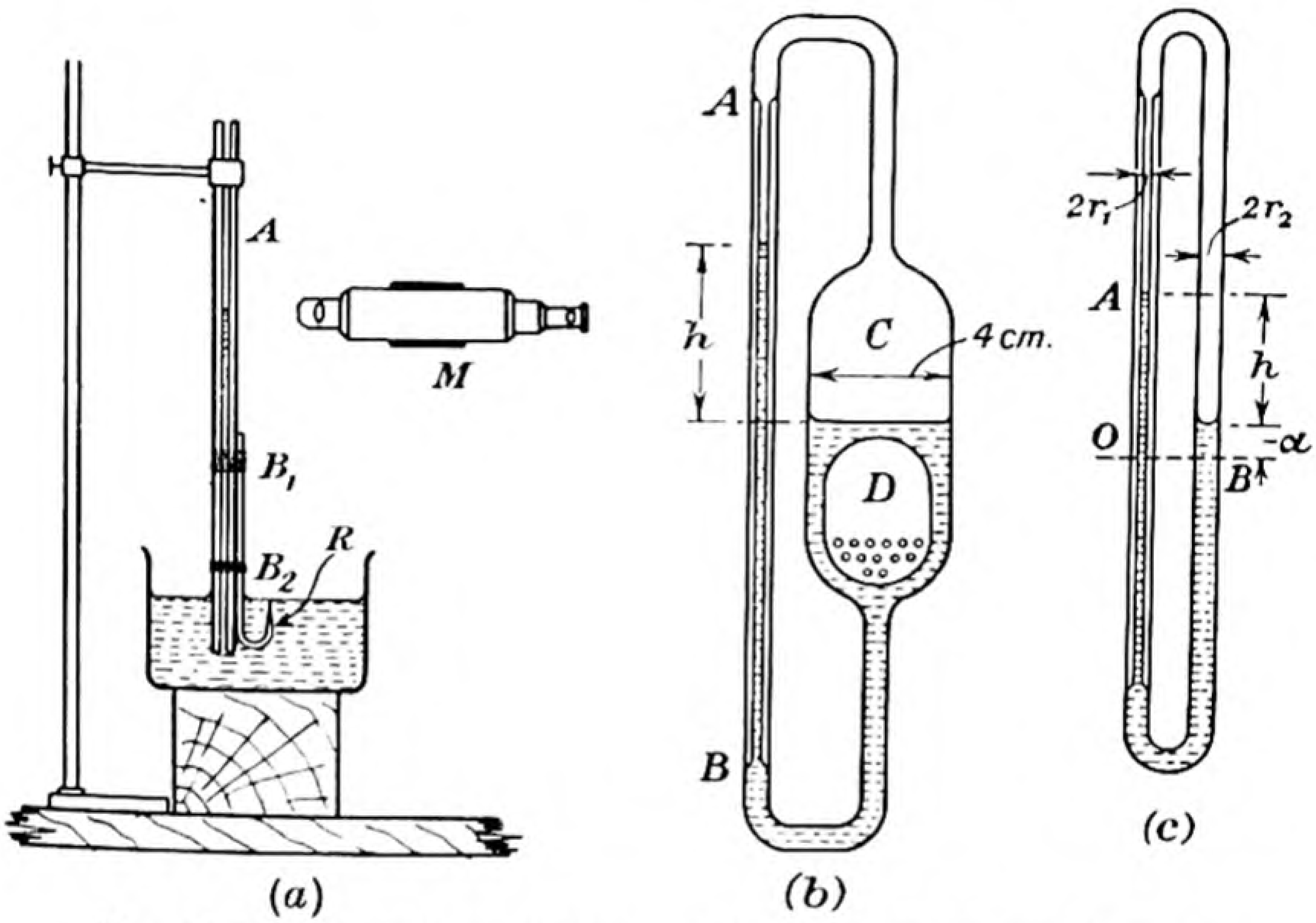


FIG. 10·18.—Measurement of surface tension by the capillary-rise method.

the rod is just about to break through the liquid surface. To measure the height of the liquid in the capillary a vernier microscope, M, should be used. The microscope is focused on the lowest point of the liquid surface in the capillary and the reading on its scale observed. The vessel containing the liquid is then removed, care being taken to see that the rod is not disturbed. The microscope is then focused on the end of the rod and the reading noted. The difference between these readings gives the height of the liquid in the capillary. These observations should be repeated. The tube is then broken at the point corresponding to the top of the meniscus and the radius found with the aid of a vernier microscope. To do this several readings of two diameters mutually at right angles are made. If the mean values of each set are equal to within about 5 per cent the mean value can be taken as a measure of r. If the discrepancy is greater than this the tube should be rejected and

another one constructed. The mean diameters of the ends of the tube should be measured before commencing the main part of the experiment. If the ends are found to be reasonably circular then the tube will probably have a circular section throughout. These end diameters should not be used in calculating γ since it is the radius at the point B, Fig. 10·12(*a*), which determines the pressure change in crossing the surface of the liquid. The value of the surface tension may then be calculated from the formula already proved.

[At this point it is convenient to ask ourselves what would happen if a tube of radius r and length less than h, where h is given by $2\gamma = g\rho hr$, were dipped in a liquid of surface tension γ and density ρ. Usually, i.e. when the length of the tube is greater than h, it is the height of the liquid in the tube which adjusts itself until the equation is satisfied. When this is no longer possible, as in the problem now contemplated, the only quantity in the above equation which is a variable is r. The liquid therefore rises to the top of the tube and there forms a surface which is concave upwards and whose radius is greater than r. Its value r_1 is given by $h_1 r_1 = hr$, where h_1 is the height of the liquid in the capillary.]

Commenting on the measurement of surface tensions by the rise of a liquid in a capillary tube method, FERGUSON and DOWSON,† state that the experiment is more than ordinarily difficult. In the first place, these authors point out that it is not easy to find a tube of sufficiently uniform cross-section to use with different liquids and, secondly, it is no small matter to calibrate, clean, and keep clean such a piece of tubing. In addition it is difficult to estimate the temperature of the meniscus with the accuracy demanded; the method is therefore useless if the variation of surface tension with temperature is the subject of an experimental determination. Further it is not easy to measure with precision the height of ascent of the liquid; 'and when we remember that, all the measurements having been made with due care, the value is not γ but $\gamma \cos \phi$, it becomes increasingly clear that the convenience of this widely used method is more apparent than real'.

Practically all these troubles are swept away if, instead of measuring the rise of the liquid in a narrow tube, the liquid be forced down to the lower end of a tube immersed vertically within the liquid, and, on a convenient manometer, the pressure required to effect this be measured.

A thermocouple placed near the lower end of the capillary is used to indicate the temperature of the meniscus.

Shorn of all corrections the method and apparatus is so similar to that designed by Jaeger and described on p. 485 that no further details will be given.

† *Trans. Far. Soc.*, **17**, 384, 1921.

RICHARDS and COOMBS† overcame some of the difficulties mentioned in the previous paragraph by constructing an apparatus of the form shown in Fig. 10·18(b). AB was a capillary tube specially selected for uniformity of bore while C was a wider tube, about 4 cm. in diameter, fused to the capillary tube. D was a glass vessel placed inside C to reduce the volume of liquid required. The apparatus was thoroughly cleaned and then filled with pure air and a sufficient quantity of the liquid whose surface tension was to be determined. The whole was immersed in a thermostat and the positions of the menisci viewed through a plate-glass window, the height h being determined with the aid of a cathetometer. The apparatus was then inverted and a second determination of the capillary rise made. The apparatus enabled the variation of surface tension with temperature to be investigated. Very consistent results were obtained.

Fig. 10·18(c) shows a modification of the apparatus used by Richards and Coombs. It consists of two vertical tubes A and B, joined together as shown. Their internal radii are r_1 and r_2 respectively. The whole apparatus is easily used in a thermostat and once the liquid has been introduced into the apparatus this can be sealed hermetically. [The apparatus can be emptied of air, if necessary.]

If h is the difference in height between the two liquid surfaces, p the pressure of the ambient air and vapour, and O a reference point at a depth α below that of the meniscus in the wider tube, we have, in the usual way,

$$p - \frac{2\gamma}{r_1} + g\rho(h + \alpha) = p - \frac{2\gamma}{r_2} + g\rho\alpha.$$

$$\therefore \frac{2\gamma}{g\rho}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) = h,$$

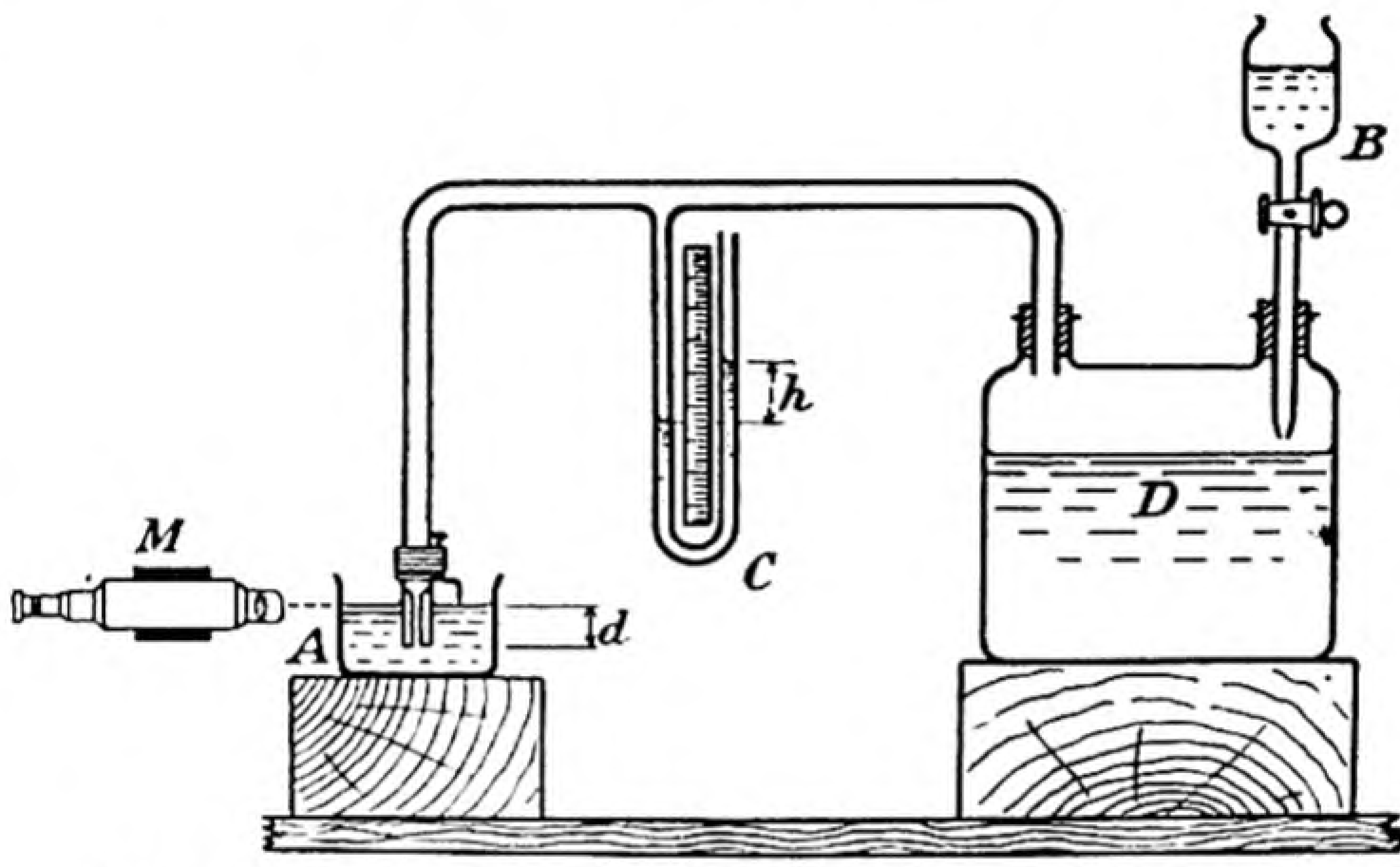
so that γ is easily obtained. The above theory assumes that the angle of contact is zero and that each tube is sufficiently narrow for the liquid surfaces to be assumed hemispherical.

(*b*) ***Jaeger's method or the method of maximum bubble pressure:*** This is based on the fact that the excess pressure inside a spherical bubble of air inside a liquid is $\frac{2\gamma}{r}$, where r is the radius of the bubble.

The experiment consists essentially in determining the maximum pressure required to produce an air bubble at the end of a vertical capillary tube immersed in the liquid whose surface tension is being

† *Jour. Amer. Chem. Soc.*, **37**, 1656, 1915.

determined. A capillary tube about 0·05 cm. in diameter is constructed as in (*a*). This is placed vertically downwards in a vessel, A, Fig. 10·19(*a*), containing the liquid whose surface tension is required. This vessel should be at least 8 cm. in diameter so that the surface of the liquid may be flat. The capillary tube is connected to a manometer, C, containing xylol, and also to a Woulff's bottle, D, fitted with a dropping funnel, B. Mercury (or water) is placed in B and permitted to run slowly into D. A difference of



[The microscope M is only used when the vessel A has been removed, for otherwise refraction at the curved surface of this vessel would vitiate the observations.]

FIG. 10·19 (*a*).—Measurement of surface tension by the method of maximum excess pressure.

pressure between the inside and the outside of the apparatus is at once shown if the apparatus is air-tight. When the pressure in D reaches a certain value bubbles are released from the end of the capillary tube dipping into A. These should be formed singly and at the rate of about one in ten seconds. The first condition is obtained by reducing the volume of air in the apparatus so that when one bubble breaks away from the end of the capillary tube, the pressure inside the apparatus is reduced to such a value that it is less than the maximum pressure required to blow the bubble; the second condition is obtained by adjusting the rate at which liquid flows into D. The maximum height h of the manometer is recorded. If ρ is the density of the liquid in the gauge, the pressure recorded by it is $g\rho h$, where g is the intensity of gravity. But this pressure difference in not entirely due to the effects of surface tension, for part is attributable to the pressure due to the fact that the orifice of the capillary is at a depth d below the surface of the liquid. If σ is the density of this liquid, this pressure amounts to $g\sigma d$, so that

the pressure difference directly attributable to surface tension is $g[\rho h - \sigma d]$. We therefore have

$$\frac{2\gamma}{r} = g(\rho h - \sigma d).$$

Hence γ may be calculated when the other variables in this equation are known.

To discover the reason why the value of r used in the above equation is equal to the radius of the capillary tube at its lower end, let us suppose that the tube is uniform in diameter and that the pressure inside the apparatus is such that the centre of the hemispherical liquid surface is at C_1—cf. Fig. 10·19(b). We are justified in assuming that this surface is part of a sphere of radius r if the capillary is narrow, and the angle of contact between the liquid and the tube zero. Suppose that the pressure inside the apparatus is increased so that the centre of the surface is at C_2, the radius still being r, but that if the surface is forced down beyond this position its radius increases. When C_3 is the centre, let the radius be $(r + \Delta r)$. The pressure difference across the surface is then less and the bubble grows since the pressure inside the apparatus is too great for the surface to be in equilibrium. Thus a bubble of air escapes, and the liquid surface will lie entirely above C_2, if the removal of one bubble is sufficient to reduce the pressure inside the apparatus below the maximum pressure necessary to cause a bubble to escape from the tube. If not, several bubbles will escape.

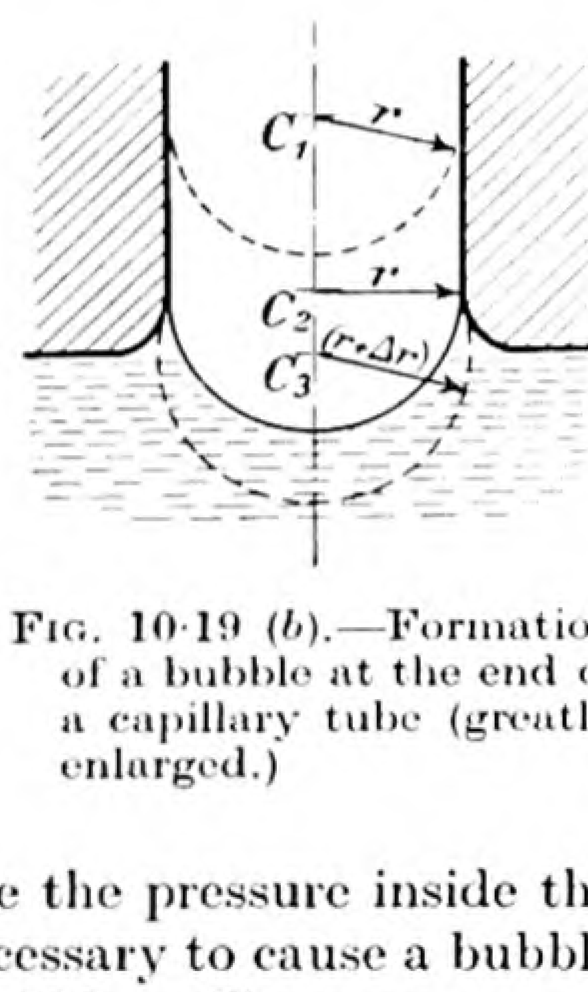


FIG. 10·19 (b).—Formation of a bubble at the end of a capillary tube (greatly enlarged.)

The great advantages of this method are that it may be applied to determine the surface tension of a molten metal, or to investigate how the surface tension of a liquid varies with temperature, or how that of a solution varies with the concentration of the dissolved substance. The method is particularly suited for such determinations as the two last, since it is not necessary to know the radius of the capillary tube. Also, since a new surface is continually being formed in the liquid the effects of contamination are reduced to a minimum, and finally the radius r can be determined before observations are made [cf. method (a)].

Unfortunately certain difficulties arise when an absolute determination of the surface tension of a liquid is being made by this method. One is seldom quite sure whether or not the size of the bubbles, when the excess pressure inside the bubble is a maximum,

is controlled by the internal or the external radius of the tube. If these radii differ considerably and the surface tension is known at least approximately, simple substitution of these values in the appropriate equation reveals the correct one.

Note on Sugden's work on the method of maximum bubble pressure.—A real difficulty, in connexion with the method of maximum bubble pressure, lies in the fact that in order to obtain a value for the surface tension of the liquid under investigation it is necessary to know how the density of that liquid varies with temperature. The requisite data is not always available and SUGDEN has so modified the original Jaeger method that such knowledge is not required. He uses two capillary tubes of different radii and determines the maximum pressure excess required to produce bubbles first from the wide capillary tube and then from the other. If the tips at which the bubbles form are at equal distances below the liquid surface, we have, with slight but obvious modification in the usual notation

$$\frac{2\gamma}{r} = g(\rho h - \sigma d) \quad \text{and} \quad \frac{2\gamma}{\mathrm{R}} = g(\rho \mathrm{H} - \sigma d),$$

so that

$$2\gamma\left[\frac{1}{r} - \frac{1}{\mathrm{R}}\right] = g\rho(h - \mathrm{H}).$$

Another advantage of Sugden's method is that a smaller container may be used, for it is not essential that the liquid surface in it should be flat since the correction term disappears from the final equation.

In the original work Sugden used capillary tubes so wide that it was not justifiable to assume that the bubbles broke away when their surfaces were hemispheres with a radius equal to that of the cross-section of the capillary tube. For further details, however, the original paper must be consulted.†

Example.—The tubes used in Sugden's method for measuring the surface tension of a liquid have diameters 0·050 cm. and 0·120 cm. If the error in the diameter of the smaller tube is 2 per cent, estimate its effect on the calculated value of the surface tension, γ.

We have, cf. above, if $p = g\rho(h - \mathrm{H})$,

$$p = 2\gamma\left(\frac{1}{r} - \frac{1}{\mathrm{R}}\right),$$

where r and R are the radii of the tubes, $r < \mathrm{R}$. Assuming the errors in the values of p and R to be negligible, we have, by differentiation,

$$0 = \delta\gamma\left(\frac{1}{r} - \frac{1}{\mathrm{R}}\right) + \gamma\left(-\frac{\delta r}{r^2}\right)$$

† *Jour. Chem. Soc.*, **121**, 858, 1922.

$$\therefore \frac{\delta\gamma}{\gamma} = \frac{\delta r}{r^2} \cdot \frac{r\mathrm{R}}{\mathrm{R} - r} = \frac{\delta r}{r} \cdot \frac{\mathrm{R}}{\mathrm{R} - r}$$

$$= \frac{2}{100} \cdot \frac{12}{12 - 5}$$

$$= 3{\cdot}4 \text{ per cent.}$$

(c) ***Ordinary balance method:*** The surface tension of a liquid which wets glass may be determined as follows. A glass plate, A, Fig. 10·20(a), (a microscope slide) is supported by means of a metal clip, C, from below the pan of a balance—the lower edge of the slide is made horizontal. The vessel, D, containing the liquid is placed on a small table below the slide. The table may be raised by means of a screw, S. The balance is equilibrated and left free to swing. The adjustable table is then screwed up till the liquid ***just*** touches the lower edge of the plate. This is shown by a sharp jerk of the pointer as the microscope slide is pulled down by surface tension. Masses are then added to the other pan of the balance until the slide is withdrawn from the liquid. Since the lower edge of the slide had been in the general level of the liquid surface there is no correction for buoyancy. If l is the length and t the thickness of the slide at its lower edge, the force due to surface tension acting on it is $2(l + t)\gamma$. This is equal to mg, where m is the mass added to the pan to restore equilibrium. Hence γ may be determined.

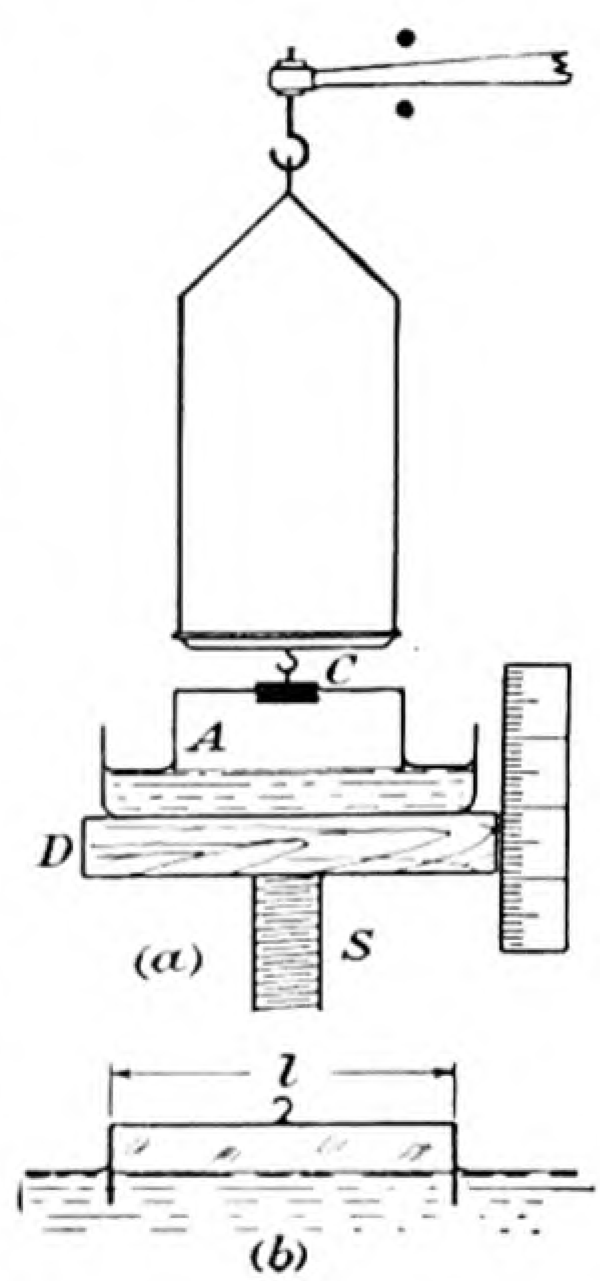


FIG. 10·20.—Surface tension by ordinary balance methods.

An alternative method is as follows. Having screwed up the adjustable table till the pointer jerks, observe the position of the table (suitable scales may be arranged as on a spherometer). Instead of restoring equilibrium as above, the table is screwed up through a distance h until the pointer is back at zero. Then the buoyancy force just balances the force due to surface tension, and if the vessel containing the liquid has a large surface area, so that h will be also the depth of immersion of the slide, then

$$2(l + t)\gamma = lth\rho g,$$

where ρ is the density of the liquid.

It must be noted that this method only yields accurate results if the liquid completely fills the containing vessel so that the surface

of the liquid may be cleaned with the aid of waxed pieces of glass, as described on p. 464.

The plate method described above may easily be adapted to determine the surface tension of a soap solution.† A glass or wire frame, as shown in Fig. 10·20(*b*), is made and is supported from below one pan of a balance, and arranged so that when the balance is equilibrated, the horizontal portion of the frame is about 0·5 cm. above the general surface of the liquid. The frame is then immersed completely and extra masses, m, added to the right-hand balance pan until the frame is in the same relative position as before. If l is the length of the horizontal portion, the weight of the film being negligible, $2\gamma l = mg$.

This method may be used for liquids such as water, the horizontal portion of the frame then being nearer to the general surface of the liquid.

(*d*) ***Thread method:*** This is a simple but interesting method for finding the surface tension of a soap solution. Two pieces of glass rod about 1·5 to 2 mm. in diameter are cleaned and one of them is then bent as in Fig. 10·21(*a*). The bent rod must then be supported horizontally in a clamp. To the points A, B, C and D, equal lengths of cotton are attached as shown, AB being about 10 cm., while AC is about 8 cm. We thus have a rectangular frame which

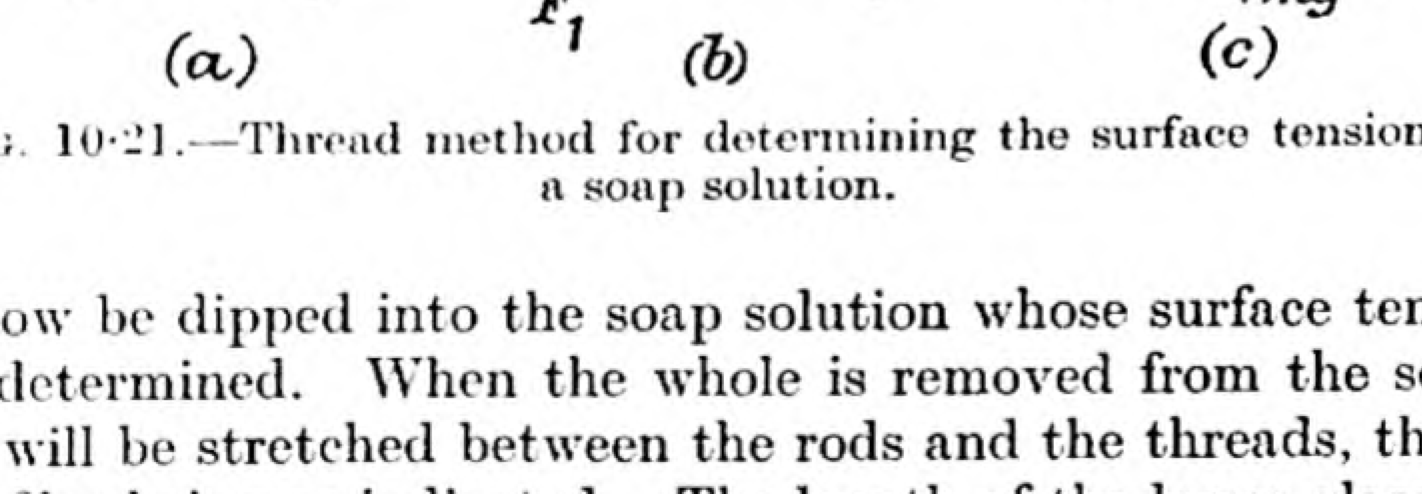


FIG. 10·21.—Thread method for determining the surface tension of a soap solution.

may now be dipped into the soap solution whose surface tension is to be determined. When the whole is removed from the solution a film will be stretched between the rods and the threads, the form of the film being as indicated. The length of the lower glass rod is then adjusted until the curvature of the threads is distinct and

† Prof. Boys recommends the following soap solution. To a litre of distilled water contained in a well-stoppered bottle add 25 gm. of sodium oleate, and let it stand for 24 hours. Then add about 300 cm.³ of glycerol, shake well, and allow to stand for a week. By means of a siphon remove the clear liquid, leaving the scum behind. Add two or three drops of liquid ammonia to the solution and store in a dark cupboard. The solution must not be warmed or filtered.

measurable, care being taken that CD is horizontal; if necessary a small loop of wire, nearly fitting the rod, may be placed on CD and its position adjusted until CD is horizontal.

Now consider the equilibrium of a small portion PQ, Fig. 10·21(*b*), of length δs, of one of the threads. Let the normals at P and Q meet at C, where $\widehat{PCQ} = \delta\theta$, and let r be the radius of curvature at P. Suppose that F_1 and F_2 are the forces at P and Q arising from the tension in the thread. The other force acting on this element, if we neglect its weight [so that the cotton used should be very thin] is due to surface tension and amounts to $2\gamma\,\delta s$; it acts normally to PQ at its mid-point. Resolving forces along and normal to the tangent at the mid-point of PQ, we have, since the forces on PQ are in equilibrium,

$$F_1 \cos \tfrac{1}{2}\,\delta\theta - F_2 \cos \tfrac{1}{2}\,\delta\theta = 0,$$

and $$2\gamma\,\delta s = (F_1 + F_2)\cos(\tfrac{1}{2}\pi - \tfrac{1}{2}\,\delta\theta).$$

Hence $$F_1 = F_2 = F \text{ (say)},$$

i.e. the tension in the thread is constant, and

$$2\gamma\,\delta s = 2F \sin \tfrac{1}{2}\,\delta\theta = F\,\delta\theta,$$

since $\delta\theta$ is small.

$$\therefore\ \frac{ds}{d\theta} = \lim \frac{\delta s}{\delta\theta} = \frac{F}{2\gamma} = \text{constant},$$

i.e. the radius of curvature is constant, or the thread takes the form of a circle of radius r, where $r = PC$. Hence

$$F = 2\gamma r. \qquad \text{(i)}$$

To show how the form of the film is related to the surface tension of the solution and the mass of CD and any wire ring attached to it, let us imagine that the film is divided into two equal portions across XY, where XY is parallel to CD—cf. Fig. 10·21(*c*). Since this portion of the film is in equilibrium,

$$2F + 4\gamma b = mg \qquad \text{(ii)}$$

where m is the mass of CD [and any wire rings attached to it], g the intensity of gravity, γ the surface tension of the solution, and $2b$ the width of the film at XY.

Eliminating F from (i) and (ii)

$$4\gamma(r + b) = mg \qquad \text{(iii)}$$

If $2h$ is the vertical distance between the rods when the film is present, and $2a$ the distance between the threads when no film is present,

$$(a - b)(2r - a + b) = h^2,$$

or $$r = \frac{1}{2}\left(\frac{h^2}{a - b} + a - b\right).$$

Substituting this value for r in (iii), we have

$$\gamma = \frac{mg(a - b)}{2(h^2 + a^2 - b^2)} \quad . \quad . \quad . \quad . \quad \text{(iv)}$$

With this very simple apparatus it is possible to obtain a very fair value for the surface tension of a soap solution, provided none is allowed to spread beyond the confines of the threads or to remain on the lower rod; that which does when the apparatus is lifted out of the solution should be removed with a piece of filter paper.

The necessary length measurements may be made with a pair of dividers and a scale in millimetres, or an image of the film may be produced by means of a converging lens on a piece of graph paper, due allowance being made for the magnification produced.

Example.—A uniform wire is bent into the form of a square of side $2a$ and one side is removed and replaced by a flexible and inextensible thread of length $(\pi + 1)a$. The arrangement is then dipped into a soap solution and removed. If the mass of the thread is negligible find, from first principles, the configuration of the thread and the area of one side of the film supported between the thread and the wire frame.

As shown on p. 491 the tension F in the thread is constant and related to the surface tension γ by the equation

$$\frac{F}{2\gamma} = \text{radius of curvature of thread,}$$

which is constant. Hence, in the problem in hand, the length of thread in contact with the film is πa, i.e. $0{\cdot}5a$ is the length of thread in contact with each 'vertical' side of the frame.

$$\therefore \text{ Area required } = (2a \times \tfrac{3}{2}a) - \tfrac{1}{2}\pi a^2 = (3 - \tfrac{1}{2}\pi)a^2 = 1{\cdot}43a^2.$$

(*e*) ***Searle's surface tension balance:*** When the lower edge of a clean, thin rectangular plate is just immersed in a liquid it experiences a pull of $2\gamma l$ downwards, where γ is the surface tension of the liquid and l the length of the glass plate. This pull is small and to measure it conveniently SEARLE designed a torsion balance shown in Fig. 10·22. A thin wire W, Fig. 10·22(*a*), is stretched between two vertical pillars and has a short length of fine copper tubing soldered to it at O. The tension in the wire can be adjusted by means of the screw R. An asymmetrical lever AOB, Fig. 10·22(*b*), is attached to the torsion wire at O by means of a small screw. A massive knob C, threaded on to OB, serves to adjust the lever to a horizontal position and since the balance is very sensitive to changes in the position of C, the mass must be securely fixed by means of the light screw D. The end A of the lever moves over a scale S (in mm.), and the lever can be adjusted exactly to a horizontal position by adding a known mass on to the lever at X_1, and by twisting the supporting wire by means of a knob or lever K attached to the wire. These adjustments are made with the plate in position and m at X_1 near to A.

The water, which we assume to be the liquid whose surface tension is to be determined, is placed in a clean vessel, E, on the top of a table which can be raised or lowered. The author† has suggested that the upper rim of the vessel E should be coated with clean paraffin wax so that water may be added until its surface is above the rim of E. Three waxed glass strips, G_1, G_2, G_3 and known as barriers, are then used to clean the water surface as described on p. 465.

The torsion balance having been equilibrated, the vessel E is

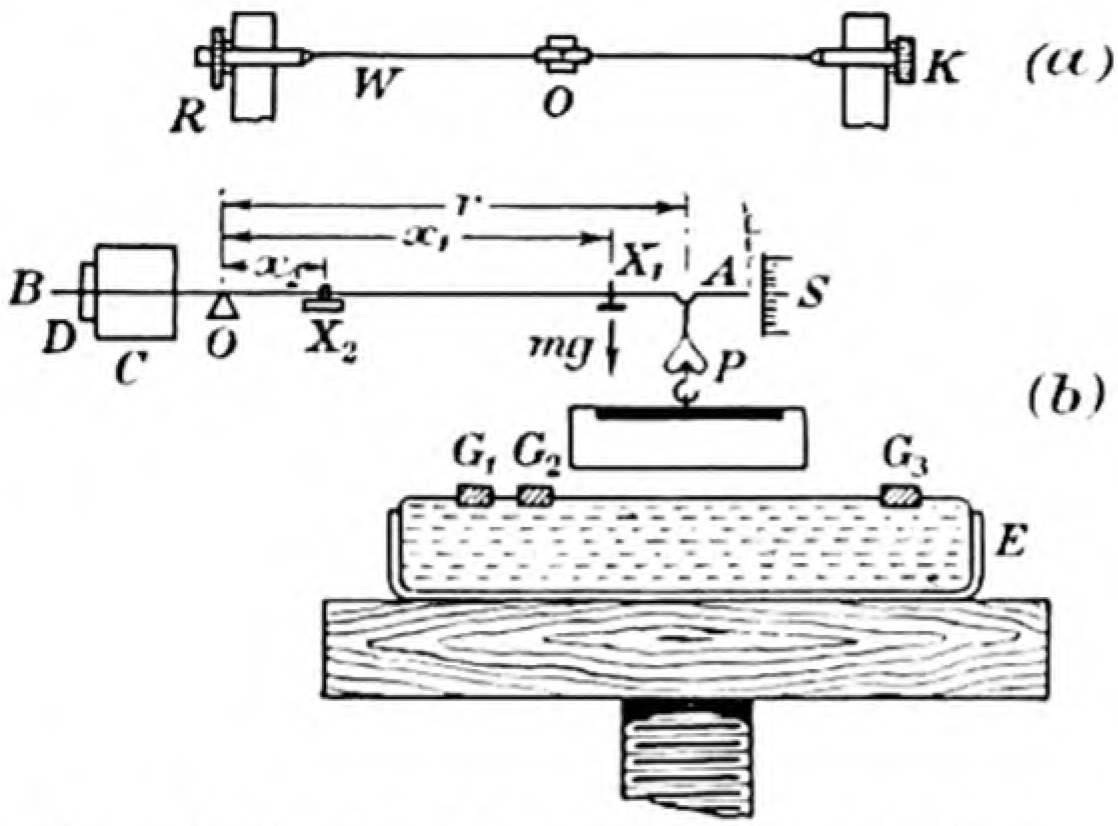


FIG. 10·22.—Searle's surface tension balance.

raised until the water surface just touches the suspended plate. At once the plate is drawn down into the water and the mass m is moved along the lever from X_1 to X_2 until the moment of the forces acting on the lever is reduced to such an extent that the glass plate leaves the water. Under these conditions there is no question of the effect of any upthrust on the glass plate for this has been made zero automatically. Then if r is the distance of the point of suspension of the plate, etc., from O,

$$2\gamma lr = mg(x_1 - x_2),$$

where x_1 and x_2 are the distances of X_1 and X_2 from O.

$$\therefore \gamma = \frac{mg(x_1 - x_2)}{2lr}.$$

[If b is the thickness of the plate, a correction can be applied by using

$$\gamma = \frac{mg(x_1 - x_2)}{2(l + b)r}.]$$

(f) Sentis' method for measuring the surface tension of a liquid: As originally described by SENTIS‡ this method requires a

† *Phil. Mag.*, **33**, 775, 1942.
‡ *Jour. de Physique*, **6**, 571, 1887.

clean capillary tube drawn out to a fine jet. This is dipped into the liquid under investigation and then withdrawn; some of the liquid which has entered the tube exudes and forms a small spherical drop at the drawn-out end of the capillary—Fig. 10·23(a). Consideration of the pressures at the points A, B, C and D shows that

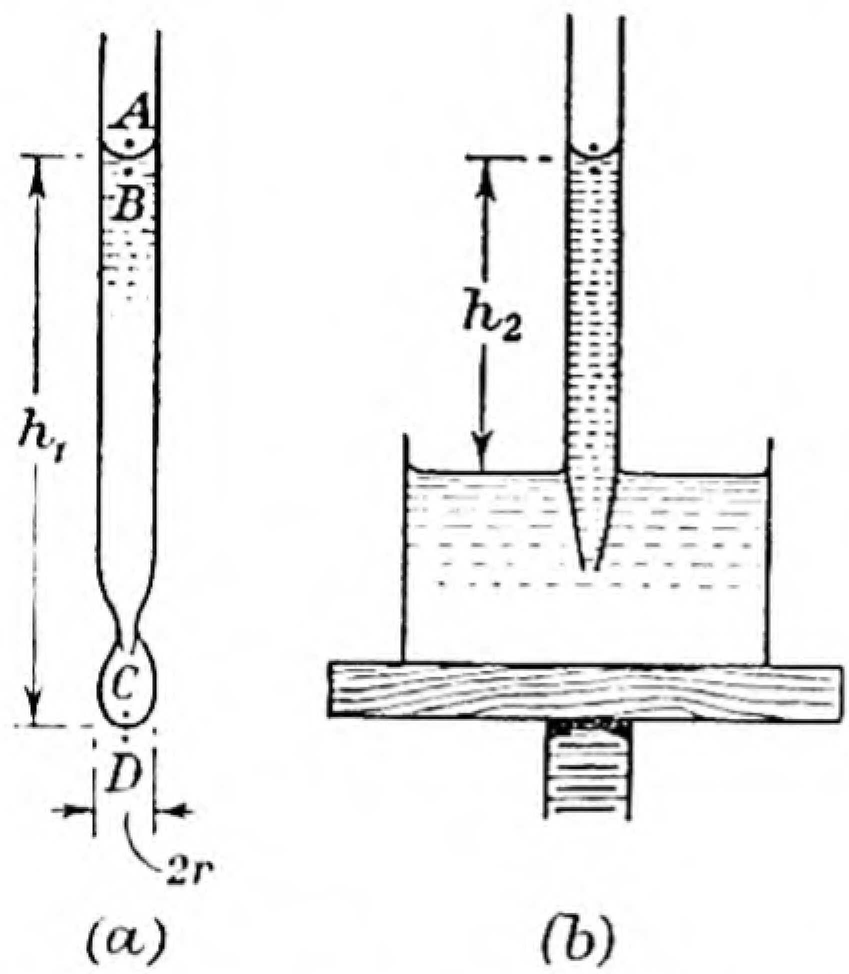


FIG. 10·23.—Sentis' method for measuring the surface tension of a liquid.

$$\frac{2\gamma}{R} + \frac{2\gamma}{R_0} = g\rho h_1.$$

where R and R_0 are the radii of curvature of the surfaces at C and at A, and the other symbols have their usual meanings. This is an exact equation.

Now place below the tube a vessel containing some of the liquid under investigation, the vessel being supported on a table which may be raised or lowered by a measured amount. Observe the position of the table when the surface of the liquid in the vessel just touches the pendent drop. Finally raise the vessel until the upper meniscus is restored to its original position in the vertical capillary—Fig. 10·23(b). Then

$$\frac{2\gamma}{R_0} = g\rho h_2.$$

so that
$$\frac{2\gamma}{R} = g\rho(h_1 - h_2) = g\rho H,$$

where H is the measured distance through which the table is raised.

It will be noted that the above equation is independent of the radius of the capillary tube and also of the angle of contact which the liquid makes with glass. This latter point is not always appreciated although Sentis himself wrote, 'Voici une méthode analogue qui est indèpendante de cette hypothése' (que l'angle de racordement est nul).

Now if the drop is truly spherical in shape below its maximum cross-section we may write $R = r$, where $2r$ is the diameter of the largest horizontal section of the pendent drop, so that

$$\frac{2\gamma}{r} = g\rho H.$$

A modification of this method is due to the author.† A criticism of the original method appears at once when the following question is asked:—Why is it necessary to 'draw down the capillary at its lower end'? One can see at once that a smaller drop will be formed when such a tube is used but it must be remembered that when the tube is thus constructed the formation of the drop is hindered if the viscosity of the liquid is appreciable, so that measurements may be taken before the truly steady conditions have been attained. This difficulty is overcome if the tube is constructed as shown in Fig. 10·24(a). A glass tube about 2 mm. external diameter is sealed on to the lower end of the capillary tube (0·5 mm. to 1 mm. in diameter) and then cut off so that its length is about 5 mm. Well-defined drops, easily measurable, are then formed very rapidly even when liquids with a high viscosity are used, but it is no longer justifiable to assume that $R = r$. To make a correction we may proceed as follows. Let P_0 be the pressure at A; then at B it is $P_0 - \dfrac{2\gamma}{R_0}$. If MN, Fig. 10·24($b$), is the plane of maximum cross-sectional area in the drop, the pressure at any point in it is

$$P_0 - \frac{2\gamma}{R_0} + (h_1 - b)g\rho,$$

where b is the distance of D below MN.

The force on MN, acting downwards, due to the above pressure is

$$\pi r^2\left[P_0 - \frac{2\gamma}{R_0} + (h_1 - b)g\rho\right].$$

Consider now the equilibrium of the lower portion of the drop which we assume to be half an ellipsoid of revolution. Its weight is $\frac{2}{3}\pi r^2 bg\rho$, and this force acts downwards. The force due to surface tension is $2\pi r\gamma$ acting vertically upwards while the air exerts a force $\pi r^2 P_0$. Hence, for equilibrium

$$2\pi r\gamma = \pi r^2\left[-\frac{2\gamma}{R_0} + (h_1 - b)g\rho\right] + \tfrac{2}{3}\pi r^2 bg\rho$$

$$= \pi r^2[\{(h_1 - h_2) - b\}g\rho] + \tfrac{2}{3}\pi r^2 bg\rho,$$

since $\dfrac{2\gamma}{R_0} = g\rho h_2$, cf. p. 494. Calling $h_1 - h_2 = H$ and writing

$$a^2 = \frac{\gamma}{g\rho},$$

we get

$$2a^2 = (H - b)r + \tfrac{2}{3}rb$$

$$= Hr - \tfrac{1}{3}br \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

† *Phil. Mag.*, **33**, 775, 1942.

It is known however, cf. p. 10, that $R = \frac{r^2}{b}$, so that $b = \frac{r^2 H}{2a^2}$, since $2a^2 = RH$. Substituting this value for b in (i), we get

$$2a^2 = Hr - \frac{r^3 H}{6a^2},$$

or

$$12a^4 - 6Hra^2 + r^3 H = 0.$$

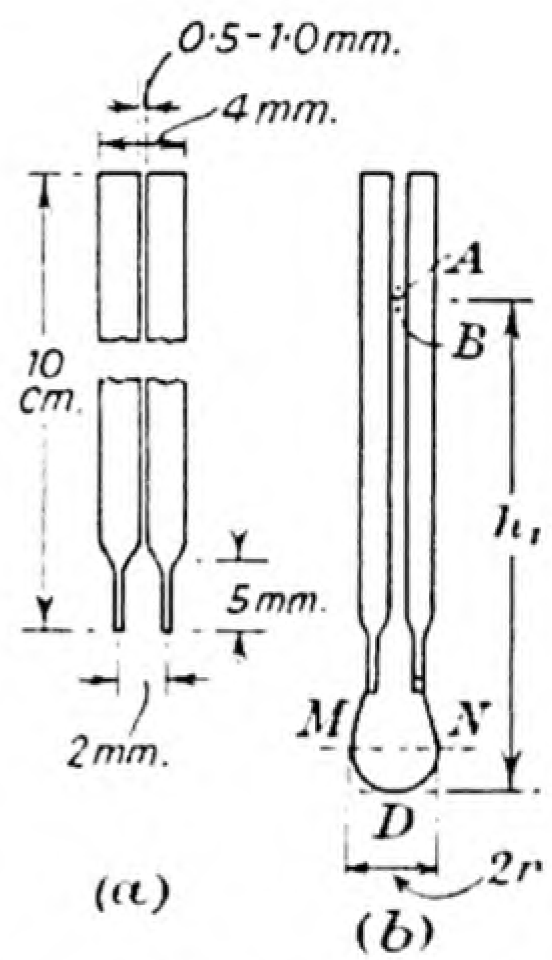


Fig. 10·24.—Author's modified form of tube for use with Sentis' method; more complete theory.

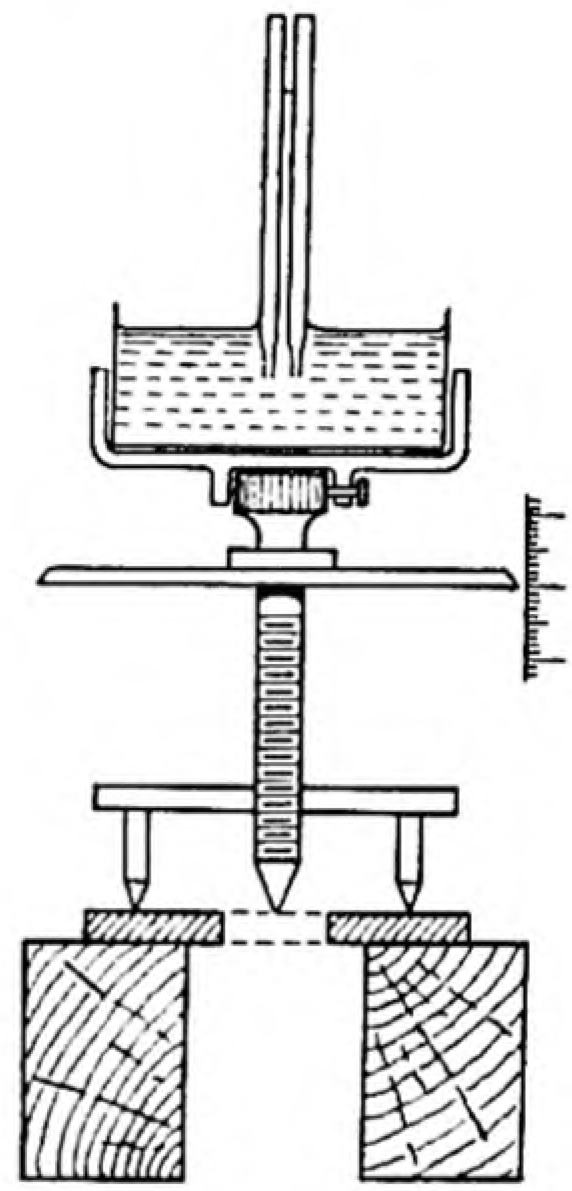


Fig. 10·25.—Determination of surface tension of a liquid by Sentis' method.

Solving this quadratic for a^2, we get

$$a^2 = \frac{6rH \pm \sqrt{36r^2H^2 - 48r^3H}}{24} = \frac{rH \pm rH\left[1 - \frac{4}{3}\cdot\frac{r}{H}\right]^{\frac{1}{2}}}{4},$$

and only the positive sign is to be retained since when $r \to 0$, we must have $2a^2 = rH$. Expanding the surd and rejecting terms in r^4, we get,

$$a^2 = \tfrac{1}{2}rH - \tfrac{1}{6}r^2 - \frac{1}{18}\cdot\frac{r^3}{H} = \frac{\gamma}{g\rho}.$$

$$\therefore\ \gamma = \tfrac{1}{2}g\rho\left[rH - \tfrac{1}{3}r^2 - \frac{1}{9}\cdot\frac{r^3}{H}\right].$$

A convenient set-up for this experiment, apart from the two microscopes, is shown in Fig. 10·25, and no further details should be required.

(*g*) ***Ferguson's method for determining surface tension:*** FERGUSON† devised the following modification of the capillary tube method for measuring the surface tension of a liquid. The great advantage of the method is that only a small volume of liquid is required. The capillary tube, arranged horizontally, is attached to a suitable pressure gauge, A, and Woulff's bottle, B, as shown in Fig. 10·26(*a*). A small quantity of the liquid under investigation

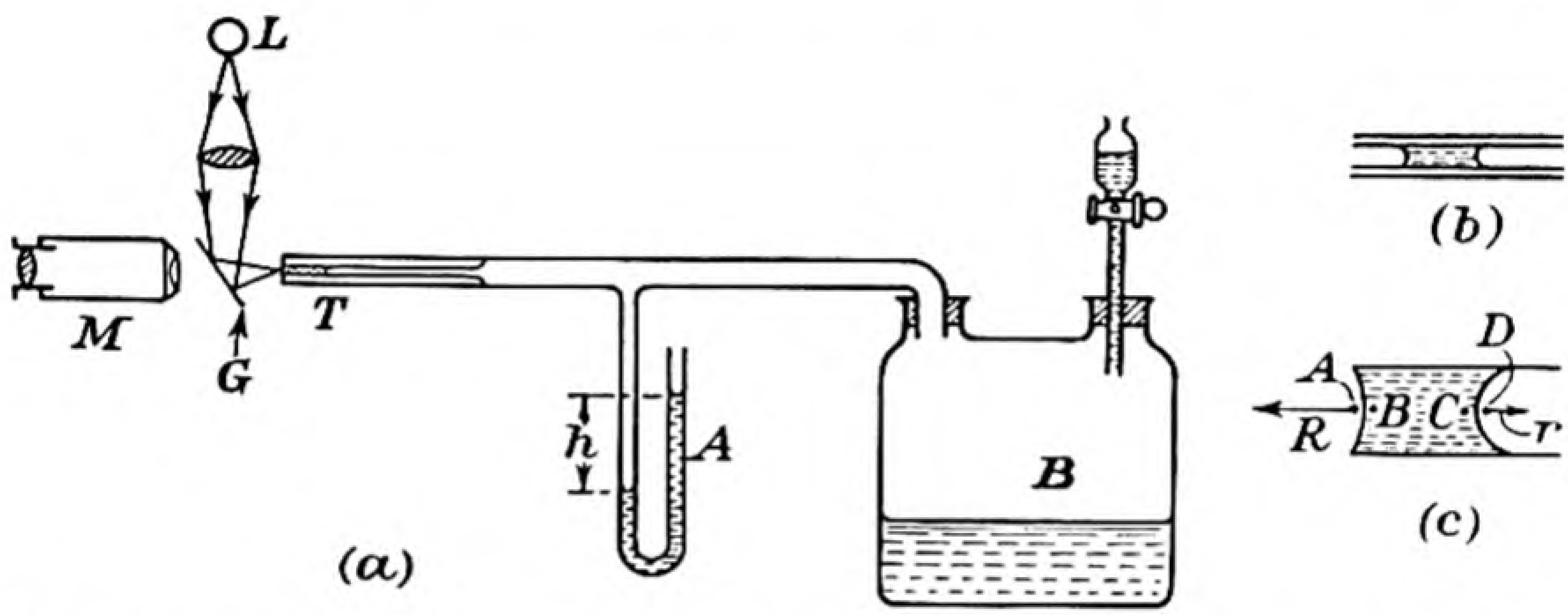


FIG. 10·26.—Ferguson's method for determining the surface tension of a liquid (especially when available in small quantities only) and the interfacial tension between two liquids.

is placed in the capillary tube, T. If the pressures on either side of the drop are equal, the drop will appear as in Fig. 10·26(*b*), the radii of curvature of the two liquid surfaces being equal but in contrary directions. Suppose now that the pressure in the Woulff's bottle is increased by allowing liquid to pass from the dropping funnel into the bottle. The smallest increase in pressure suffices to cause the drop to move outwards to the end of the capillary tube; when greater increases in pressure occur equilibrium of the drop is obtained by the outer surface becoming less curved, i.e. the radius of curvature is greater. For suppose the pressure excess inside the apparatus is $g\rho h$, where ρ is the density of the liquid in the gauge and h is the difference in height of the liquid surfaces therein, then at B, Fig. 10·26(*c*), a point just inside the liquid, the pressure excess above that at A is $-\dfrac{2\gamma}{\mathrm{R}}$, where γ is the surface tension of the liquid and R is the radius of curvature at B. The pressures at B and C are necessarily equal. At D, a point just outside the other surface of

† *Proc. Phys. Soc.*, **36**, 37, 1923; cf. also Appendix.

the drop, the pressure exceeds that at C by $\frac{2\gamma}{r}$, where r is the radius of curvature at C. Therefore, pressure excess at D over that at A is $2\gamma\left[-\frac{1}{R}+\frac{1}{r}\right]$, and this is $g\rho h$.

Suppose that the pressure is increased till $R \rightarrow \infty$, i.e. one surface of the drop is flat. Then if $h = H$ when $R \rightarrow \infty$,

$$g\rho H = \frac{2\gamma}{r}.$$

Thus γ may be found, and it will be noted that the density of the liquid under investigation does not appear in the formula. This is an additional advantage to the method.

To test the degree of flatness of the first surface of the drop, an image of a small opal lamp, L, is focused on the end of the capillary tube, T, with the aid of a short focus converging lens. A piece of thin glass, G, placed with its plane at 45° to the axis of the tube, permits the operation to be carried out without obstructing the view of the end of the tube in a microscope M. When the surface is flat it appears to be uniformly illuminated. Otherwise an image of the opal lamp is seen and this gradually broadens as the surface becomes plane. The method is very sensitive so that the equilibrium position may be ascertained accurately.

[N.B. In the above argument it has been assumed that the liquid completely wets, cf. p. 463, the walls of the capillary tube.]

This method is readily adapted to determine the interfacial tension between two liquids. Suppose that α and β are two liquids in the horizontal capillary tube. Suppose that a pressure excess, $g\rho h$, inside the apparatus is sufficient to make plane the surface of the liquid β at the open end of the capillary tube. Then if zero contact angles are assumed,

$$\gamma_{\alpha\beta} + \gamma_{\alpha} = \tfrac{1}{2}g\rho h r,$$

where $\gamma_{\alpha\beta}$ is the interfacial tension required, and γ_{α} is the surface tension of the liquid α.

(*h*) ***Anderson and Bowen's method:*** This method† depends on the measurement by optical means of the radius of curvature of the meniscus in a vertical tube immersed in the liquid and the height of the centre above (or below) the surface of the liquid outside. It is thus free from any correction due to the departure of the shape of the meniscus from that of a true hemisphere; moreover no knowledge of the angle of contact is required.

The curve connecting the radius of curvature thus determined

† *Phil. Mag.*, **31**, 143, 1916.

with the radius of the tube at the top of the meniscus, for a number of tubes of different diameters, enables the angle of contact to be determined.

A glass vessel, A, Fig. 10·27(a), with a plane base, had cemented to its base a right-angled glass prism, P. Parallel sodium light from a collimating lens, L_1, was reflected by the prism along the axis of a vertical glass tube, T, dipping into the liquid in A, and the image formed by refraction of the light at the meniscus was observed by means of a low-power microscope, M (scale read to 0·001 cm.)—cf. also Fig. 10·27(b).

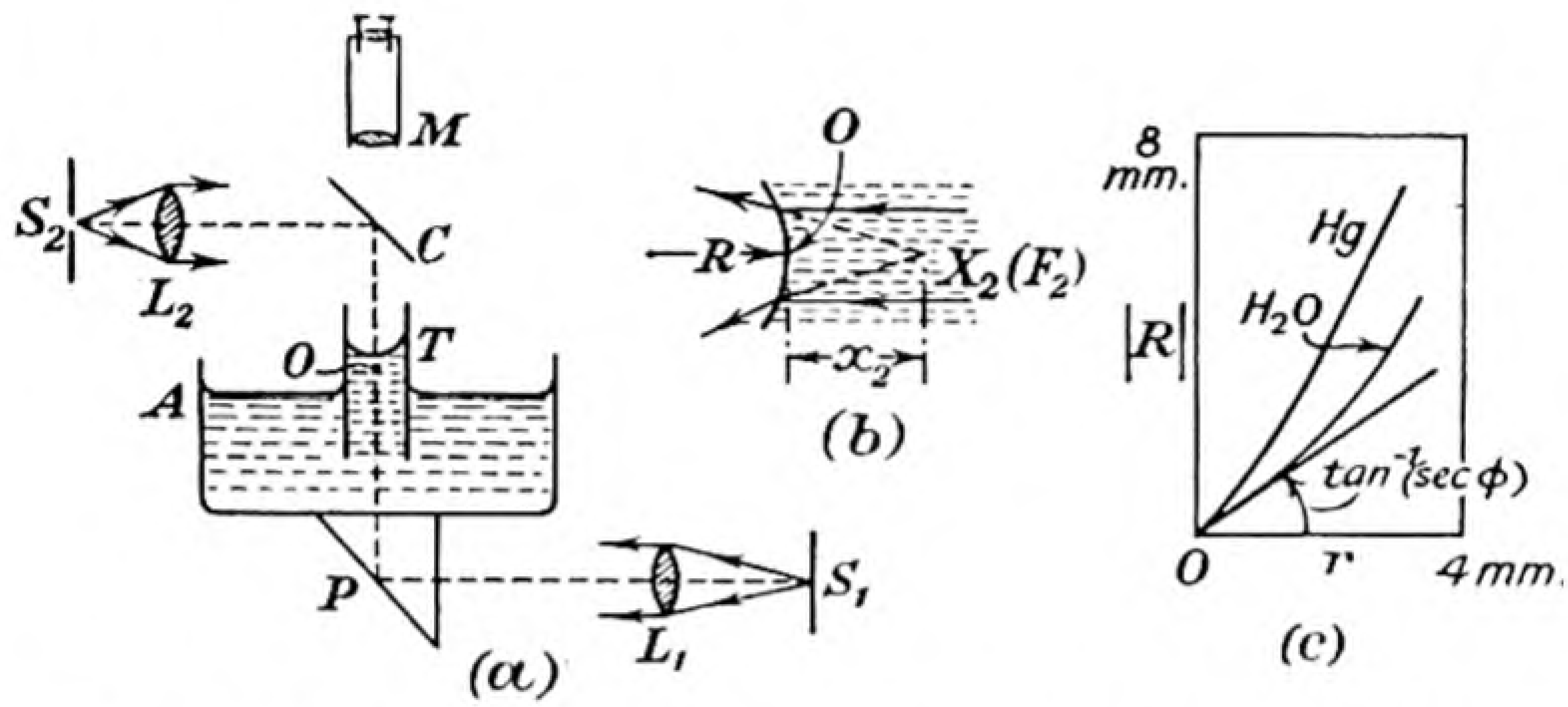


FIG. 10·27.—Anderson and Bowen's apparatus for measuring the surface tension of a liquid.

Let R be the radius of curvature of the meniscus. Then if x_2 is the distance of the image $X_2(=F_2)$, formed by the refraction of the parallel beam at the curved surface of the liquid, from the lowest point on the meniscus

$$\frac{1}{x_2} - 0 = \frac{1-\mu}{R},$$

where μ is the refractive index of the liquid for sodium light.

Since R is essentially negative, x_2 is positive.

$$\therefore\ |x_2|\,(\mu - 1) = |R|.$$

To determine x_2 it was necessary to know the position of O. Direct focusing on O was avoided by using light from a point source S_2 and collimating this light by means of a converging lens L_2 and then reflecting this on to the meniscus by means of a cover glass so that an image at a distance $\frac{1}{2}\,|R|$ from the meniscus was seen. If ξ is the distance between the image formed by refraction and that formed by reflexion, then

$$\xi = \tfrac{1}{2}\,|R| + \frac{|R|}{(\mu - 1)},$$

or
$$|R| = 2\xi . \left[\frac{\mu - 1}{\mu + 1}\right].$$

If h is the height to which the liquid rises in the tube—this was determined by shifting the microscope laterally and focusing on the level surface—then
$$\gamma = \tfrac{1}{2} g \rho h\, |R| = g \rho h \xi \left[\frac{\mu - 1}{\mu + 1}\right].$$

For opaque liquids such as mercury it is necessary to find R by a reflexion method and the uppermost point in the surface of the mercury must be accurately sighted.

To apply the method to find angles of contact, experiments must be carried out with tubes of different radii (r). Then a plot of r (as x) against $|R|$ (as y) will give a curve whose angle of slope at the origin is $\tan^{-1}$ (sec ϕ), where ϕ is the angle of contact. The reason for this statement is that when the tubes are narrow $|R| = r \sec \phi$, cf. p. 476. For water, glycerine, olive oil and turpentine, the slope of the curves at the origin was unity, pointing to zero contact angle. For mercury on glass ϕ was found to be 139°—cf. Fig. 10·27(c).

(i) ***Large sessile drops and bubbles:*** To determine, at room temperature, the surface tension of mercury, γ, and its angle of contact with glass, ϕ, Quincke† made observations on a sessile

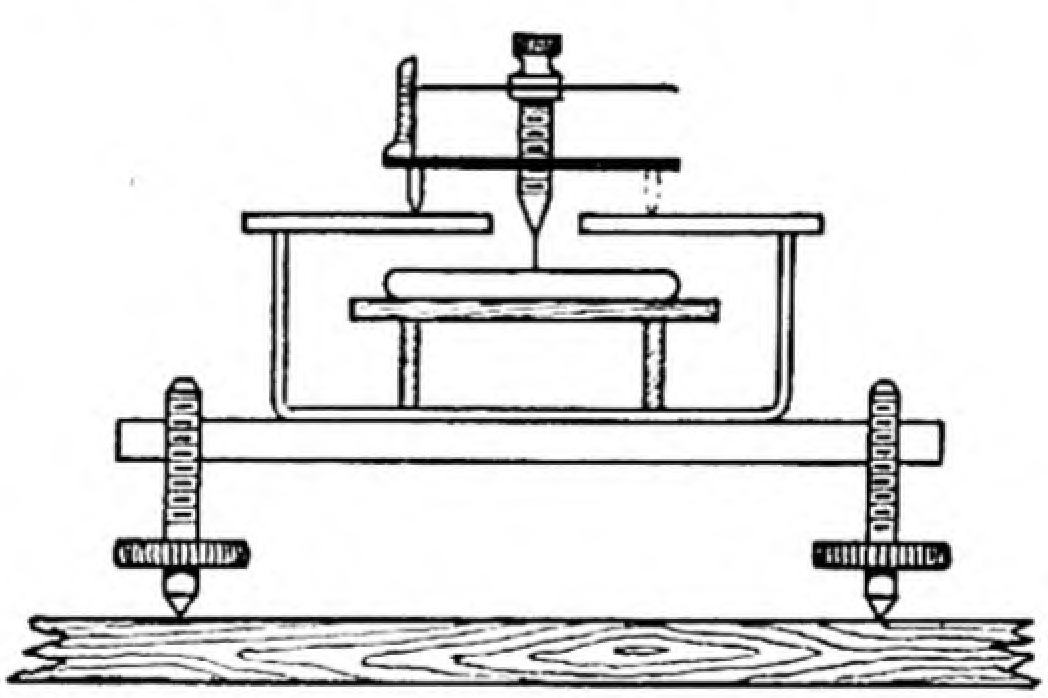


Fig. 10·28.—Surface tension of mercury from observations on a large sessile drop.

drop of mercury formed on a glass plate. The experiment may be repeated in the following manner. A piece of plate glass, Fig. 10·28, rests on a steel cylinder inside a rectangular glass box supported on a table provided with levelling screws. The upper surface of the glass plate must be cleaned scrupulously and then made level with the aid of a drop of mercury at least 2 cm. in diameter and which has been formed upon it, the mercury being filtered by

† *Pogg. Ann.* **105** 1, 1858.

allowing it to run through a very fine capillary tube. More mercury is then added in the same way until the drop is at least 8 cm. in diameter. To determine γ and ϕ, the total height of the bubble, H, and the depth of the maximum cross-section below its upper surface, h, must be measured. To ascertain the exact point where the contour of the bubble is vertical the following optical arrangement, due to COOK, may be used. Beside the telescope or low-power microscope used to view the drop is placed a lamp with a horizontal slit in front of it. This slit is adjusted to be on the same level as the axis of the telescope. When this axis, the horizontal slit and the point on the curved surface at which a tangent plane to the drop is vertical are all in the same horizontal plane, a bright 'star' is observed at the centre of the cross-wires of the telescope.

To determine H and h a spherometer, with a glass pin, drawn out to a very fine point and attached to the central leg of the instrument, is placed on a glass plate which rests on the glass box. The pin is lowered until it just touches the upper surface of the mercury, and its position noted. The mercury drop is removed and the microscope raised and 'reached' forward until the tip of the pin is seen. The vertical shift of the microscope is h. The central leg of the spherometer is then lowered until the pin just touches the top of the plate; H is obtained from the spherometer readings. Then since

$$h^2 = \frac{2\gamma}{g\rho}, \quad \text{and} \quad H^2 = \frac{2\gamma(1 - \cos\phi)}{g\rho}, \quad \text{[cf pp. 481–2]},$$

so that

$$1 - \cos\phi = \frac{H^2}{h^2},$$

both γ and ϕ may be determined.

(*j*) ***The drop-weight method:*** One of the commonest methods for measuring the surface tension of a liquid consists in determining the mass of a drop which falls slowly from the tip of a vertical tube. If it were justifiable to assume that the drop breaks away as soon as it has acquired a cylindrical surface near the end of the tube, so that the radius of cross-section of the drop near the tube is equal to the external radius of the tube from which it falls, then mg, the weight of the drop, would be given by

$$mg = \pi\gamma r,$$

if account of the excess pressure within the drop is taken into consideration.

To establish this equation let us assume that r is the radius of the cylindrical portion, AB, of the drop—cf. Fig. 10·29(a). Just before the drop breaks away it is in static equilibrium under the action of

(i) its weight mg, directed downwards,
(ii) an upward force $2\pi\gamma r$, due to surface tension,
(iii) a force $p(\pi r^2)$, directed upwards and due to the pressure p of the atmosphere.
(iv) a force $\left(p + \frac{\gamma}{r}\right)\pi r^2$, acting downwards on its surface AB; the term γ/r arises since the pressure inside a cylindrical surface exceeds the external pressure, cf. p. 459.

These forces are indicated in Fig. 10·29(b). Since the drop is formed slowly we assume that it is in static equilibrium and, by equating the above forces, obtain $mg = \pi\gamma r$.

[If the term γ/r in (iv) is omitted, we get

$$mg = 2\pi\gamma r,$$

an equation which, many years ago, was used (incorrectly of course) in connexion with the drop-weight method—cf. p. 505, however.]

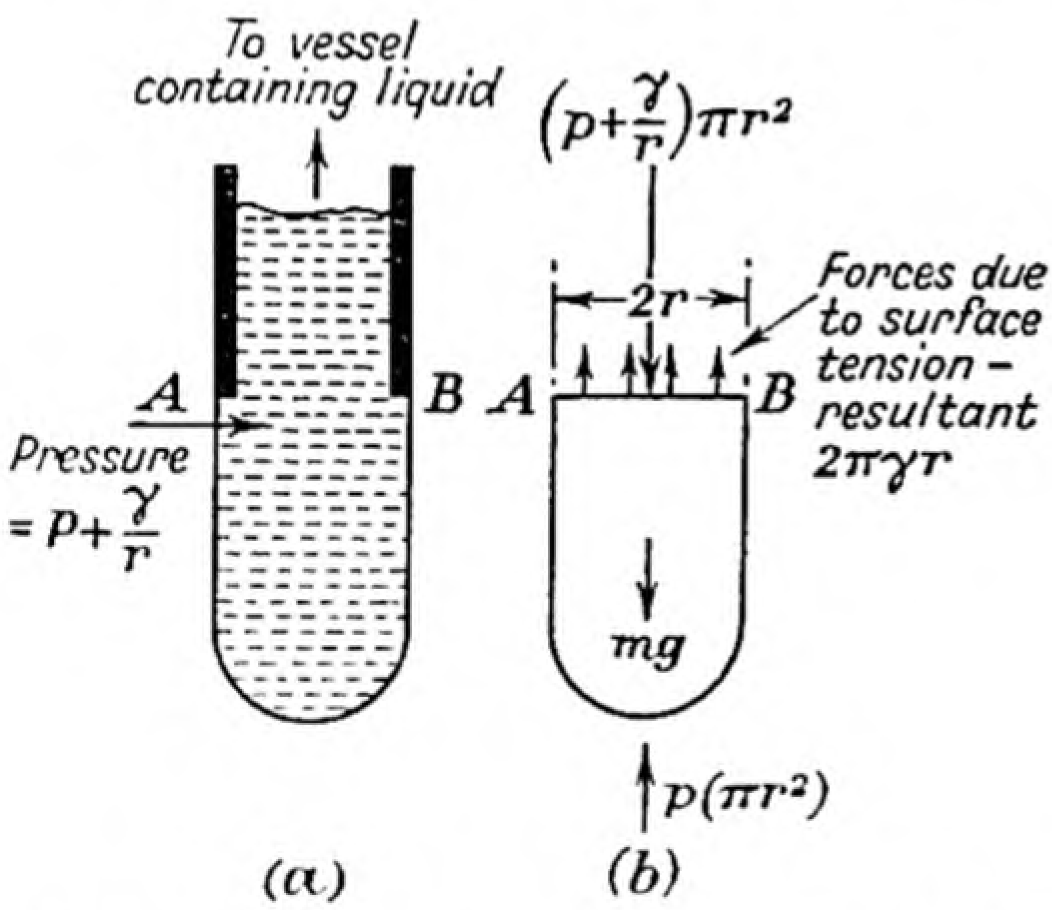


FIG. 10·29.—The drop-weight method for measuring surface tension.
(a) Shows the drop and it *is assumed* to break away when this stage is reached.
(b) Shows the forces on the drop.

This equation cannot be valid since the edge of the drop is rarely vertical and also because the whole of the drop does not fall. A cinematographic study of a falling drop by Gaye and Perrot† shows why no simple formula can be found connecting the variables; Fig. 10·30 shows four successive stages in the process after the drop has become unstable. Two drops are formed—the smaller is known as ***Plateau's spherule***—and some of the liquid remains behind. The mass of the two drops together is determined in this method of finding γ.

† *Arch. Sci. Phys. Nat.* (*Geneva*), **15**, 178, 1903.

The theory of the method has not been worked out in detail but HARKINS and BROWN,† following the pioneer work of RAYLEIGH, have bridged any gap in the theory by carrying out experiments with liquids for which γ was known accurately. They find that one of the essential conditions for success in this method is that the drop shall be formed, at least in the stages immediately preceding its detachment, with great slowness. Four minutes is about the minimum time required for the formation of one drop, although this can be shortened considerably by allowing the first stages in the formation

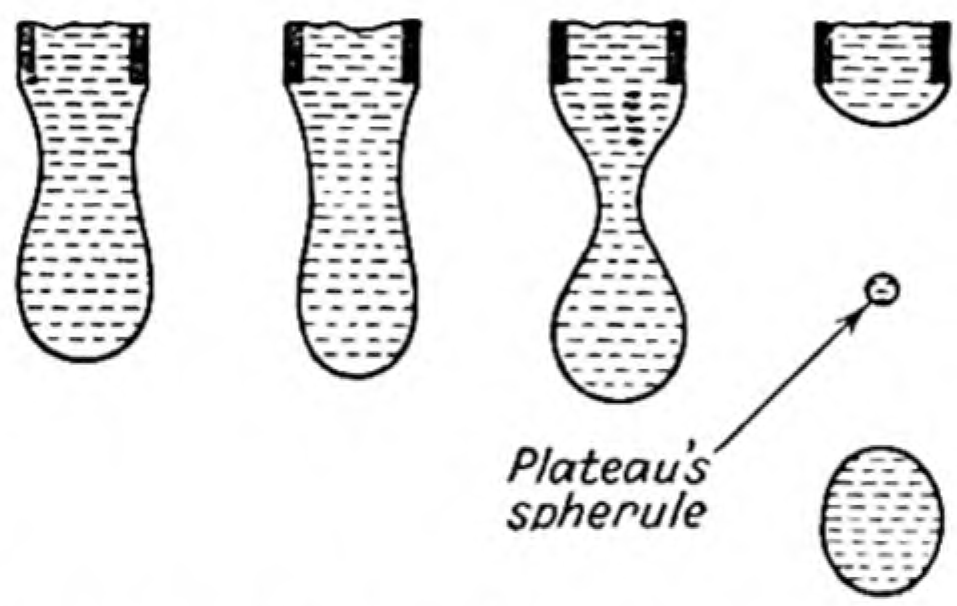


FIG. 10·30.—The slow formation of a drop.

of the drop to occur rapidly and permitting the final stages to proceed with the necessary slowness. These investigators also paid especial attention to the production of the glass tips. A Jena glass capillary tube was clamped in a high-speed precision lathe and rotated; a very fine carborundum disc was also rotated at high speed, the adjacent edges of the disc and tube were made to move in opposite directions. The ground sides were then polished with rouge in oil. A thick (2 cm.) cylindrical brass block, 10 cm. in diameter, was then perforated with a hole perpendicular to one of its faces, and of such a diameter that the cylindrically ground end of the tube would fit into it very exactly. The brass round the tip was then removed and filled with molten Wood's metal which was permitted to harden round the tip. The glass, Wood's metal, and brass were then ground away together by rubbing the plane surface on a plane steel block covered with wet carborundum powder.

When Harkins and Brown had determined experimentally the values of mg for tubes of different diameters and liquids of different surface tensions, they wrote

$$mg = f(r, \gamma, \rho, V),$$

where ρ was the density of the liquid and V the volume of a drop. Now each term in the function f will be of the type

$$\kappa r^x \gamma^y \rho^z V^w,$$

† *Jour. Amer. Chem. Soc.*, **41**, (i), 499, 1919.

and its dimensions must be those of weight. Hence

$$[\mathrm{MLT}^{-2}] = [\mathrm{L}^x(\mathrm{MT}^{-2})^y(\mathrm{ML}^{-3})^z\mathrm{L}^{3w}].$$

Equating powers of like dimensions we obtain

$$1 = y + z, \quad 1 = x - 3z + 3w, \quad \text{and} \quad -2 = -2y.$$

These give

$$y = 1, \quad z = 0, \quad \text{and} \quad 1 = x + 3w.$$

∴ Each term in f will be of the type

$$\kappa r^x \gamma \mathrm{V}^{\frac{1-x}{3}} = \kappa r\gamma\left(\frac{r^{x-1}}{\mathrm{V}^{\frac{x-1}{3}}}\right)$$

$$\therefore \; mg = r\gamma\Phi\left(\frac{r}{\mathrm{V}^{\frac{1}{3}}}\right) = r\gamma\mathrm{F}\left(\frac{\mathrm{V}}{r^3}\right),$$

where the function F is determined experimentally. [Φ is a less convenient function.]

The following table gives corresponding values of $\frac{\mathrm{V}}{r^3}$ and $\log\left(\frac{1}{\mathrm{F}}\right)$.

$\mathrm{V}r^{-3}$	$\log\left(\frac{1}{\mathrm{F}}\right)$	$\mathrm{V}r^{-3}$	$\log\left(\frac{1}{\mathrm{F}}\right)$
10·29	$\bar{1}$·3799	1·310	$\bar{1}$·4230
8·190	$\bar{1}$·2874	1·211	$\bar{1}$·4218
6·662	$\bar{1}$·3943	1·124	$\bar{1}$·4203
5·522	$\bar{1}$·4004	1·048	$\bar{1}$·4177
4·653	$\bar{1}$·4051	0·980	$\bar{1}$·4153
3·975	$\bar{1}$·4092	0·912	$\bar{1}$·4125
3·443	$\bar{1}$·4128	0·865	$\bar{1}$·4099
2·995	$\bar{1}$·4152	0·816	$\bar{1}$·4065
2·637	$\bar{1}$·4186	0·771	$\bar{1}$·4038
2.241	$\bar{1}$·4208	0·729	$\bar{1}$·4009
2·093	$\bar{1}$·4224	0·692	$\bar{1}$·3978
1·884	$\bar{1}$·4235	0·658	$\bar{1}$·3949
1·706	$\bar{1}$·4242	0·626	$\bar{1}$·3916
1·555	$\bar{1}$·4243	0·597	$\bar{1}$·3883
1·424	$\bar{1}$·4239	0·570	$\bar{1}$·3856

In capable hands the drop-weight method gives results correct to within 0·1 per cent, provided that the above correction table is used. The common practice of comparing the surface tensions of liquids by dropping liquids (at speeds greater than those mentioned above) from the same tip and assuming that the ratio of the masses of the drops is the ratio of the surface tensions is fallacious and exceedingly untrustworthy. N. K. Adam says, 'The use of the table is so simple that there is now no excuse for the old plan of simply comparing the weights of drops, and hoping the result will not be far wrong.'

Now Quincke was one of the first persons to use the drop-weight method and it is interesting to consider why he used the equation

$$mg = 2\pi r\gamma,$$

for as Ferguson says,† 'this physicist was not prone to make elementary errors'.

Now on the statical theory of a drop cylindrical round the line of contact with the tip, the above equation is in error because the pressure-excess in the interior of the drop, due to curvature, is neglected. But if when the drop is about to break away it is assumed that its waist is a region of anticlastic curvature such that the two curvatures are equal and opposite, the pressure-excess is zero and we may write

$$mg = 2\pi R\gamma,$$

where R is the radius of cross-section of the waist. It was such conditions which Quincke contemplated and it is probable that this investigator's use of the above formula caused the appearance of

$$mg = 2\pi r\gamma,$$

and its use in circumstances never considered by Quincke.

Now let us consider an improved drop-weight method for the comparison of surface tensions. In this method, due to R. C. BROWN, 1948, the flat tip of definite radius employed in the drop-weight method of Harkins and Brown is replaced by an inverted cone. Surface tension determinations may then be carried out provided the cone is standardized by means of a liquid of known surface tension. The advantage of this new method is that no correction factors are needed.

Let us therefore consider the detachment of a liquid drop from a conical tip such as that shown in Fig. 10·31(a). In a way to be described later, liquid is caused to trickle down the sides of the inverted cone to form a pendent drop. Eventually this drop becomes unstable and falls; the average mass of the *detached drop* is found in the usual way. When the drop is unstable, let V_1 be the volume of the solid of revolution below the plane AB, which is defined by the line of contact of the liquid surface with the cone. This volume is determined by θ, the semi-angle of the cone, and the parameters γ, ρ and g, which have their usual meanings. We may therefore write

$$V_1 = \gamma^x \rho^y g^z \psi_1(\theta),$$

where $\psi_1(\theta)$ is an unknown dimensionless function. Dimensional analysis gives for the constants x, y and z the values $\frac{3}{2}$, $-\frac{3}{2}$ and $-\frac{3}{2}$ respectively.

† *Jour. Sci. Instr.*, 6, 163, 1929.

$$\therefore\ V_1 = \left(\frac{\gamma}{g\rho}\right)^{\frac{3}{2}} \psi_1(\theta).$$

Furthermore b, the radius of curvature of the drop at the vertex when the drop becomes unstable will be determined by θ, γ, ρ and g. Dimensional analysis gives

$$b = \left(\frac{\gamma}{g\rho}\right)^{\frac{1}{2}} \psi_2(\theta).$$

Now the theoretical work of Bashforth and Adams shows that this equation implies that for a fixed value of θ all drops at the unstable stage are similar in shape. For such similar drops the volume V_2 of the liquid below the cone is a constant fraction of the

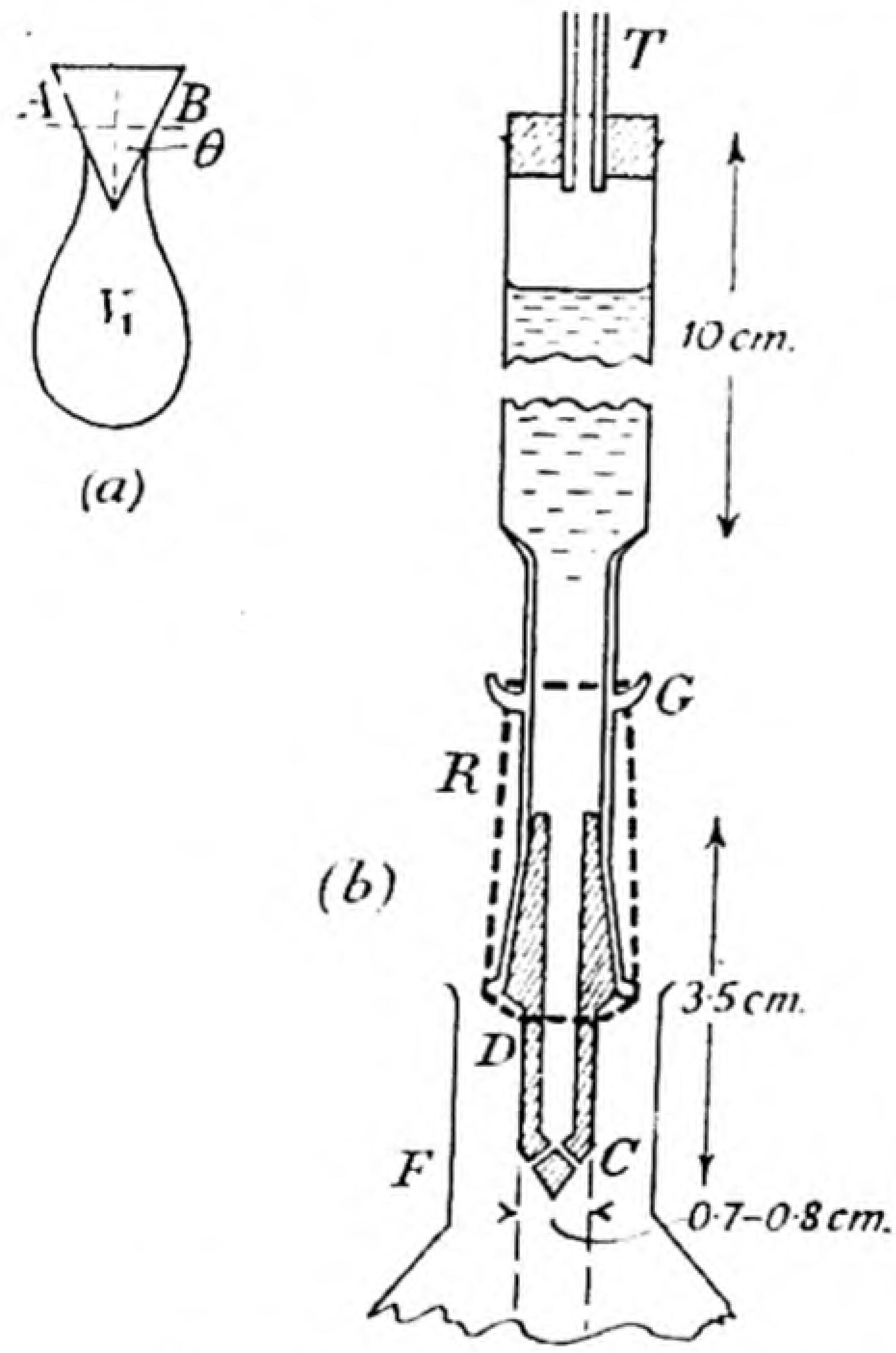


FIG. 10·31.—An improved drop-weight method for measuring surface tensions.

volume V_1. So far the theory is exact; but now we make an assumption which appears to be justified by the work of Harkins and Brown, viz. that the volume V of the detached drop is a constant fraction of V_2 and hence of V_1. Hence, if m is the mass of the detached portion we have

$$m = V\rho = \left(\frac{\gamma}{g}\right)^{\frac{3}{2}} \rho^{-\frac{1}{2}} \psi_3(\theta),$$

where $\psi_3(\theta)$ is a constant fraction of $\psi_1(\theta)$.

For two different liquids, 1 and 2, dropping away from the same conical tip, we have

$$\frac{\gamma_1}{\gamma_2} = \left(\frac{m_1}{m_2}\right)^{\frac{2}{3}} \left(\frac{\rho_1}{\rho_2}\right)^{\frac{1}{3}}.$$

A simplified version of the apparatus used by Brown is indicated in Fig. 10·31(b). The brass cone C is constructed and should be coated with platinum. The cone is extended by the collar D over which a glass tube G, shaped as shown, fits snugly. A rubber band holds these two parts of the apparatus together and the liquid does not escape via the conical joint.

T is a piece of fine capillary tube whose length is adjusted so that the pendent drops form slowly. F is a small conical flask containing some of the liquid and used to catch a known number of the drops.

Since it is essential to use liquids which wet the cone ($\phi = 0$), benzene should be used as the standardizing liquid and then the apparatus may be used to find the surface tensions of solutions of alcohol in water, of water to which a trace of 'wetting agent' has been added, etc.

(k) ***The method of ripples:*** It has already been shown, cf. p. 473, that the velocity, c, of ripples is given by

$$c^2 = \frac{g\lambda}{2\pi} + \frac{2\pi\gamma}{\lambda\rho}.$$

Many experimentalists have made observations on the velocity of ripples on a liquid surface to determine a value for γ. Only an outline of the method as described by R. C. Brown† will be given. The water, for which γ was required, was contained in a shallow brass tank, the edges of whose sides had been previously waxed, so that the water surface could be swept clean with the aid of waxed strips of glass—cf. p. 465. The ripples were produced by means of a dipper, D, Fig. 10·32(a), rigidly attached to a reed, R_2, which was energized by the alternating current passed through the loud-speaker unit U_2, the length of the reed being adjusted so that its natural frequency coincided with that of an alternating current supply. To measure the wave-length of the ripples a stroboscopic arrangement, shown in Fig. 10·32(b), was used. Light from a pointolite lamp, P, was focused by means of a lens, L_1, on a horizontal slit, S_1. By means of a concave mirror, M_1 (1 metre radius of curvature), the light from S_1 was focused at S_2, a point in the first focal plane of a lens, L_2. In this way a beam of parallel light was directed on to a plane mirror, M_2, inclined at such an angle to the horizontal that a

† *Proc. Phys. Soc.*, **48**, 312, 1936.

parallel beam of light was incident upon the disturbed surface of the water.

Now the concave mirror M_1 was attached to a reed, R_1, tuned to the frequency of the a.c. supply and energized by the current through a second loud-speaker unit U_1, in series with U_2. Hence

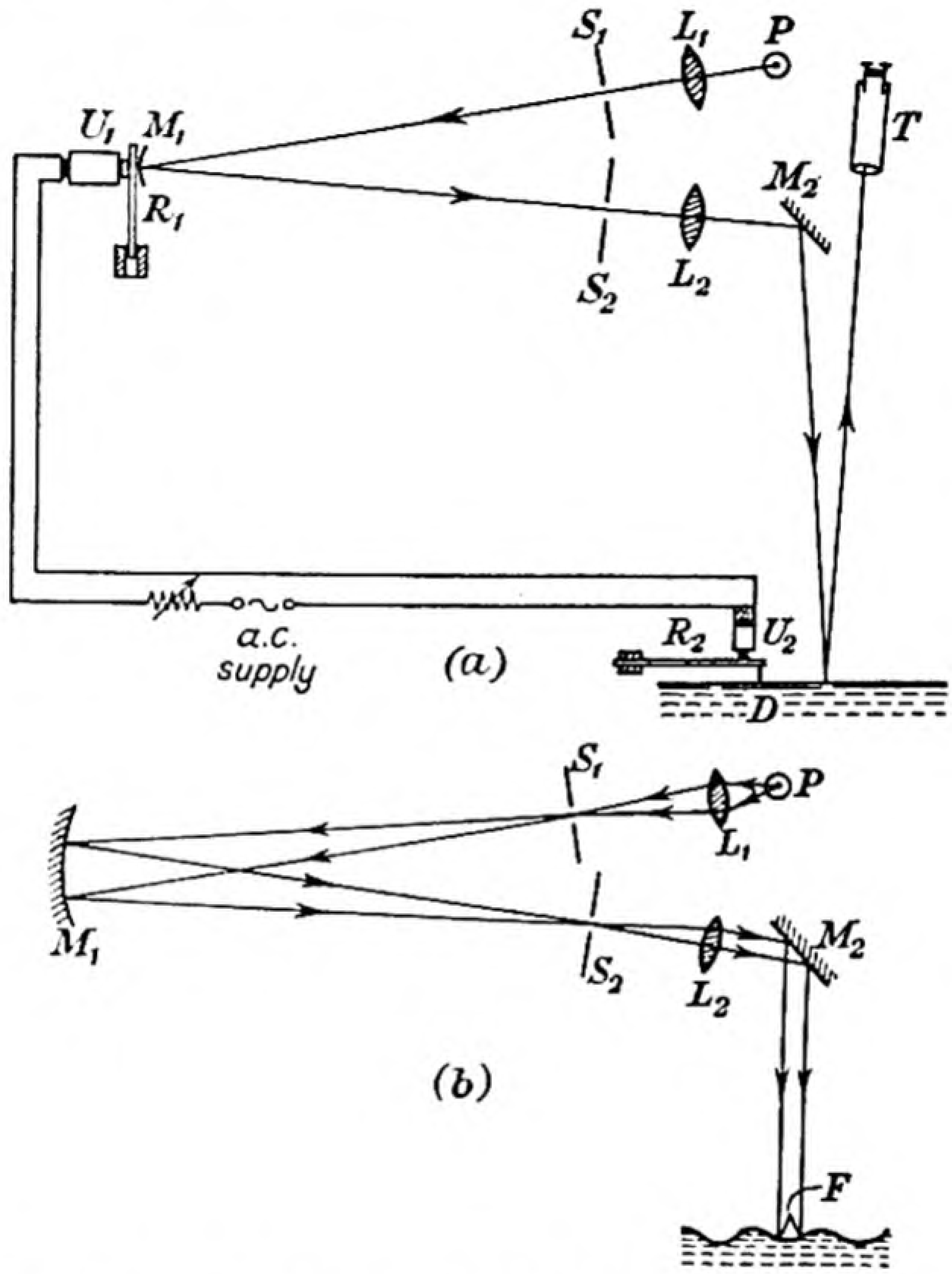


FIG. 10·32.—Surface tension by the method of ripples (or capillary waves).

for every ripple produced by D, i.e. during one complete vibration of R_2, the beam of light from M_1 passed twice across S_2 so that the intermittent illumination of the water surface had a frequency twice that of the dipper. By means of a telescope, T, the intermittent light reflected from the water surface was examined; the image consisted of a series of straight lines parallel to the edge of the dipper. The distance between successive lines was determined by shifting the telescope laterally. Owing to the relationship between the frequencies of the dipper and the reflected beam the distance was half the wave-length required. The value for γ was calculated with the aid of the equation given on p. 507.

(*l*) ***Soap solutions:*** The surface tension of a soap solution can be obtained at once from observations on the excess pressure within a soap bubble of known radius. A special form of manometer is necessary since the excess pressure is seldom more than a few millimetres of water.

The principle of a suitable manometer is as follows. Let A, Fig. 10·33(*a*), be a cylindrical vessel of cross-sectional area A and let B be a side tube attached to A and inclined at a small angle ψ to the horizontal. Let this apparatus contain a liquid of density ρ and suppose that when the pressure in A is atmospheric, p, the liquid in

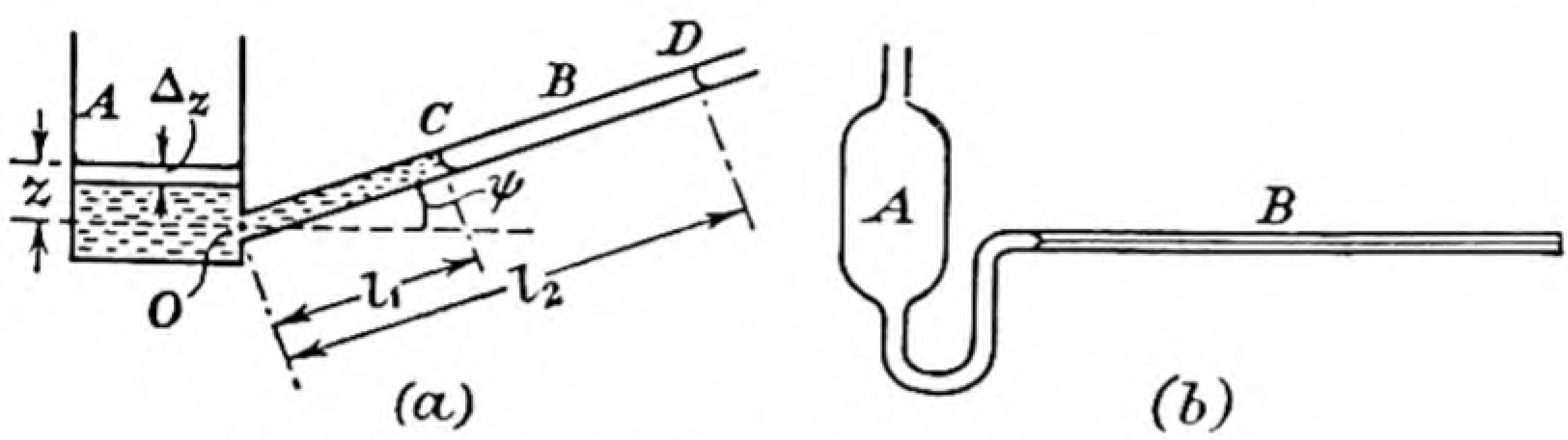


FIG. 10·33.—A sensitive manometer for measuring small pressure differences.

A, is at a height z above a datum level through O; let the liquid in B occupy a length l_1 of that tube. When the pressure on the liquid surface in A becomes $p + \Delta p$ so that this surface falls a distance Δz, let the liquid in B move to D so that CD $= l_2 - l_1 = \lambda$ (say). Since the volume of manometric liquid is invariable, we have

$$\mathrm{A}(\Delta z) = a(l_2 - l_1) = a\lambda,$$

where a is the cross-sectional area of the tube B. Now a consideration of the pressure at O in the first stage leads to the equation

$$p + \alpha + g\rho l_1 \sin\psi = p + g\rho z,$$

where α is a correction term due to any small pressure differences which may exist across the liquid surfaces and due to surface tension. If each of the tubes is uniform in cross-section α will be constant.

Similarly, in the second stage, we have

$$p + \alpha + g\rho l_2 \sin\psi = (p + \Delta p) + g\rho(z - \Delta z),$$

so that

$$\Delta p = g\rho\lambda\left[\frac{a}{\mathrm{A}} + \sin\psi\right].$$

By making a small and A large the fraction $\frac{a}{\mathrm{A}}$ can be made quite small but even with a manometer tube of radius 0·1 cm. and a vessel of radius 1 cm., $\frac{a}{\mathrm{A}}$ is 0·010, which is comparable with $\sin\psi$ when ψ is 2°.

In practice the manometer takes the form shown in Fig. 10·33(*b*) so that it may easily be cleaned. The tube B is normal to the axis of A and the required inclination ψ is obtained by tilting the stand to which the instrument is fixed.

A convenient apparatus for forming a soap bubble of suitable size is shown in Fig. 10·34. A is a box, each edge about 10 cm., two opposite sides of which are made of glass; the remainder are brass. One of the vertical metal sides is drilled and then shaped so that a circular metal plate P may be fitted to it and held in position with screws. P is provided with an aperture through which passes a

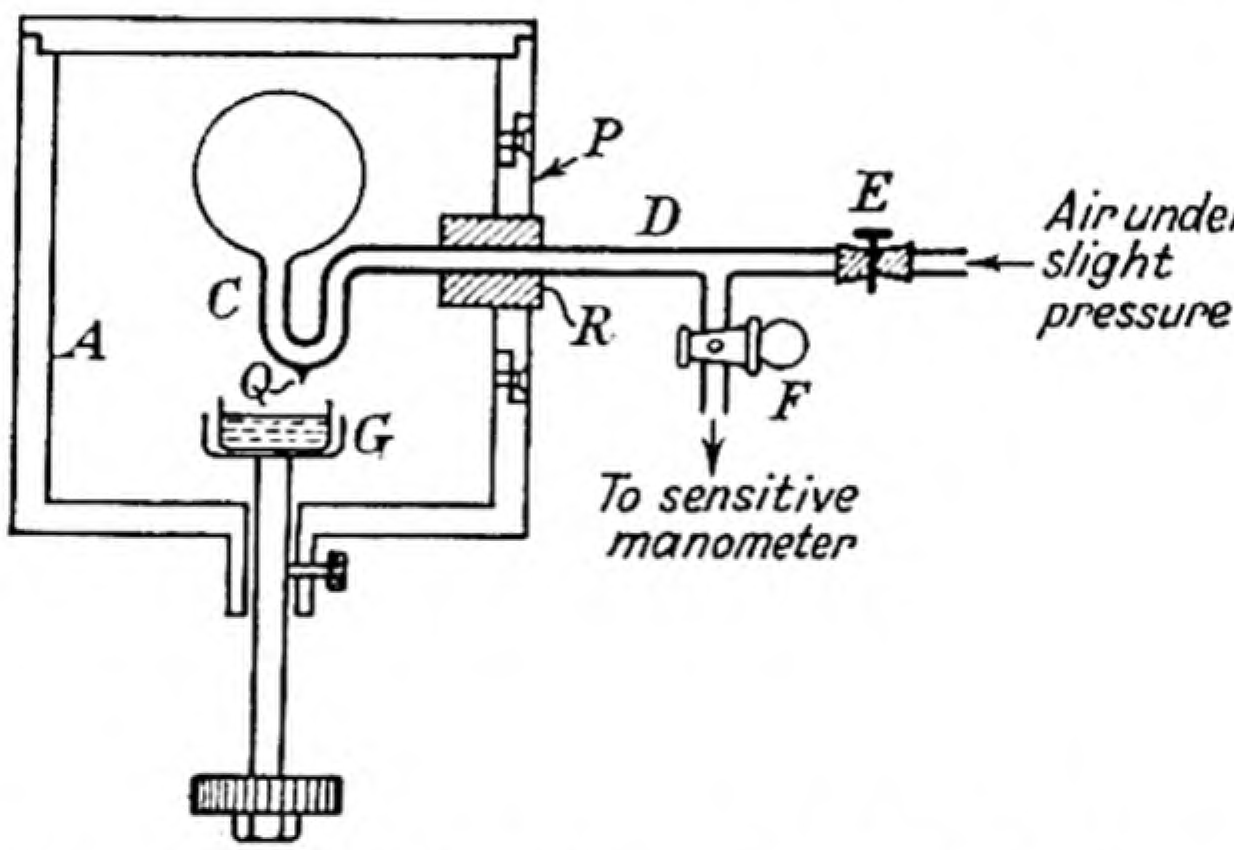


Fig. 10·34.—Experimental method for determining the surface tension of a soap solution from observations on the excess pressure within a bubble.

small rubber bung carrying a glass tube about 3 mm. in diameter and of the shape shown. The tube CD is connected via the rubber tubing and clip E to a supply of air under slight pressure, while a side tube F leads to a sensitive manometer. When a soap bubble is to be formed at the end of C the stop-cock in F is closed so that the gauge liquid shall not be expelled. The solution under investigation is placed in a small glass container at G which can be raised or lowered in the manner indicated.

To produce a bubble the tube CD is rotated about a horizontal axis until C points downwards and the liquid in G is then raised so that C dips into it. When G is lowered a soap film is formed across the end of C and this film, by blowing, is extended to form a bubble of the required size. The tube CD is rotated through two right angles so that the bubble occupies the position shown in the diagram. If the bubble is used with C pointing downwards a small blob of liquid will almost invariably hang from the bubble and destroy its sphericity. This want of sphericity is avoided when C is turned upwards and the glass point Q tends to collect any solution draining from the bubble; eventually this drop falls into G.

Now if D is the diameter of the bubble, we have

$$\frac{4\gamma}{r} = g\rho\lambda\left[\frac{a}{A} + \sin\psi\right] = \frac{8\gamma}{D}.$$

Hence
$$\lambda = \frac{8\gamma}{g\rho\left[\frac{a}{A} + \sin\psi\right]}\frac{1}{D},$$

so that if a series of observations on bubbles of different sizes is made and a graph of $\frac{1}{D}$ against λ plotted, a value for γ can be deduced from the slope of the graph.

To measure D the bubble should be illuminated from a distance. A converging lens of about 25 cm. focal length is then arranged to form, on a screen, an image of the bubble's profile, in the plane of the diagram. The necessary calculation is too simple to be given here.

Some remarks on methods for measuring surface tension.—All methods for measuring surface tension fall into one or other of two classes; consequently they are called ***static*** and ***dynamic methods***. In the former observations are made on a stationary surface, which has existed for some time, and depend on one or two principles. The most accurate static methods require the measurement of the pressure difference which exists across a curved surface of a liquid; once this pressure is known a value for the surface tension can be obtained provided that two principal radii of curvature of the surface are also known. Methods involving the use of capillary tubes, the sessile drop method and the drop weight method are all examples of static methods. Sometimes the drop weight method is regarded as a dynamic method but since HARKINS and BROWN showed that unless the drops are formed very slowly no reliable result can be obtained, this method is effectively a static one. The less accurate of the static methods involve observations on the extension or shape of a liquid film.

The dynamic methods depend on the fact that certain vibrations imparted to a liquid may cause periodic extensions and contractions of its surface; when these arise surface tension supplies the restoring force which is essential for the vibrations to take place. The oscillations of hanging drops, cf. Ex. 10·14 p. 530, or of jets issuing from an elliptical orifice and the propagation of ripples are the principal dynamic methods. Such methods may give a value for the surface tension of a solution which differs from that obtained by a static method since a new surface is constantly being formed and

may not be sufficiently old for adsorption† to have reached equilibrium. The work of R. C. BROWN, cf. p. 507, who viewed progressive ripples stroboscopically, shows, however, that the ripple method yields the 'static' surface tension. This is usually attributed to the fact that there is practically no renewal of the surface and very little interchange of material between the surface and the liquid itself. All known dynamic methods suffer from the fact that, in their present stage of development, the time which elapses between the formation of a surface and the measurement of surface tension is practically unknown.

MISCELLANEOUS PROBLEMS ON SURFACE TENSION

The force between two plates separated by a thin layer of a liquid.—Suppose that a drop of liquid rests between two parallel glass plates and that the area of each plate in contact with the drop is large. It is found experimentally that a large force is required to separate them in a direction normal to the plane of the plates, but they slide over one another comparatively easily. The force required to separate them normally may be calculated as follows.

Let 2R be the diameter of the circle which the drop, Fig. 10·35(a),

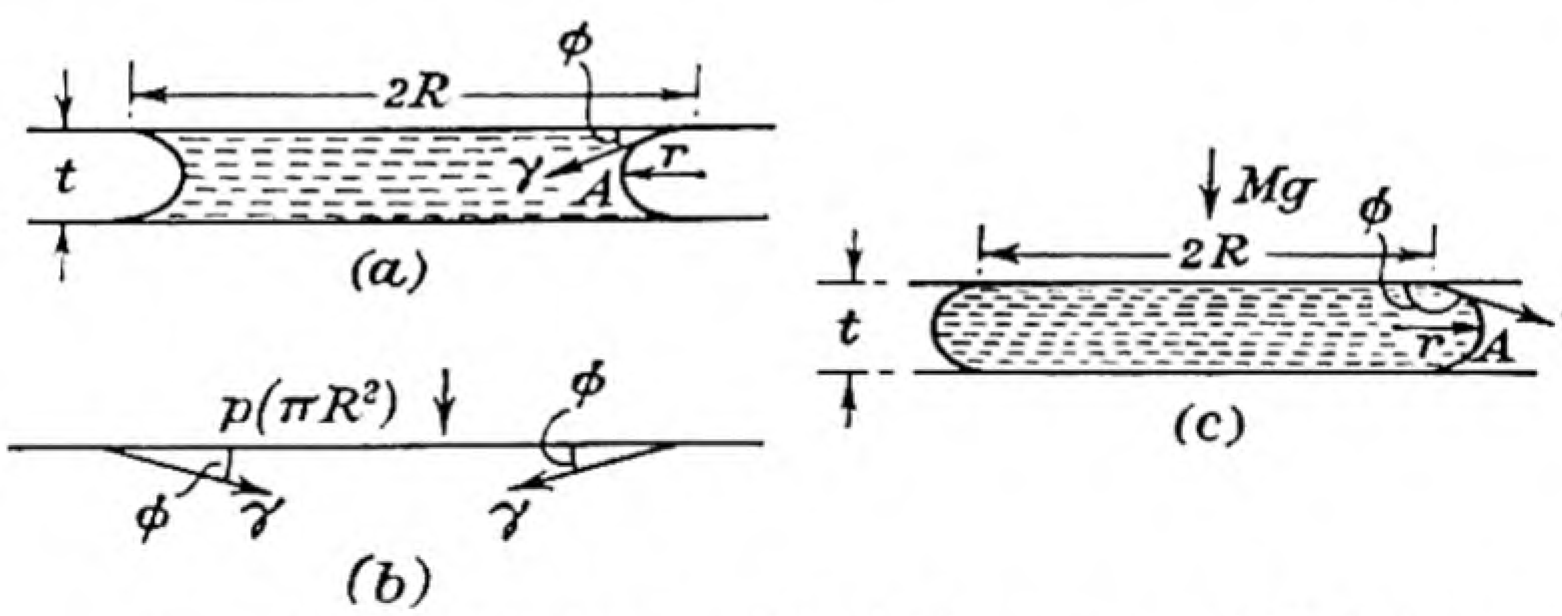


FIG. 10·35.—Thin layers of liquid between parallel plates.

makes with the upper plate, t the distance between the plates, γ the surface tension of the liquid and ϕ the angle of contact between the liquid and the glass. The pressure within the liquid is less than atmospheric by an amount $p = \gamma\left[\frac{1}{r} - \frac{1}{\mathrm{R}}\right] \simeq \frac{\gamma}{r}$, where r is the radius of curvature at A. Now, if the plates are very close together

$$r = \frac{t}{2\cos\phi} = \tfrac{1}{2}t\sec\phi.$$

† Adsorption is the term applied to the concentration of one constituent of a solution at its surface; the rule for adsorption in the case of a liquid-liquid solution is that if the solution has a surface tension less than that of the solvent, the solute is concentrated at the surface when it is newly formed.

∴ Resultant force (acting downwards) on upper plate, cf. Fig. 10·35(*b*), is

$$p(\pi R^2) + 2\pi R\gamma \sin\phi = \pi R^2.\left(\frac{2\gamma}{t\sec\phi}\right) + 2\pi R\gamma\sin\phi,$$

$$= 2\pi R\left[\frac{\gamma R}{t\sec\phi} + \gamma\sin\phi\right].$$

∴ Force required to separate the plates is $2\pi R\left[\frac{\gamma R}{t\sec\phi} + \gamma\sin\phi\right]$. The term $\gamma\sin\phi$ is very small compared with that containing t, and may be neglected.

If a drop of mercury is to be maintained between two parallel plates close together, then the upper plate must be loaded, the lower one being at rest on a table. The pressure within the mercury is greater than atmospheric by an amount $p = \frac{\gamma}{r} = \frac{2\gamma}{t}\cos(\pi - \phi)$, cf. Fig. 10·35(*c*).

∴ Resultant force on upper plate due to mercury is

$$\pi R^2 p - 2\pi R\gamma\sin(\pi - \phi) = \pi R^2.\left[\frac{2\gamma}{t\sec(\pi - \phi)}\right] - 2\pi R\gamma\sin\phi,$$

and this acts upwards. If M is the mass of the load on the upper plate required to hold it in position, and g the intensity of gravity, then

$$Mg = 2\pi R\left[\frac{\gamma R}{t\sec(\pi - \phi)} - \gamma\sin\phi\right].$$

Again the term $\gamma\sin\phi$ is small and may be neglected.

On the force required to withdraw a plate from a liquid surface.—When a flat plate is placed on the surface of a liquid at rest and then slightly raised, it is found that a layer of liquid adheres to the plate. As the force raising the plate is increased so the amount of liquid adhering to the plate continues to increase until, when the force reaches a value F (or mg, say), the plate suddenly breaks away from the surface. Just before this occurs let z be the height of the plate above the general level of the liquid surface. If p_0 is the atmospheric pressure, A the area of the plate (say a circle of radius R), and ρ the density of the liquid then the pressure in it at a point just below the plate is $(p_0 - g\rho z)$, i.e. the force pulling the plate downwards is

$$p_0 A - (p_0 - g\rho z)A = Ag\rho z.$$

This is equal and opposite to F, so that $m = A\rho z$.

Now let us assume that R is large, so that if the liquid wets the plate, the trace, in a vertical plane, of the meniscus is everywhere

a semi-circle of radius $\frac{1}{2}z$. Then at a point in the liquid where the horizontal section of the raised liquid is least, the pressure will be less than atmospheric by an amount which is very nearly given by

$$\gamma\left(\frac{1}{r} - \frac{1}{R}\right) = \frac{\gamma}{r} = \frac{2\gamma}{z} \quad \left(\text{if } \frac{r}{R} \to 0\right).$$

This pressure difference must be $\frac{1}{2}g\rho z$, if the meniscus has the shape we have assumed, so that

$$m = A\rho z = 2A\sqrt{\frac{\gamma\rho}{g}},$$

i.e.

$$F = mg = 2A\sqrt{\gamma\rho g}.$$

Thus if m is measured, the plate being suspended from an equilibrated balance, a value for the surface tension γ may be found.

Example.—A vertical glass capillary tube with a slightly conical bore just dips with its apex upwards into a liquid of density ρ, surface tension γ and angle of contact with glass zero. A very small hole is left at the apex so that air can escape from the tube. If a is the maximum radius of cross-section of the tube and l its length, show that the liquid rises to a height h given by

$$h = \frac{l}{2} \pm\sqrt{\frac{l^2}{4} - \frac{2\gamma l}{g\rho a}},$$

where g is the intensity of gravity.

Discuss the equilibrium of these two possible positions and also find a value for h when l is very large.

Suppose that the liquid takes up the position shown in Fig. 10·36(a) and that r is the radius of curvature of the meniscus. Then

$$r = \frac{a}{l}(l - h) \qquad \text{and} \qquad \frac{2\gamma}{r} = g\rho h.$$

$$\therefore\ h^2 - lh + \frac{2\gamma l}{g\rho a} = 0,$$

which gives

$$h = \frac{l}{2} \pm \sqrt{\frac{l^2}{4} - \frac{2\gamma l}{g\rho a}}.$$

$$\therefore\ (h_2, h_1) = \frac{l}{2} \pm \frac{l}{2}\sqrt{1 - \frac{8\gamma}{g\rho a l}}.$$

When l is very large

$$\sqrt{1 - \frac{8\gamma}{g\rho a l}} \simeq 1 - \frac{4\gamma}{g\rho a l}.$$

$$\therefore\ [h]_{l\to\infty} = \frac{l}{2} - \frac{l}{2}\left(1 - \frac{4\gamma}{g\rho a l}\right) = \frac{2\gamma}{g\rho a}.$$

The positive sign is rejected since $[h]_{l\to\infty}$ is the height to which the liquid rises in a uniform capillary of radius a. This suggests that it is the lower height h_1 which is the stable one.

To establish this fact conclusively consider the liquid when its meniscus is at a height h, r being the radius of the corresponding cross-section of the tube; the liquid is not necessarily in equilibrium.

Let $g\rho h = p$; $\dfrac{2\gamma}{r} = \dfrac{2\gamma l}{a(l-h)} = p'$.

For equilibrium, $p = p'$. If $p > p'$ the meniscus will fall; if $p < p'$ the meniscus will rise, i.e. p will increase.

Now plot p and p' both against h; the result is shown in Fig. 10·36(b).

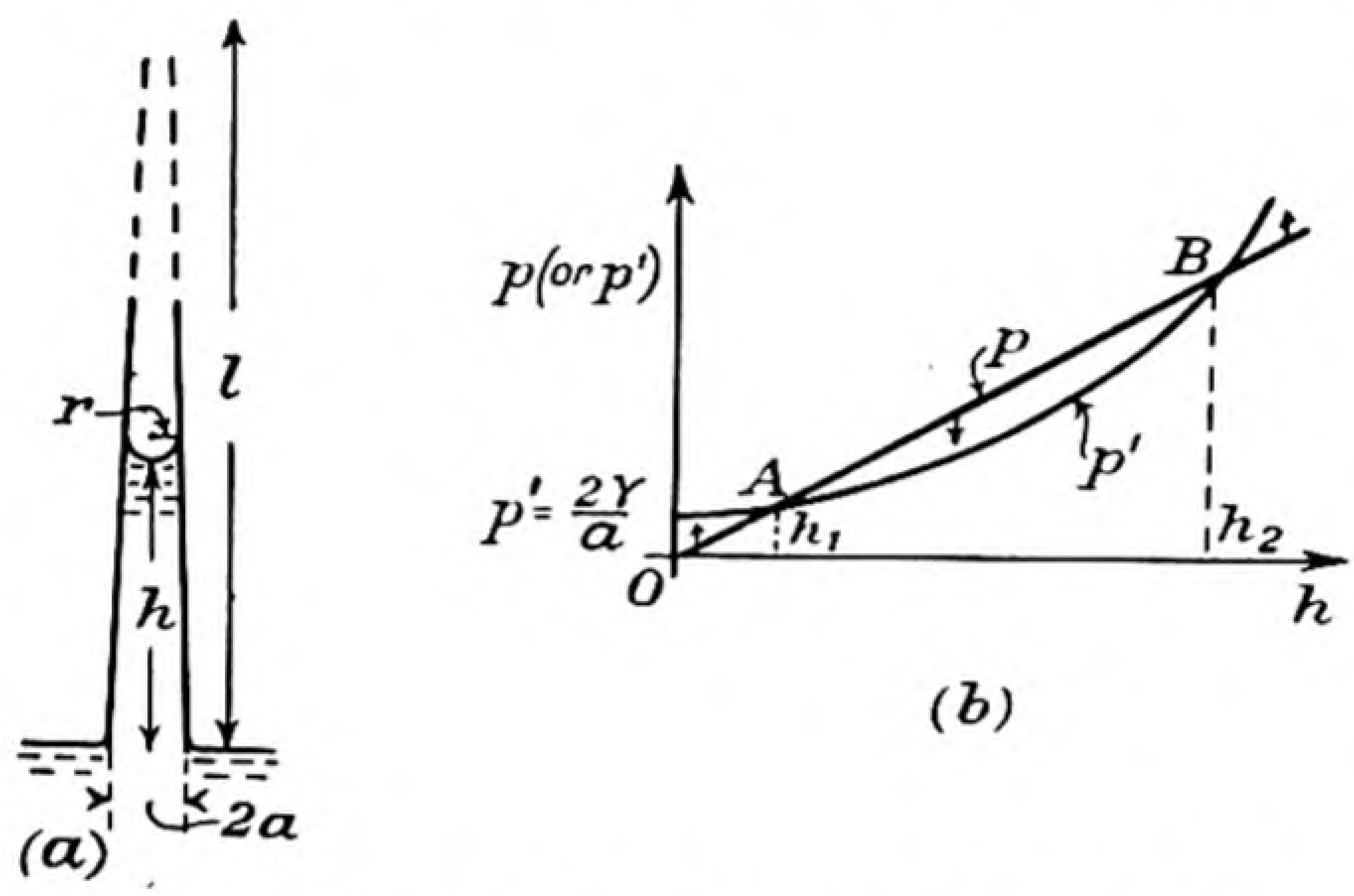


FIG. 10·36.—Rise of a liquid in a conical capillary tube.

In general there will be two points of intersection, viz. A and B, corresponding to the roots h_1 and h_2 of the equation

$$p = p'.$$

Between A and B for any given h, $p > p'$ so that a meniscus placed at this level would fall; this means that surface tension effects are not large enough to support the liquid column.

Below A, $p < p'$ and above B, $p < p'$, so that in each of these two instances the meniscus would rise.

Now A corresponds to a stable position of the liquid in the tube because a displacement to either side is followed by a return of the meniscus to its undisturbed position. On the other hand B corresponds to an unstable position as a small displacement to either side is followed by a greater displacement.

The existence of two equilibrium positions depends on whether or not the p' curve crosses the straight line for p. Imaginary roots occur if

$$\frac{2\gamma}{g\rho a l} > \tfrac{1}{4},$$

while if the tube is uniform p' is parallel to the h-axis and there is only one position of equilibrium.

The variation of surface tension with temperature.—In 1915, FERGUSON† proposed that for ***non-associated***‡ liquids, such as

† *Phil. Mag.* **31**, 37, 1916; *Trans. Far. Soc.*, **19**, 407, 1923.

‡ Associated liquids, such as water, formic acid (H.COOH), acetic acid ($CH_3.COOH$), ethyl alcohol ($C_2H_5.OH$), etc. have strong dipole moments.

benzene and carbon tetrachloride, the variation of surface tension with temperature θ, as measured on a centigrade scale could be represented by the equation

$$\gamma = \gamma_0(1 - b\theta)^n,$$

where b and n are constants for a given liquid, γ_0 is the surface tension at 0° C., and since the surface tension becomes zero at the critical temperature θ_c, one may reasonably expect to find that $b = \theta_c^{-1}$. To determine the constants b and n, Ferguson proceeded as follows. By differentiation, we have

$$\frac{d\gamma}{d\theta} = -bn\gamma_0(1 - b\theta)^{n-1}.$$

$$\therefore\ \gamma\frac{d\theta}{d\gamma} = -\left[\frac{1 - b\theta}{bn}\right] = -\frac{1}{bn} + \frac{1}{n}(\theta).$$

If, therefore, we plot $\theta = x$, $y = \gamma\frac{d\theta}{d\gamma}$, we should obtain a straight line whose slope is $\left(\frac{1}{n}\right)$ and whose intercept on the y-axis is $-\frac{1}{bn}$. The following are some of the results obtained.

	n	$b \times 10^3$	$b^{-1} = \theta_c$	θ_c °C. (observed)
Ether	1·248	5·155	194	193·8
Benzene C_6H_6 . .	1·218	3·472	288	288·5
Carbon tetrachloride CCl_4	1·206	3·553	281·5	283·1

For fourteen different liquids examined the variations in n were small; the mean value of n for them was 1·210.

In 1893 van der Waals proposed that

$$\gamma = A\left(1 - \frac{T}{T_c}\right)^m,$$

where A and m are constants for a given liquid, T the temperature on the absolute scale, and T_c the critical temperature. This was an empirical formula but van der Waals predicted that m would be a constant. Now following Verschaffelt (1925) it may be shown that Ferguson's and van der Waals equations are equivalent. For

$$\gamma = \frac{A}{T_c^{\ m}}(T_c - T)^m = \frac{A}{T_c^{\ m}}(\theta_c - \theta)^m = \frac{A\theta_c^{\ m}}{T_c^{\ m}}\left(1 - \frac{\theta}{\theta_c}\right)^m.$$

But $A\left(\frac{\theta_c}{T_c}\right)^m$ is, like γ_0, a constant, so that the identity is established provided $m = n$.

Now according to thermodynamical reasoning, cf. Vol. II, the relation between surface energy density ϵ, surface tension γ and absolute temperature T, is

$$\epsilon = \gamma - T.\frac{\partial \gamma}{\partial T} = \gamma - T.\frac{\partial \gamma}{\partial \theta}.$$

Using Ferguson's equation we have

$$\frac{\partial \gamma}{\partial \theta} = -bn\gamma_0(1 - b\theta)^{n-1} = -n\frac{\gamma_0}{\theta_c}\left(1 - \frac{\theta}{\theta_c}\right)^{n-1}.$$

$$\therefore \text{ At } \theta = \theta_c, \text{ since } n > 1, \qquad \frac{\partial \gamma}{\partial \theta} = 0,$$

i.e. the γ, T curve is tangential to the T-axis at $T = T_c$.

$$\text{Now} \qquad \frac{\partial \epsilon}{\partial T} = \left(\frac{\partial \gamma}{\partial T} - \frac{\partial \gamma}{\partial \theta}\right) - T\frac{\partial^2 \gamma}{\partial \theta^2}$$

$$= -T\left[-n\frac{\gamma_0}{\theta_c}\left(-\frac{1}{\theta_c}\right)\left(1 - \frac{\theta}{\theta_c}\right)^{n-2}\right]$$

At θ_c this becomes ∞, since $n \simeq 1{\cdot}2$, i.e.

$$\left[\frac{1}{\left(1 - \frac{\theta}{\theta_c}\right)^{2-n}}\right]_{\theta = \theta c} \to \infty$$

Thus the ϵ, T curve is normal to the T-axis at $T = T_c$.

Also $$(\epsilon)_{\theta_c} = 0 - 0 = 0.$$

The graphs shown in Fig. 10·37 indicate how γ and ϵ may be expected to vary with temperature.

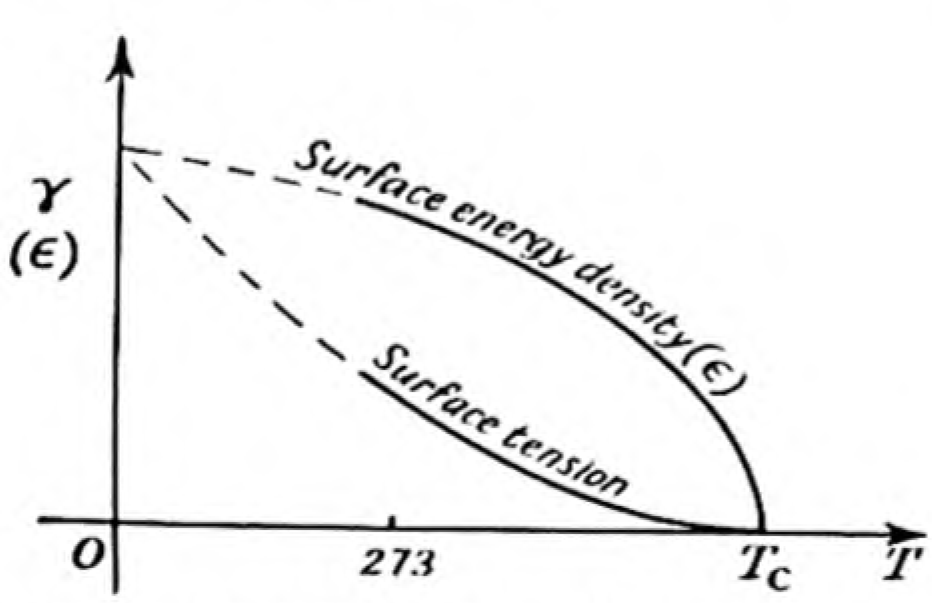


FIG. 10·37. Variation of γ and ϵ with temperature.

Molecular volume and the parachor.—The molecular volume of a liquid is obtained by dividing the molecular weight in grams by the density of the substance. As early as 1842 KOPP noted that in general the molecular volumes of organic compounds,

evaluated at their normal boiling point, were additive functions of the atomic volumes of the constituent elements. This was the first instance of a physical property to show marked additivity.

In 1923 MACLEOD discovered that for non-associated liquids the relation

$$\frac{\gamma^{\frac{1}{4}}}{\rho} = \mathrm{C}$$

is valid over a wide range of temperature, where γ is the surface tension of the liquid, ρ the difference between the density of the liquid and that of its saturated vapour at the temperature of the experiment and C is a constant. In 1924 SUGDEN pointed out that if both sides of the above equation are multiplied by M, the molecular weight in grams, then

$$\frac{\mathrm{M}\gamma^{\frac{1}{4}}}{\rho} = \mathrm{MC} = [\mathrm{P}],$$

where [P] is a constant, termed by Sugden the ***parachor***. The name implies that [P] is a comparative volume, $\pi\alpha\rho\acute{\alpha}$ = next to, $\chi\acute{\omega}\rho\alpha$ = space. At temperatures which are not too high, ρ may be taken as the density of the liquid so that $\frac{\mathrm{M}}{\rho}$ is the molecular volume, Ω. Thus

$$[\mathrm{P}] = \Omega\gamma^{\frac{1}{4}},$$

and it appears at once that the parachor represents the molecular volume of a liquid at such a temperature that its surface tension is 1 dyne.cm.$^{-1}$. Hence, instead of following Kopp and comparing molecular volumes at their normal boiling points, we may compare them under such conditions that their surface tensions are the same. Since the attractions between neighbouring molecules in a liquid determine its surface energy density and hence its surface tension and also the so-called internal pressure of a liquid, it follows that if the surface tensions of two liquids are the same then their internal pressures are at least approximately equal. Thus a comparison of the parachors of two liquids at such temperatures that their free surface energies are equal may be expected to provide a better basis for the comparison of molecular volumes than did Kopp's method. In this connexion it is essential to remind ourselves that since volume is a function of temperature, the main difficulty in applying the principle that molecular volumes are additive, is to discover the correct temperature at which to compare molecular volumes. By using Macleod's† relation we are able, as Sugden first showed, to compare these volumes under such conditions that the effect of

† *Trans. Far. Soc.*, **19**, 38, 1923.

temperature is neutralized since $\frac{\gamma^{\frac{1}{4}}}{\rho}$ is very nearly independent of temperature. Sugden found, however, that the value of the fraction increased slowly with rise of temperature for liquids which are considered to be ***associated***,† e.g. methyl alcohol, ethyl alcohol, acetic acid, amines, etc. For mercury the fraction increases by about 2 per cent as the temperature changes from room temperature to that of boiling mercury; for tin over a similar range of temperature the change is 5 per cent.

By comparing the parachors of different compounds in the liquid state Sugden showed that the parachor is pre-eminently an additive property. Thus (*a*) ***isomeric‡ compounds*** have parachors of the same value and (*b*) the differences between the parachors of successive members of an homologous series are the same in different series. Moreover, when the parachors of individual atoms and of certain groups of atoms are determined, the agreement between experimental and calculated values is most striking.

It was soon found that the parachor depends not only on the constituent elements in a compound but also on their mode of linking together. Thus the formation of a ring structure increases the parachor by an amount determined by the number of atoms in the ring; likewise the effect of a double bond on the parachor depends on whether or not the bond is non-polar as in ethylene [C_2H_4] or semi-polar as in trimethylamine-oxide [$(CH_3)_3NO$]. The parachor of a compound may be expressed as the sum of a number of atomic and structural constants which are independent of the nature of the compounds. Thus, although the real meaning of the parachor is unknown, it has been used in determining the structure of compounds and in deciding between alternative structures which may have been suggested. Structures determined in this way are not always in agreement with chemical evidence or with the conclusions from reliable physical methods, e.g. by the use of X-rays or absorption spectra.

The surface tension of molten metals.—The accurate determination of the surface tension of a molten metal offers exceptional difficulties, for the surfaces are very easily contaminated. Some of the best work on this subject is by BIRCUMSHAW [1926]. He used the method of maximum bubble pressure as modified by Sugden—cf. p. 488. The fact that two tubes, dipped to the same depth in the molten metal, are used is now a distinct advantage since no term containing the depth of immersion or the density of the molten metal appear in the final equation from which the surface tension is calculated. In the actual research Bircumshaw

† Their molecules have strong dipole moments.
‡ i.e. of the same molecular weight.

applied corrections, as worked out by Sugden, to allow for the fact that just before the bubbles break away they are not truly hemispherical in shape.

For all molten metals the surface tensions are found to be reasonably high and for copper and cadmium it is probable that the surface tension increases with temperature.

Surface films of insoluble substances on water.—When a small quantity of a practically insoluble and non-volatile substance is placed upon a liquid surface and that liquid, like water, has a high surface tension, then one of two things may be observed. The substance may either spread over the water surface or remain as a compact mass. It is now known that the necessary and sufficient condition for the substance to spread is that its molecules must attract the water more than they attract one another. If the extent of the surface is great enough the spreading substance forms a film one molecule thick and is known as a ***monomolecular surface film***. If the surface is not large enough for the whole substance to exist as such a film, then it is found that the major part exists as a monomolecular surface film interspaced with minute but visible droplets of much greater thickness.

These monomolecular films are found, in general, to have a very simple structure so that a study of their behaviour reveals to us facts concerning the size, shape and other properties of individual molecules.

Mainly historical.—Surface films on water were known in ancient times, for the peoples of those days comprehended that ships in a tempestuous sea could be protected by calming the surface waves by adding a small quantity of oil. Mineral oils, as distinct from vegetable oils, are not very efficient in this respect. When the conception of surface tension had become understood, it was found that an oil film on water reduced the surface tension considerably. RAYLEIGH, in 1890, was the first to measure the minimum quantity of an oil necessary to reduce the surface tension of water by a definite amount. It was well known that fragments of camphor introduced on to a water surface move rapidly over the surface; the concentration in the surface layer of the dissolved camphor is not uniform and since the surface tension of a solution varies with concentration, a differential surface thrust is experienced by each camphor particle until the surface is completely saturated with that substance. The surface tension is then about 62 dyne.cm.$^{-1}$. Rayleigh found that $0{\cdot}81 \times 10^{-3}$ gm. of olive oil, spread over a water surface $5{\cdot}5 \times 10^{3}$ cm.^{2} in area, is sufficient to stop the movements of camphor particles on the water surface; by adding this amount of oil to the water its surface tension had been

reduced by about 16 dyne.cm.$^{-1}$ and this had been caused by a layer of oil 16×10^{-8} cm. thick.

A year later this work was followed by a most important discovery by Fraülein POCKELS. She discovered that the extent of a surface film could be adequately controlled by means of a 'barrier', which is a piece of glass-strip coated with paraffin wax. The barrier must rest on and touch the water at all points along its side and extend along the whole width of the surface. On p. 465 it is emphasized how necessary such barriers are in making clean a water surface; such barriers are the 'corner-stones' of all accurate methods of experimenting with thin films of water. In 1899 Rayleigh repeated and considerably extended the work of Fraülein Pockels. His work confirms the fact that by bringing the barriers closer together and so diminishing the area occupied by a known mass of olive oil on a water surface, the surface tension remains almost invariable until a certain critical area is reached; then the value of the surface tension suddenly drops by approximately half the value for clean water. From the known volume of the oil placed on the surface and the critical value of the area when the reduction in surface tension occurs, the thickness of the film at this stage may be calculated. It is found to be about 10×10^{-8} cm. and from the similarity between the magnitude of this thickness and the known diameters of molecules derived from the simple kinetic theory, Rayleigh deduced that under these circumstances the film is one molecule thick.

The monomolecular nature of surface films. Differential surface tension or 'surface pressure'.—Let A, Fig. 10·38, be a light floating strip of length l, resting on the surface of a liquid contained in a trough with rigid boundaries. The strip is normal

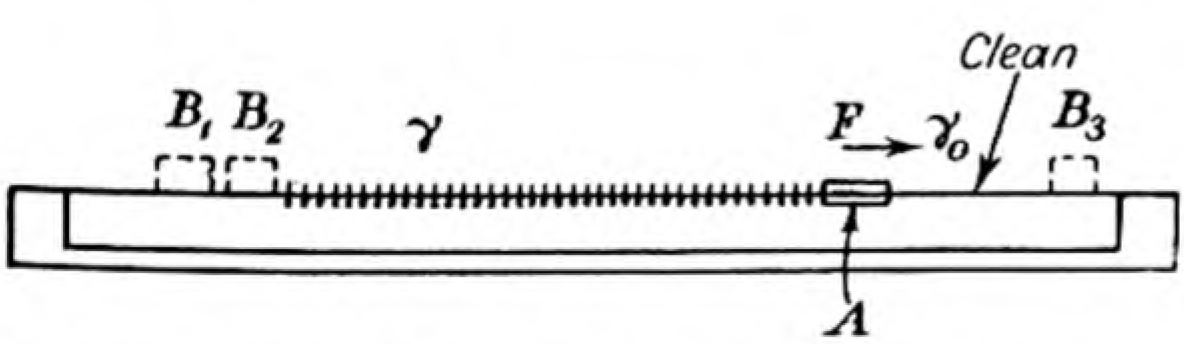


FIG. 10·38.—Differential surface tension or 'surface pressure'.

to the plane of the diagram. Let the surface to the right of A be clean, while on the left of A the surface is covered by a film whose area may be altered by moving the barrier B_2. All three barriers must be used to prepare and maintain the clean surface. Let F be the outward force experienced by A and let γ_0 and γ be the surface tensions of the clean liquid and of the film-covered surface. If A is displaced by the force F a distance δx to the right,

the work done by the force must equal the decrease in the total free-energy of the system, if thermal energy sufficient to keep the temperature constant is supplied. Thus

$$\begin{aligned} F\,\delta x &= \text{decrease in total free-energy} \\ &= \text{decrease in free-energy of the `clean surface' + the} \\ &\quad\ \text{decrease in free-energy of the `unclean surface'} \\ &= \gamma_0 l\,\delta x - \gamma l\,\delta x, \end{aligned}$$

the negative sign indicating an increase in the total free-energy of this portion of the system. Hence

$$\frac{F}{l} = \gamma_0 - \gamma.$$

Let $\frac{F}{l} = \psi$; we shall call ψ the ***differential surface tension*** although it is often called the surface pressure.

ADAM has called attention to the close analogy which exists between ψ, the force per unit length exerted by a surface film on a floating barrier, and the osmotic pressure of a solution; the float simulates a semi-permeable membrane and it is through the float that the differential surface tension makes itself manifest.

Langmuir's work on monomolecular or unimolecular films.—LANGMUIR† made a great step forward in the study of surface films when he used, instead of 'oils', pure substances of known composition. The substances, liquids as well as solids, were spread by dissolving them in benzene; a 'drop' was allowed to

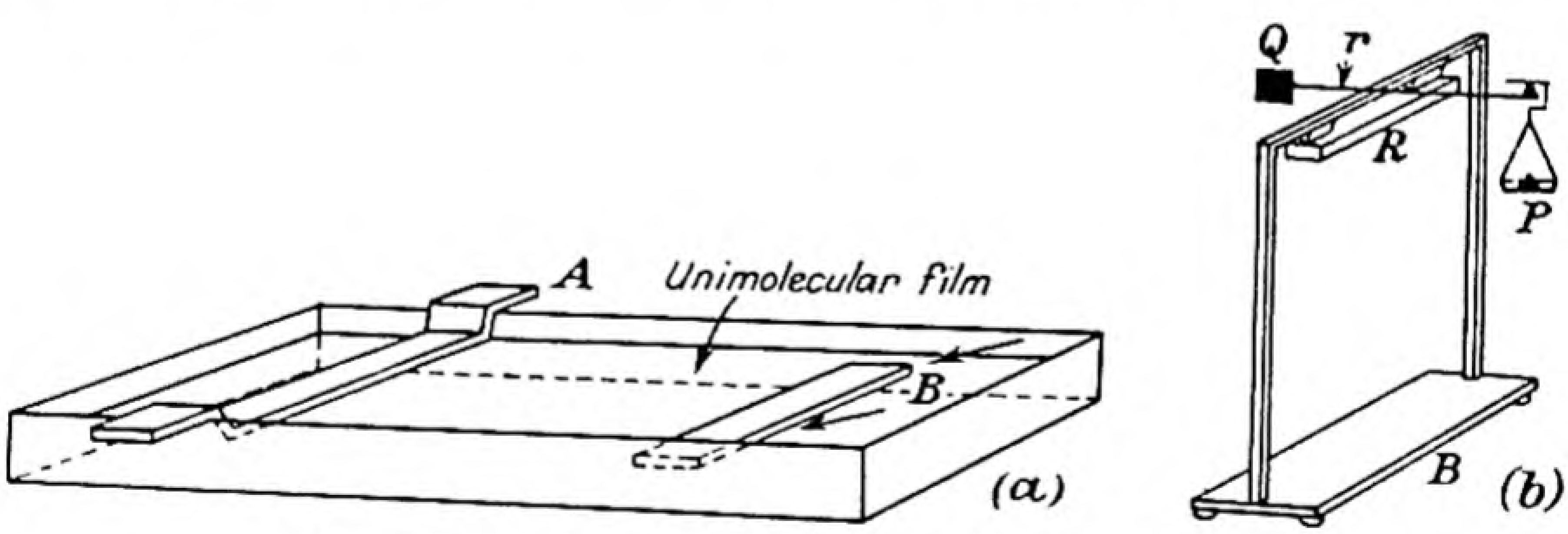


FIG. 10·39.—Langmuir's trough (diagrammatic).

fall on the water surface when the solution spread rapidly and in a few seconds the solvent (benzene) evaporated. The apparatus used by Langmuir to measure ψ is shown in Fig. 10·39(a). It consists of a long narrow shallow trough partly filled with water. A is a waxed cardboard strip which rests on the water surface near to one end of the trough. At the other end of the trough and floating

† *J.A.C.S.*, **39**, 1848, 1917.

on the water is a waxed cardboard strip, B, whose length is nearly equal to the width of the trough. By moving the barrier A the extent of the surface covered by the 'oil' can be varied and the film is prevented from spreading between B and the sides of the trough by two suitably directed air-blasts, as indicated by the arrows.

The strip B is suspended by means of a light glass framework from two knife-edges resting on a rigid platform, R, Fig. 10·39(*b*). The top of the framework carries a lever with a scale-pan, P, at one end and a counterpoise, Q, at the other.

To carry out the experiment, a small load of known mass is placed in P so that the float is displaced from its position of rest. The barrier A is then moved towards B until the floating barrier B is restored to its zero position. In this way the areas of the film for a given set of values of ψ is determined, the value of ψ being deduced from the mass in P and the dimensions of the apparatus.

With saturated fatty acids and alcohols Langmuir established quite definitely the existence of a critical area at which ψ becomes finite; this corresponds to the stage in Rayleigh's experiments at which the surface tension is diminished. Langmuir expresses his results graphically by plotting the area of the surface occupied by one molecule of the film material and the corresponding value of the differential surface tension ψ. The area occupied by a single molecule is calculated from a knowledge of the mass of 'oil' spread over a known area, its molecular weight and Avogadro's constant. Fig. 10·40 is a typical curve. When the area occupied by one molecule is greater than the amount represented by the point Q on the diagram, i.e. 21×10^{-16} cm.^{2}, the value of ψ is too small to be detected, i.e. there is no measurable change in the surface tension of water. As the area occupied is diminished, the value of ψ increases slowly at first and then more rapidly until a stage represented by the point H is reached. As the area is further diminished the value of ψ remains almost constant, but it is in this stage that the film can be detected visually for it begins to crumple up so that strain lines appear on the surface.

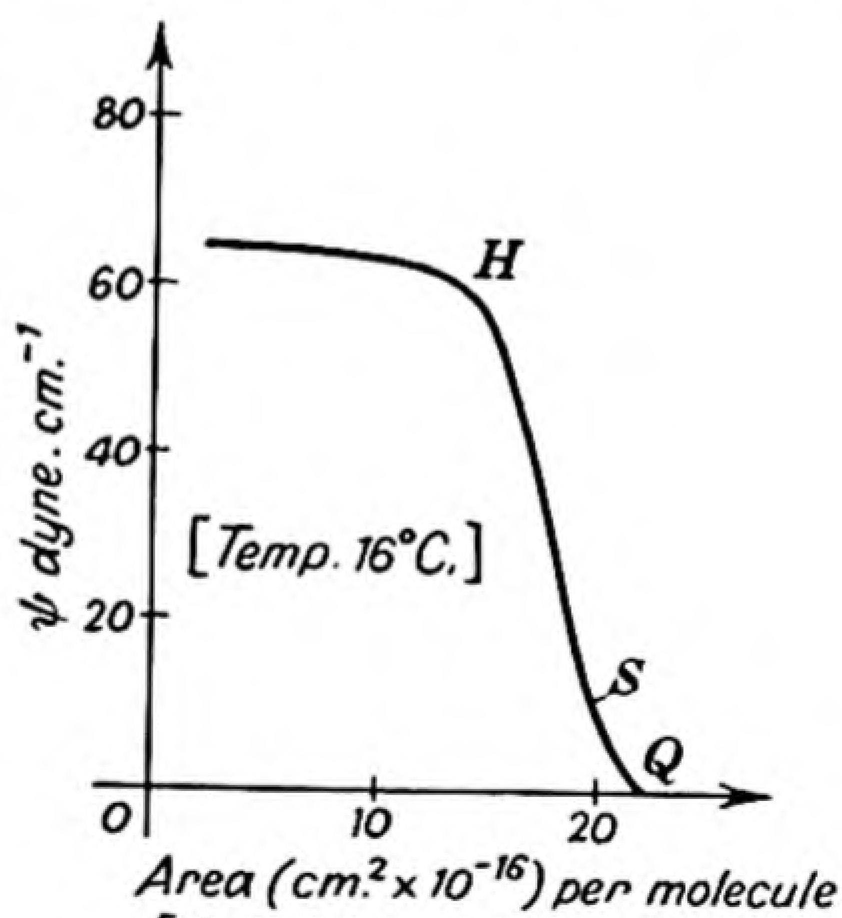


FIG. 10·40.—Some of Langmuir's results with thin films.

In the region between Q and a not-too-well defined point S, the film is considered to be in the 'liquid state', for particles of dust on the surface can be seen to move about quite freely. At S the film becomes 'solid' for, from then on, as the area is reduced, the dust particles are stationary.

The condensed film.—Langmuir found that for the series of fatty acids from palmitic acid ($C_{15}H_{31}.COOH$) to cerotic acid ($C_{25}H_{51}.COOH$) the surface area occupied by one molecule at the stage represented by Q in Fig. 10·40 is always approximately (21×10^{-16}) cm.^{2}. The fact that the critical area does not change as the length of the hydrocarbon chain is varied in the above series of normal, saturated fatty acids proves that the molecules are orientated steeply to the surface and it is possible that the orientation of the chain is vertical.

Now palmitic acid, with a total number of 16 carbon atoms, has a molecular volume of 300 cm.^{3} mole.$^{-1}$, so that each molecule of this acid has a volume

$$300 \div 6{\cdot}02 \times 10^{23} = (498 \times 10^{-24})\ \text{cm.}^3.$$

According to Langmuir its cross-section is $20{\cdot}5 \times 10^{-16}$ cm.^{2}, so that its length in a direction normal to the surface is 24×10^{-8} cm., if the density of the substance is independent of whether it exists as a film or in bulk. A molecule of palmitic acid therefore appears to be about five times as long as it is thick. Such results are in accord with those found by X-ray studies and thus the view that the long chains in these molecules are either normal or very steeply orientated to the surface on which they rest is confirmed.

The work of N. K. Adam and his collaborators.—About 1920 ADAM began a series of investigations on surface films whereby the work of Langmuir was not only confirmed but also greatly extended. At first Adam used an apparatus similar to Langmuir's except that waxed glass strips were used for the barriers. By 1926 ADAM and JESSOP had designed an apparatus of greater sensitivity and reliability. The trough is made of brass, although one of silica would be preferable, and both it and the barriers are coated with a hard paraffin wax, whose purpose is to provide a non-wetting layer across which the water cannot spread. It is most important that the barriers and tops of the trough shall not become wet, for under such conditions the passage of the film beyond the area to which it is supposed to be confined cannot be prevented.

To prevent the passage of the film beyond the float, the clearance between this and the sides of the trough—about 1 cm. on each side—is blocked with very light platinum ribbons. To measure the force acting on the float the arrangement shown in Fig. 10·41 is used. A rigid vertical framework carries two horizontal torsion

wires, W_1 and W_2. The lower wire carries a mirror, M, and is connected to the float and upper wire by means of silver wire PQR bent to take the form indicated. AB is the float and the platinum ribbons attached to it are shown—the rest of the trough and many details of the refinements used by Adam are omitted.

When a force F acts on the float the mirror M is rotated about a horizontal axis and by applying a twist to the upper wire by means of a torsion head the turning effect of the force F is counterbalanced and the float and mirror M are restored to their zero

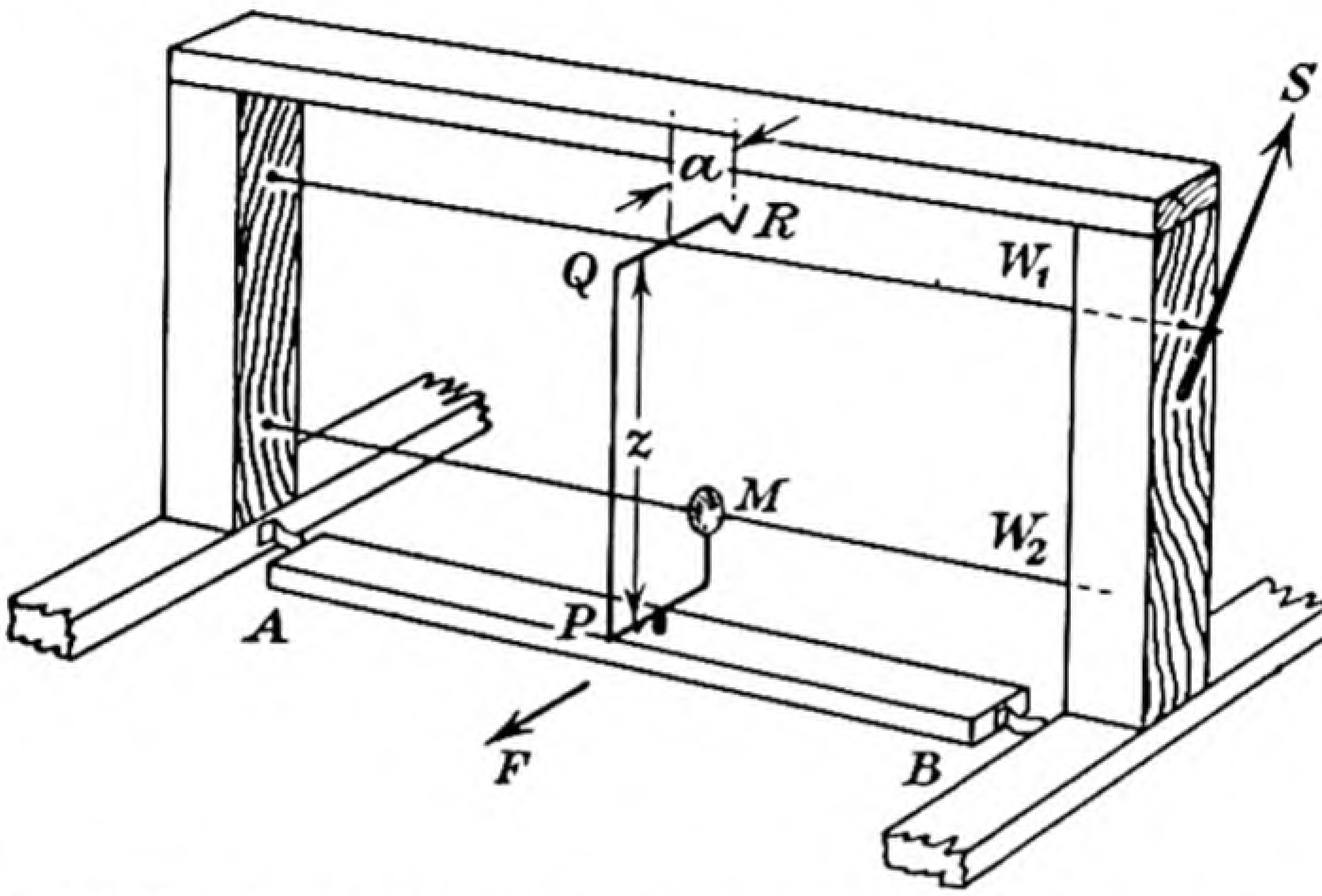


FIG. 10·41.—The float and torsion wires of a surface-film trough as designed by Adam.

positions. The amount of rotation given to the wire W_1 is indicated by a pointer S moving over a circular scale in degrees—not shown in the diagram.

To calibrate the instrument a known mass μ is placed in the hook R at a distance a from the axis of W_1. If z is the distance PQ and $\widehat{PQR} = \frac{\pi}{2}$, then the force, X, on the float is given by $Xz = a\mu g$. Thus X—or better $\frac{X}{l}$—is known, where l is equal to the length of AB plus the clearance on one side. If S is moved through an angle ϕ to restore the system to its zero position, then a rotation of S through 1° corresponds to a force $\frac{X}{l\phi}$ on unit length of the float AB. A force per unit length as small as 0·01 dyne.cm.$^{-1}$ can be detected in this way.

Fig. 10·42(a) is a typical 'force-area' curve for a film of a fatty acid on distilled water at room temperature. It is similar to those obtained by Langmuir except that the point Q, Fig. 10·40, is not

so definite as Langmuir thought it to be. The line BC is slightly inclined to the vertical axis and represents the compression of the condensed film. By extrapolating CB to cut the horizontal axis in E, the area per molecule for zero compression is obtained. At C the film collapses.

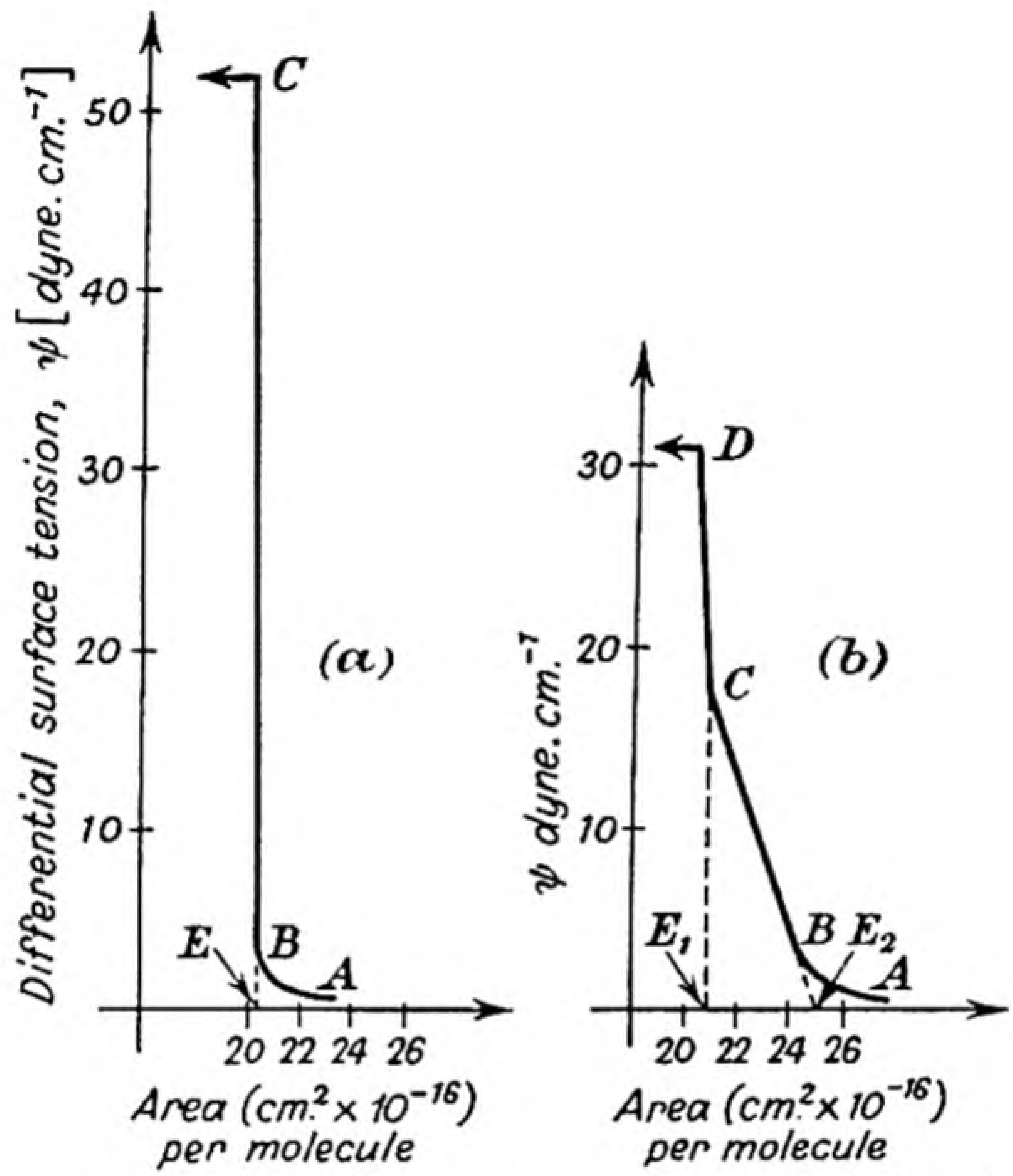
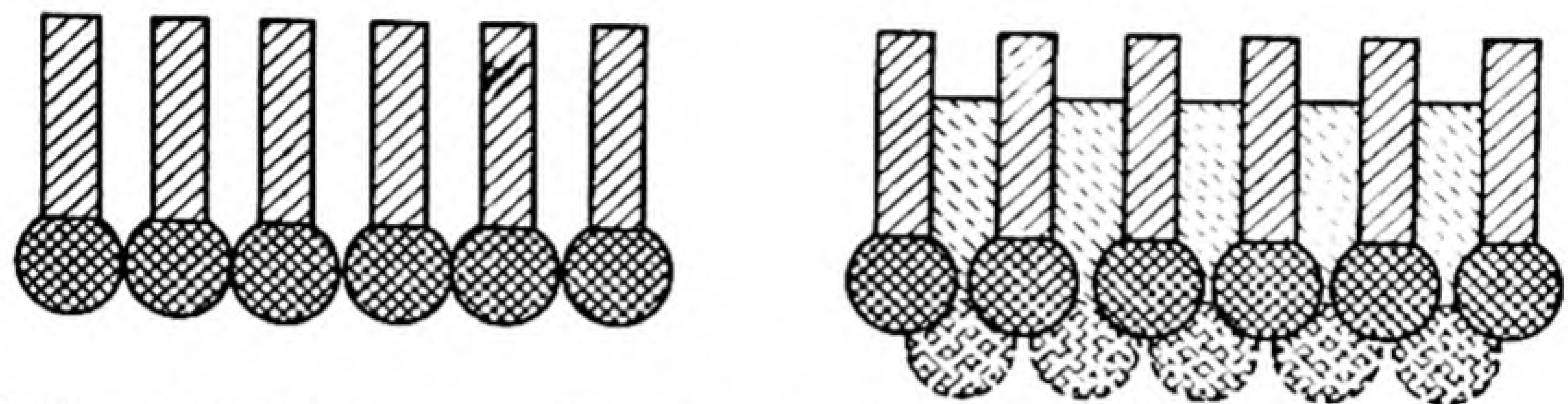


FIG. 10·42.—Typical 'force-area' curves for monomolecular films of a fatty acid (*a*) on water, (*b*) on dilute HCL.

Films on dilute hydrochloric acid.—Langmuir had already discovered the following remarkable property of unimolecular films on dilute hydrochloric acid and Adam made a more complete study of the phenomenon. The force-area curve for a film of fatty acid on 0·1N. to 0·01N.hydrochloric acid is shown in Fig. 10·42(*b*). The upper portion CD represents the film in its state of closest packing and the area per molecule, as indicated by E_1, where DCE_1 is a straight line, is still $20{\cdot}5 \times 10^{-16}$ cm.^{2} and all the fatty acids give this value as for films on pure water. The portion BC does not apparently occur with films on water and on extrapolating to E_2 the value $25{\cdot}1 \times 10^{-16}$ cm.^{2} is found to be independent of the length of the hydrocarbon chain in the fatty acid used.

To explain these results Adam puts forward the following theory. Since the liquid on which the film rests is acid there will be a tendency for the COOH or carboxyl groups to be repelled from the

surface so that until the stage represented by the first kink in the curve at C is reached, the carboxyl groups or 'heads' of the molecules are packed together as closely as possible; such packing is shown in Fig. 10·43(a). When the degree of compression on the film is raised the molecules rearrange themselves by pushing some of the heads into the solution. When this interlocking takes place, cf. Fig. 10·43(b), the hydrocarbon chains become closely packed and



(a) *Diagrammatic representation of the structure of a unimolecular film, when the chains are not quite in contact with one another.*

(b) *A unimolecular film under greater compression: the chains are now in contact.*

FIG. 10·43.—Adam's theory of the structure of monomolecular films.

under these conditions the area occupied per molecule of the fatty acid has its normal value.

A modern use for monomolecular films.—In recent years use has been made of cetyl alcohol, $CH_3(CH_2)_{14}CH_2OH$ or $C_{16}H_{33}OH$, as a monomolecular film on large surfaces of water. It is found that the presence of such a film greatly impedes the rate of evaporation of the water. The economical value of such operations in tropical climates is very great.

The calming of waves by oil.—In a storm at sea, the waves may be calmed to a large extent by pouring a vegetable oil on the surface of the water; such a fact was known in classical times. PLINY, who was suffocated in the eruption of Vesuvius in 79 A.D., wrote, 'All sea is calmed by oil, and that is why divers sprinkle it on their faces'. PLUTARCH, later in the same century or early in the second century A.D., asked, 'Why, when the sea is sprinkled with oil, clearness and calm result?' He attributes these effects to the slipping of the wind over the surface of the water and tells how a diver takes olive oil into his mouth and blows it out when he is below the surface of the water.

The modern explanation of the calming effect is that an oil film does not reduce the height of large gravity waves but damps out the small ripples which otherwise appear on the larger wave. If the oil is not used the ripples, caused by the wind, lead to a condition in which the breaking of the waves becomes dangerous to shipping, for the effect of the wind on the ripples is cumulative. AITKEN

(1883) showed that much of the action of the oil is due to its regularizing the motion imparted by a high and gusty wind to the water surface. A clean surface, with no surface skin, is blown in different directions and with varying force if the wind is gusty; the excitation of the irregular, interfering, ripples is almost entirely attributable to this cause. On the other hand, an oil film, which is not readily compressed, distributes the motion more uniformly over large areas, thus greatly diminishing the excitation of the ripples. According to HARDY (1926), who expresses the above conclusions somewhat differently, the oil makes the sea so very smooth that the wind 'cannot catch upon it'. When there is no oil film a great wave carries countless ripples and wavelets each of which enables a wind to exert a direct thrust on the surface. It is to the suppression of these ripples and wavelets that the characteristic smoothness is due, and when they are not present the chief 'catch' of the wind upon the sea is lost.

Oil films also act to some extent by damping ripples already formed. POCKELS showed that a coherent film, insufficient to cover more than a small fraction of an available surface, damps ripples produced mechanically in a trough; the extent of the damping increases with an increasing proportion of the surface covered until the covering is complete. Further compression of the film, once the whole surface is covered, does not appear to produce any further damping effect; this shows that the diminution of surface tension alone, which does not commence until the film covers the whole surface, is not the cause of the damping.

In connexion with this phenomenon N. K. ADAM writes:—'It is scarcely necessary to say that good spreading power is essential for efficient damping of waves. Fish oils are supplied to ships, and to their life-boats, for this purpose. Mineral oils are not good; but in an emergency they might be improved by melting a few stearine candles, and mixing with the oil; the carboxyl groups, (COOH), in the stearine provide the necessary adhesion to the water.'

EXAMPLES X†

10·01. Distinguish between ***intrinsic surface energy*** and ***surface tension.***

Give the theory of *two* accurate methods of determining the surface tension of water at room temperature.

10·02. Show that the excess pressure within a soap bubble of radius r is $4\gamma r^{-1}$, where γ is the surface tension of the soap solution.

Calculate the difference in the levels in the two limbs of an open U-tube manometer containing oil of density 0·75 gm.cm.$^{-3}$, when one

† In the questions taken from papers set at London University the symbol T for surface tension has been altered to γ.

end is attached to a tube at the other end of which is a small spherical soap bubble of diameter 2 mm. Assume $\gamma = 25$ dyne.cm.$^{-1}$.

[1·36 cm.]

10·03. A capillary tube is placed in a vertical position with one end below the surface of a liquid. Show that when the liquid has risen in the tube and equilibrium is established, the height of the meniscus in the tube above the level surface is independent of the shape of the tube below the meniscus.

Two plane glass plates in contact along a vertical edge and inclined to one another at a small angle θ are placed with their lower ends in water. Find an expression for the height to which the water rises at a distance x from the edge.

10·04. A soap bubble ($\gamma = 26$ dyne.cm.$^{-1}$) is slowly enlarged from a radius of 1 cm. to a radius of 10 cm. Calculate, by two methods, the work done in the above operation. If the operation were performed more quickly, discuss whether or not more work would have to be done.

[$6{\cdot}46 \times 10^4$ erg.]

10·05. Show that if the pressures inside and outside a liquid surface are p and π respectively, then $(p - \pi) = \gamma(1/r_1 + 1/r_2)$ where γ is the surface tension and r_1 and r_2 the principal radii of curvature. Apply this to the case of a spherical soap bubble.

A soap bubble of radius r is blown on a tube with a solution whose surface tension is γ. The atmospheric pressure is P. The tube on which the bubble is blown is then connected to a vessel of volume V filled with air at atmospheric pressure. Show that the bubble will shrink until its radius is $\frac{1}{2}r$ if

$$2\pi r^2\gamma + \tfrac{7}{12}\pi r^3\mathrm{P} = 4\gamma \mathrm{V} r^{-1}.$$

10·06. Two soap bubbles of radii a and b coalesce to form a single bubble of radius r. If the external pressure is P, prove that the surface tension of the solution from which the bubbles are formed is

$$\tfrac{1}{4}\mathrm{P}(r^3 - a^3 - b^3) \div (a^2 + b^2 - r^2).$$

10·07. A vertical U-tube contains liquid. One limb of this tube is open to the atmosphere while a soap film is formed across the end of the other limb. A side tube is attached to the U-tube so that, by blowing, the soap film may be distended. Show that for different bubbles the product of the radius of the bubble and the difference in height of the liquid levels in the U-tube is constant. If the liquid in the U-tube is water and the above constant is 0·123 cm.^{2}, calculate a value for the surface tension of the soap solution. [30·2 dyne.cm.$^{-1}$.]

10·08. Define *surface energy* (ϵ), *surface tension* (γ).

Theoretical considerations enable us to establish the relation

$$\epsilon = \gamma - \mathrm{T}\frac{d\gamma}{d\mathrm{T}},$$

where T is the temperature on the absolute scale. Describe and explain the experiments which would have to be carried out to determine the value of ϵ for water at 40° C.

10·09. A thin circular rubber band, when unstretched, has a radius of 3·8 cm. It is placed on a soap film and the film within the ring is then broken. The new radius of the ring is found to be 3·9 cm. The ring is then cut at one place, and it is found that a force of 0·28 gm.-wt. is required to increase its natural length by 1 cm. Calculate a value for the surface tension of the soap solution from which the film was formed.

[22·7 dyne.cm.$^{-1}$.]

10·10. Water, surface tension 72 dyne.cm.$^{-1}$, rises to a height of 20 cm. in a vertical capillary tube of uniform bore. The above tube is removed from the water, dried and its upper end sealed. The tube, of total length 40 cm., is then held vertically with its lower end just touching a water surface. If the atmospheric pressure is equal to that of a vertical column of water 10 metres high, to what height will the water rise in the tube? [0·75 cm.]

10·11. Give the essential theory of the capillary rise method for measuring the surface tension of a liquid.

Discuss the relative merits of this method and its various modifications.

Describe in detail a method that can be used for a small quantity of a hygroscopic and volatile liquid.

10·12. Discuss the relation between *surface energy* and *surface tension*.

A soap solution of surface tension γ is used to form a film between a horizontal rod of length l and a length of weightless inextensible thread attached to each end of the rod. A weight W is attached at the mid-point of the thread. Prove that (a) the shape of each half of the thread is circular, (b) the tension in the thread is equal to

$$\frac{l\gamma}{\cos\theta_2 - \sin\theta_1},$$

where θ_1 is the angle which the tangent to the thread at each end makes with the horizontal, and $2\theta_2$ is the angle between the tangents to the thread at the point where W is attached. (L. Sch. adapted.)

10·13. A hollow glass cylinder of internal radius 1 cm. and wall thickness 0·5 mm. is open at both ends and is suspended vertically from one arm of a balance which is then equilibrated. A vessel containing liquid is brought up so that the surface of the liquid just touches the edge of the suspended glass cylinder. Explain what happens.

The vessel of liquid is then raised until the balance is again in equilibrium. Given that the surface tension of the liquid is 30 dyne.cm.$^{-1}$, its angle of contact with glass zero, and its density 1·02 gm.cm.$^{-3}$, calculate the depth of immersion of the cylinder. [1·22 cm.]

10·14. The time of oscillation, T, of a drop of liquid hanging from the end of a vertical tube is proportional only to the radius of the drop, r, its density, ρ, and the surface tension, γ. Prove that the relation between these variables must be

$$T = \kappa\sqrt{\frac{\rho r^3}{\gamma}},$$

where κ is a constant. $\left[N.B.\quad \kappa = \frac{1}{\sqrt{2}}\pi.\right]$

10·15. A spherical bubble of radius 0·10 cm. is blown in an atmosphere whose pressure is 10^6 dyne.cm.$^{-2}$. If the surface tension of the liquid comprising the film is 50 dyne.cm.$^{-1}$, to what pressure must the surrounding atmosphere be brought in order that the radius of the bubble may be doubled? Assume isothermal conditions and that there is no diffusion through the bubble. [$1·24 \times 10^5$ dyne.cm.$^{-2}$.]

10·16. Two grams of mercury are placed between two horizontal plane sheets of glass and these are pressed together until the mercury forms a circular disc of 7 cm. radius. Assuming that the disc is of uniform thickness, the surface tension of mercury is 435 dyne.cm.$^{-1}$,

its angle of contact with glass is 140° and its density 13·6 gm.cm.$^{-3}$, calculate a value for the thrust, due to the mercury, on the upper plate.

10·17. Outline three methods of measuring the surface tension of a liquid, indicating in each case the quantities which have to be measured. Describe in detail one of these methods which you consider to be specially suitable for investigating the variation of surface tension with temperature.

10·18. Describe and explain how the surface tension of a liquid may be measured by forcing bubbles of air through it. Discuss whether the result obtained in this way should be the same as that given by the capillary tube method.

10·19. Assuming that for mercury, $\gamma = 435$ dyne.cm.$^{-1}$ and $\rho = 13{\cdot}6$ gm.cm.$^{-3}$, calculate the maximum length of a mercury pellet which can remain at rest in a vertical glass capillary tube of radius 0·015 cm., if the 'advancing' and 'receding' angles of contact for mercury on glass are 150° and 110° respectively. [2·28 cm.]

10·20. Explain why very small drops of liquid are spherical while large drops are usually not so.

How may large spherical drops be obtained?

A sphere of water of radius R cm. is sprayed into 1000 small drops of equal volume. Derive an expression for the minimum amount of work required to do this. [$36\pi R^2\gamma$ ergs.]

10·21. Establish an expression for the difference of pressure across an element of the curved surface of a liquid in terms of surface tension and the principal radii of curvature of the element. Apply the result to determine the capillary displacement, h, in a narrow vertical tube of circular section. What happens if the liquid rises and the unimmersed length of the tube is less than h? (G)

10·22. Describe (*a*) a method suitable for the comparison of the surface tensions of a solution at different concentrations, and (*b*) a method for the determination of the surface tension of a liquid when only a few drops are available. (G)

10·23. A conical glass tube 12 cm. long, 0·12 cm. in diameter at one end and 0·04 cm. at the other is fixed vertically with its wider end just touching the surface of water, for which the surface tension is 72 dyne.cm.$^{-1}$. To what height will the water rise in the tube?

[2·82 cm.]

10·24. Describe and explain an experiment to determine the surface tension of a liquid by measuring the pull necessary to detach a horizontal circular plate which makes contact with the surface of the liquid.

Two circular glass plates, 4 cm. diameter, are separated by a film of water 0·1 mm. thick. Assuming that the surface tension of water is 73 dyne.cm.$^{-1}$, find the normal pull in gm.-wt. which must be exerted in order to separate the plates. Explain the method of calculation.

(G) [374 gm.-wt.]

10·25. A jet of liquid emerges from a cylindrical tube, the transverse section of which is slightly elliptical. Owing to the action of surface tension the jet oscillates about the circular cylindrical form. Find the form of relationship between the period of oscillation T, the diameter of the jet d, and the density ρ and surface tension γ of the liquid.

[$T = k\sqrt{\rho d^3\gamma^{-1}}$]

10·26. What is meant by the surface tension at the interface between two media, e.g. between water and air? Give some account of its experimental measurement.

The velocity of gravity waves on deep water is given by

$$c^2 = \frac{\lambda g}{2\pi}.$$

Investigate the correction to be applied to this formula when the influence of surface tension on the velocity of the waves is taken into account. (G)

10·27. A vertical glass capillary tube with a slightly conical bore just dips with its apex upwards into a liquid of density ρ, surface tension γ, and angle of contact with glass zero. A very small hole is left at the apex so that air can escape from the tube. If a is the maximum radius of cross-section of the tube and l its length show that the liquid rises to a height

$$h = \frac{l}{2} \pm \sqrt{\frac{l^2}{4} - \frac{2\gamma l}{g\rho a}},$$

where g is the intensity of gravity.

10·28. A large drop of a liquid lies on a horizontal surface in air. Take the origin at the centre of the upper surface and show that at any point P in a vertical plane section the ordinate y and the slope ψ of the tangent to the intersection of the plane and surface are related by the equation

$$g\rho y^2 = 2\gamma(1 - \cos\psi),$$

where ρ is the density of the liquid and γ its surface tension. Assume that the curvature in the horizontal section through P is very small.

Describe and explain how this formula can be applied to determine the surface tension and angle of contact in certain cases. Discuss briefly the measurement of angles of contact. (S)

10·29. Explain the method of dimensional analysis for solving physical problems. Give an account of its uses and limitations. Find how the time of oscillation of a liquid drop depends upon the relevant variables. (S)

10·30. For a certain large sessile drop of mercury on glass it was found that the depth of the longest diameter below the highest point in the drop is $\frac{\sqrt{3}}{2}$ times the maximum height of the drop. Show that the angle of contact for mercury on glass is $\pi - \cos^{-1}(\frac{1}{3})$.

10·31. Show that the equation of the surface of contact of two liquids can be written in the form:

$$\gamma\left(\frac{1}{R_1} + \frac{1}{R_2}\right) - \rho g y = \text{constant},$$

where y is measured vertically from some convenient level, R_1 and R_2 are the principal radii of curvature of the surface, γ is the surface tension and ρ the difference of density of the two liquids. When the surface is a cylinder with a horizontal axis, show that a plane perpendicular to the axis cuts it in the curve

$$2\gamma(1 - \cos\psi) = \rho g y^2,$$

the origin being suitably chosen, and ψ being the inclination of the tangent at y to the horizontal. A bubble of air is blown beneath a horizontal glass plate in a transparent liquid. Explain, giving a

criticism of the method, how the surface tension of the liquid and the angle of contact with glass can be determined from observations on the bubble. (S)

10·32. A drop of water of radius r is formed at the lower end of a vertical capillary tube. The height of the water in the tube above the lowest point of the drop is h_1. When the drop and lower end of the tube are immersed in a beaker of water at such a depth that the level in the capillary tube is unaltered this level is at a height h_2 above the level in the beaker.

Show that

$$2\gamma/g\rho = r(h_1 - h_2) - r^2/3,$$

where γ is the surface tension of water and ρ its density.

Describe a method suitable for the determination of the surface tension of a liquid when only a few drops are available. (S)

10·33. What is the experimental evidence for saying that substances like stearic acid can form solid films on liquid surfaces?

How do these experiments lead to an estimate of the area of cross-section of a molecule in the film?

10·34. A small quantity of mercury rests between two horizontal glass plates. What load must be placed on the upper plate so that the plates may be everywhere 0·1 mm. apart if the area of each plate in contact with the mercury is 40 cm.^{2}? The surface tension of mercury may be taken as 430 dyne.cm.$^{-1}$ and its angle of contact with glass as 135°. [2·48 kgm.]

Calculate the work done under isothermal conditions in blowing a soap bubble of 500 cm.^{3} capacity, the surface tension of the soap solution being 30 dyne.cm.$^{-1}$ and the barometric height 76 cm. of mercury. [$1{\cdot}83 \times 10^4$ erg.]

10·35. A glass vessel with a flat top was filled with water, except that a large flat air bubble remained under the central part of the top. This bubble was found to be 0·542 cm. deep over its central portion. The widest part of the bubble was 0·152 cm. below the under surface of the glass top. Calculate values for the surface tension of water and its angle of contact with glass. [74·5 dyne.cm.$^{-1}$, 5·1°]

10·36. A uniform capillary tube of internal diameter $2r$ is held vertically while a wide dish containing liquid of density ρ is slowly raised until the surface of the liquid just touches the lower end of the tube. The elevated column of liquid reaches equilibrium when its height is h. Ignoring small corrections, derive expressions in terms of the quantities given and g for (*a*) the work done by the capillary forces during the capillary ascent and (*b*) the increase in gravitational potential energy which occurs as a result of the ascent. Explain why these expressions are not equal. [(*a*) $\pi r^2h^2g\rho$, (*b*) $\frac{1}{2}\pi r^2h^2g\rho$.]

10·37. A drop of liquid rests on a horizontal plate which it does not wet. The ratio (p) of any two given linear dimensions of the drop, *e.g.* the diameter of its largest horizontal section and the depth of this section below the highest point of the drop, depends on four quantities, viz. the volume (V) of the drop, the density (ρ) and the surface tension (γ) of the liquid and the acceleration due to gravity (g). Let

$$p = aV^w\rho^x\gamma^y g^n,$$

where a, w, x, y and n are constant numbers. Use dimensional analysis to find w, x and y in terms of n.

Two drops are formed of different liquids A and B, and their volumes

are adjusted until each has the same value of p. Use the following data (which are all expressed in consistent units) to calculate a value for the surface tension of B.

Liquid	Volume of drop	Density	Surface tension
A	6·75	0·99	72
B	1·55	0·88	—

The equality of the values of p might be tested with the help of optical projection. Suggest a possible procedure. [Precise details of the optical system are not required.] (L. Sch.) [n, n, $-n$. 24·0]

10·38. A piece of wire is bent into the form of a square ABCD whose sides are of length 4·0 cm. The side AD of the square is removed and the two free ends of the wire are joined by a light flexible inextensible thread of length $2(\pi + 1)$ cm. A soap film is then formed between the wire framework and the thread. Describe the configuration of the thread.

The thread is now pulled at its midpoint through a distance of 2·0 cm. towards the centre of the line AD. Assuming the expansion of the film takes place under isothermal conditions and the surface tension of the soap solution is 40 dyne.cm.$^{-1}$, calculate values for (a) the increase in surface area of the film, (b) the work done in stretching the film and (c) the force required to maintain the film in its new position.

[(a) 6·3 cm.^{2}, (b) 251 erg., (c) 160 dyne.]

CHAPTER XI

VISCOSITY AND THE NEWTONIAN FLOW OF FLUIDS

Qualitative tests for viscosity.—From observations on the magnitude of the surge of an oil in a container when the latter is rocked, a skilled craftsman is able to differentiate between oils which are light or heavy, thin or thick. Such terms are not used to denote the relative density of the oils under examination but to indicate, at least in part, their usefulness for oiling machinery or for other purposes. Sometimes a skilled worker judges the suitability of an oil from the manner in which it 'runs' when poured on to a glass plate or from the resistance offered to the motion of a stirrer. Such arbitrary methods are unsatisfactory, scientifically, since, in general, more than one physical property of the oil is involved in the test applied, e.g. surface tension and density will undoubtedly be contributory factors. Before any absolute method could be elucidated it was essential that there should be some hypothesis concerning the behaviour of liquids when there is relative motion between adjacent layers. Such an hypothesis was first formulated by NEWTON.

Newton's hypothesis of viscous flow.—The whole theory and practice of viscometry is based on this hypothesis, which is as

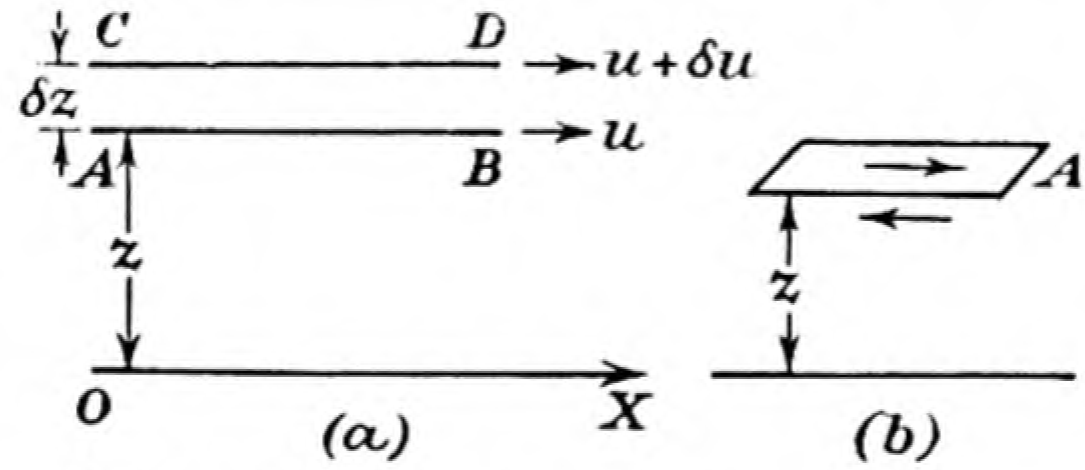


FIG. 11·01.—Newton's law of viscous flow.

follows. Suppose that a fluid is moving without turbulence over a plane OX, Fig. 11·01(*a*), the fluid in contact with the plane being at rest. In planes AB and CD, at distances z and $z + \delta z$ from OX, let u and $u + \delta u$ be the velocities of flow. Then $\dfrac{du}{dz}$ is called the ***velocity gradient*** at points in the plane AB. [If u is also a

function of other variables, then $\frac{\partial u}{\partial z}$ must be used to denote the velocity gradient.]

Now consider a plane area at a height z above a fixed plane, cf. Fig. 11·01(b). The fluid above the plane exerts a tangential force F on the plane tending to urge it forward; the fluid below exerts a force F in the opposite direction. According to Newton's hypothesis the tangential stress across the plane is proportional to the velocity gradient in a direction normal to the plane, i.e.

$$\frac{\mathrm{F}}{\mathrm{A}} \propto \frac{\partial u}{\partial z}, \qquad \text{or} \qquad \frac{\mathrm{F}}{\mathrm{A}} = \eta \frac{\partial u}{\partial z},$$

where η is a constant known as the ***viscosity of the fluid.*** The dimensions of η are $[\mathrm{ML^{-1}T^{-1}}]$, so that in the c.g.s. system the unit is the gm.cm.$^{-1}$sec.$^{-1}$, and this is sometimes called the ***poise*** in honour of Poiseuille. A derived unit, the ***centipoise***, is also widely used.

Mathematicians and engineers call $\frac{\eta}{\rho}$ the ***kinematic viscosity***, ν, of a fluid, where ρ is its density. The dimensions of ν are $[\mathrm{L^2T^{-1}}]$, a fact which is easily remembered for it is possible to express the kinematic viscosity of a fluid in acre.year.$^{-1}$! The unit of kinematic viscosity is the ***stokes***.

It must be emphasized that Newton's hypothesis is only valid for relatively small values of u, the velocity of flow; also, there is no direct experimental proof of the validity of this hypothesis, but equations deduced from it are confirmed experimentally in the case of many fluids so that for them the validity of Newton's hypothesis cannot be doubted. The criterion which determines the upper limit for the velocity of flow of a liquid through a tube will be discussed later.

On the forces, due to viscosity, acting on a small rectangular element of volume in a fluid for which the stream lines are all parallel to the plane containing one face of the given element.—Let δx, δy, δz, Fig. 11·02, be a rectangular element, the flow of fluid being parallel to the x-direction. At the ends of the element defined by the planes x, $x + \delta x$, let the pressures be p and $p + \delta p$, respectively. If u is the velocity of flow at points in the lower face of the element, then the tangential force on this face and due to viscosity acts in the direction of x decreasing and is given by $\eta \frac{\partial u}{\partial z} \delta x \, \delta y$. On the upper $\delta x \, \delta y$ face the viscous force is $\eta \frac{\partial}{\partial z}\left(u + \frac{\partial u}{\partial z} \delta z\right) \delta x \, \delta y$ and this is directed along the positive direction of x. The forces due to pressure on the '$\delta y \, \delta z$' ends of the element

are $p\,\delta y\,\delta z$ and $(p + \delta p)\,\delta y\,\delta z$, the direction of these forces being in the positive and negative directions of x respectively. Since the fluid element moves without acceleration in the x direction the force in this direction must be zero, so that

$$\eta \frac{\partial}{\partial z}\left(u + \frac{\partial u}{\partial z}\,\delta z\right)\delta x\,\delta y - \eta\frac{\partial u}{\partial z}\,\delta x\,\delta y + p\,\delta y\,\delta z - \left(p + \frac{\partial p}{\partial x}\,\delta x\right)\delta y\,\delta z = 0,$$

i.e.
$$\eta\frac{\partial^2 u}{\partial z^2} - \frac{\partial p}{\partial x} = 0.$$

This equation has been derived to indicate the method of attack frequently used in investigating problems on viscous flow and also

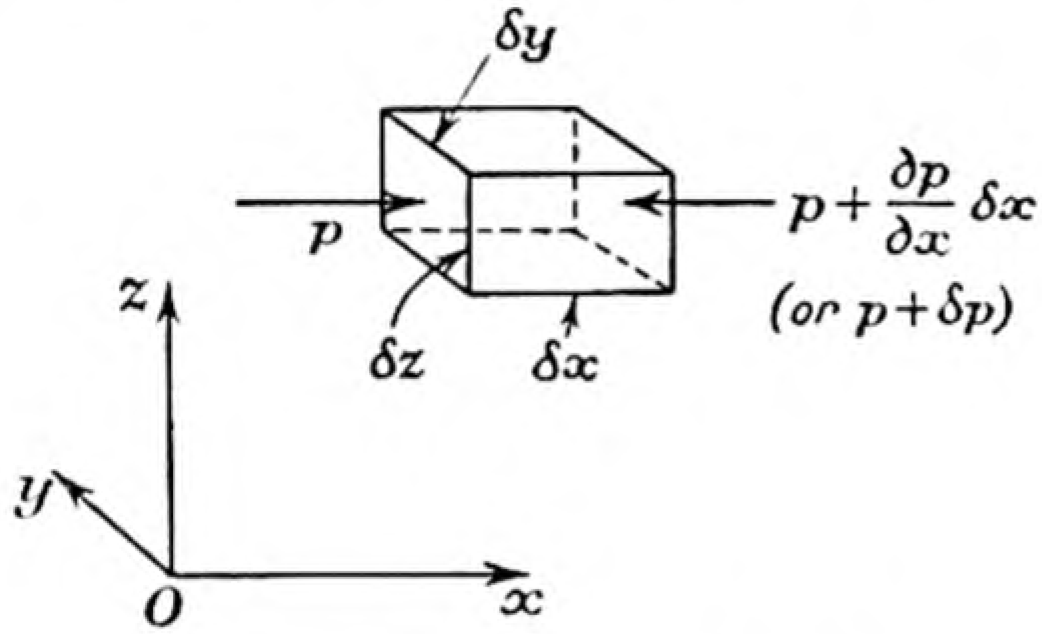


FIG. 11·02.—The equation $\eta\dfrac{\partial^2 u}{\partial z^2} - \dfrac{\partial p}{\partial x} = 0.$

to show that in fluids, where the motion is due entirely to the existence of a pressure gradient, it is impossible for u to be a linear function of z. For then $\dfrac{\partial p}{\partial x}$ would be zero and no motion would be possible under the conditions postulated.

On the steady flow of an incompressible liquid through a horizontal capillary tube.—Consider a right cylindrical horizontal tube of radius a and length l, through which an incompressible liquid is driven, without turbulence, by the application of a steady pressure difference P between the ends of the tube. It will be assumed:—

(*a*) that the flow is everywhere parallel to the axis of the tube,
(*b*) that the flow is steady, initial disturbances due to acceleration from rest having been damped out,
(*c*) that the liquid in contact with the wall of the tube is at rest,
(*d*) that the liquid flows when it is subject to a very small shearing force.

These assumptions require that the pressure gradient shall be constant along the tube. Let the axis of the tube be taken as the x-axis, so that the ends of the tube are defined by $x = 0$ and $x = l$, the pressures there being p_1 and p_2, where $p_1 > p_2$. Then $p_1 - p_2 = \mathrm{P}$. Since the pressure gradient is constant, we have

$$\frac{\partial p}{\partial x} = \alpha, \text{ say.}$$

$$\therefore\ p = \alpha x + \beta,$$

where β is a constant of integration. Now at $x = 0$, $p = p_1$, so that $\beta = p_1$, and at $x = l$, $p = p_2$, so that $p_2 = \alpha l + p_1$. Hence

$$\frac{\partial p}{\partial x} = -\frac{p_1 - p_2}{l} = -\frac{\mathrm{P}}{l}.$$

Let us now evaluate the forces acting on a small portion of the liquid confined between two cylindrical surfaces, coaxial with the

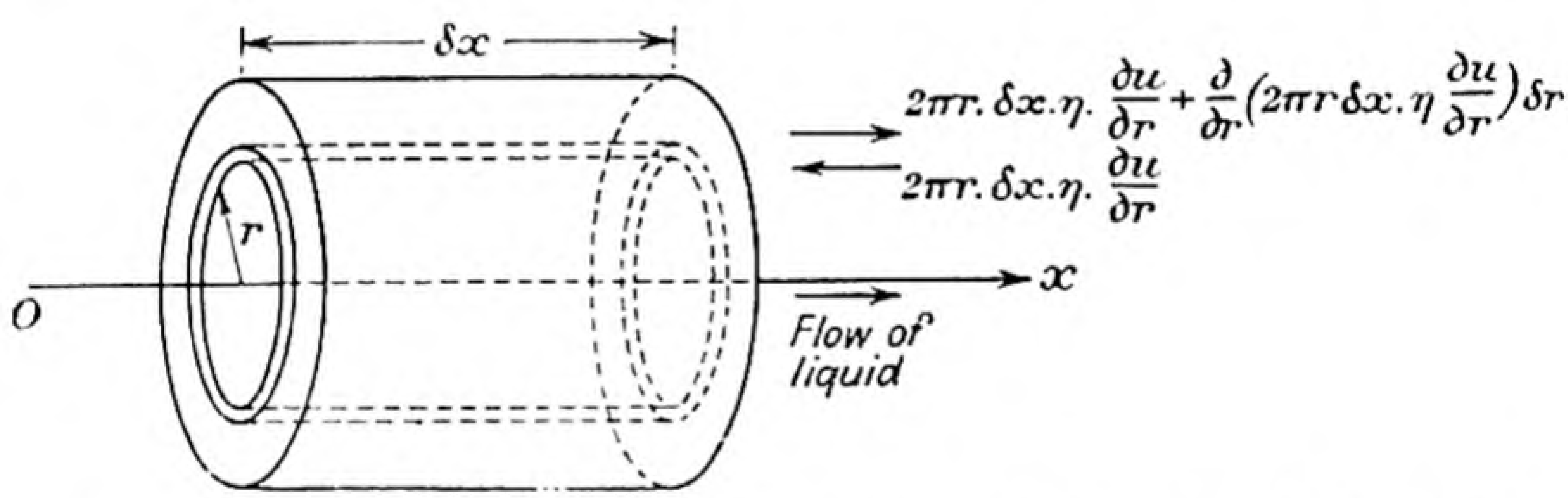


FIG. 11·03.—Flow of an incompressible liquid through a horizontal capillary tube.

axis of the tube, having radii r and $r + \delta r$, cf. Fig. 11·03, and terminated by planes at distance δx apart and at right angles to the axis of the tube. Moreover, the tube must be horizontal so that gravity does not influence the flow of the liquid.

The liquid nearer to the axis than is the cylindrical element, exerts on the inner annular surface of the element a force $2\pi r\, \delta x\, \eta \frac{\partial u}{\partial r}$ directed in the negative direction of the x-axis. The liquid outside the annular element exerts on its outer surface a force

$$2\pi r\, \delta x\, \eta \frac{\partial u}{\partial r} + \frac{\partial}{\partial r}\left(2\pi r\, \delta x\, \eta \frac{\partial u}{\partial r}\right) \delta r,$$

in the opposite direction. Moreover, the forces on the element

due to pressure within the liquid are $p.2\pi r\,\delta r$, in the direction of x-increasing, and

$$\left[2\pi r\,\delta r\left(p+\frac{\partial p}{\partial x}\,\delta x\right)\right],$$

and this is in the direction of x-decreasing. Since the liquid moves without acceleration the total force acting on it is zero, i.e.

$$\frac{\partial}{\partial r}\left(2\pi r\,\delta x\,\eta\,\frac{\partial u}{\partial r}\right)\delta r-2\pi r\,\delta r\,\delta x\,\frac{\partial p}{\partial x}=0,$$

i.e.

$$\frac{\partial p}{\partial x}=\frac{\eta}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial u}{\partial r}\right),$$

or

$$-\frac{\mathrm{P}}{l}\frac{r}{\eta}=\frac{\partial}{\partial r}\left(r\,\frac{\partial u}{\partial r}\right).$$

After two integrations this becomes

$$u=-\frac{\mathrm{P}r^2}{4\eta l}+\mathrm{A}\ln r+\mathrm{B},$$

where A and B are integration constants. Since the velocity is not infinite at $r=0$, $\mathrm{A}=0$. The condition that $u=0$, when $r=a$, gives

$$u=\frac{\mathrm{P}}{4\eta l}(a^2-r^2).$$

Now Q, the volume of liquid flowing through the tube in ***unit time***, is given by

$$\mathrm{Q}=\int_0^a u.2\pi r\,dr=\frac{2\pi\mathrm{P}}{4\eta l}\int_0^a(a^2-r^2)r\,dr=\frac{\pi a^4\mathrm{P}}{8\eta l}=\frac{\pi a^4(p_1-p_2)}{8\eta l}.$$

If V is the volume flowing through the tube in time t, we have

$$\mathrm{V}=\frac{\pi a^4(p_1-p_2)t}{8\eta l}.$$

This is known as ***Poiseuille's equation.***

Note on the calibration of a tube for use in a viscometer.—Let us assume that a capillary tube with squared ends and of length l is divided into n sections, cf. Fig. 11·04(a), the length of the k-th section being $(\Delta l)_k$. Let a_k be the mean radius of cross-section for this portion of the tube which is shown in Fig. 11·04(b). Let p_1 and p_2 be the pressures at the entrance and exit ends of the capillary tube, while p', p'', . . . are the pressures at the exit ends of

the first, second, . . . sections. Then if V is the volume of liquid flowing through the tube in time t, we have, if a is the effective radius of the tube,

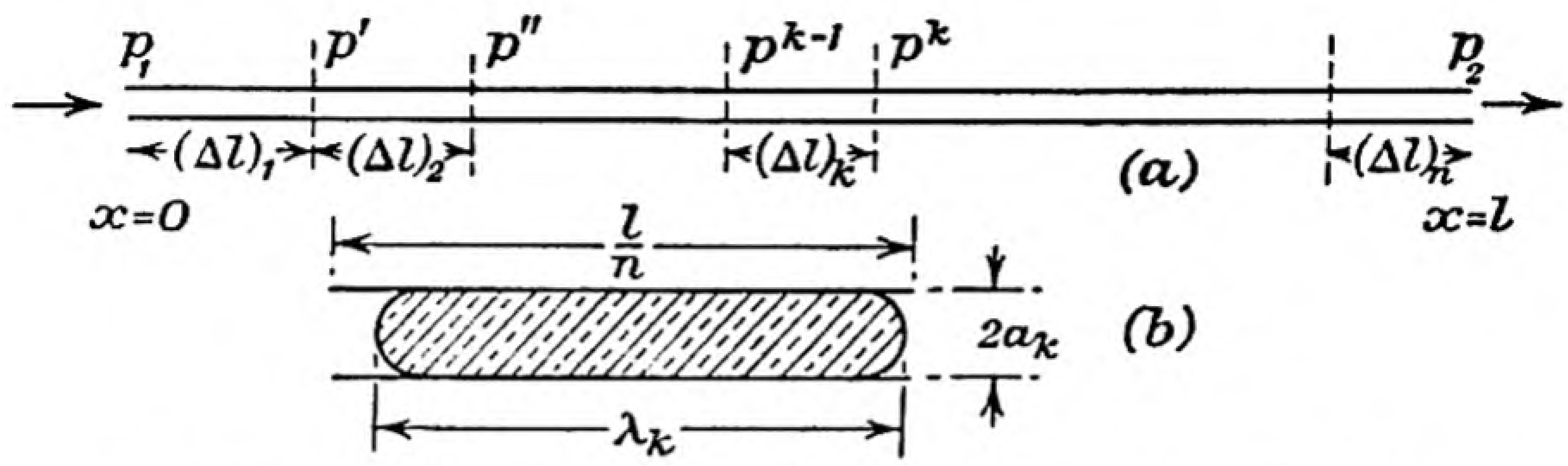


FIG. 11·04.—Calibration of a capillary tube for use in a viscometer.

$$V = \frac{\pi a^4(p_1 - p_2)t}{8\eta l}$$

$$= \frac{\pi a_1{}^4(p_1 - p')t}{8\eta(\Delta l)_1} = \frac{\pi a_2{}^4(p' - p'')t}{8\eta(\Delta l)_2} = \dots$$

$$\dots = \frac{\pi a_k{}^4(p^{k-1} - p^k)t}{8\eta(\Delta l)_k} = \dots \text{ to } n \text{ terms}$$

$$\therefore\ V = \frac{\pi t}{8\eta}\frac{p_1 - p'}{\dfrac{(\Delta l)_1}{a_1{}^4}} = \frac{\pi t}{8\eta}\frac{p' - p''}{\dfrac{(\Delta l)_2}{a_2{}^4}} = \dots$$

$$\dots = \frac{\pi t}{8\eta}\frac{(p^{k-1} - p^k)}{\dfrac{(\Delta l)_k}{a_k{}^4}} = \dots \text{ to } n \text{ terms}$$

$$\therefore\ V = \frac{\pi t}{8\eta}\frac{p_1 - p' + p' - p'' + \dots + p^{k-1} - p^k + \dots \text{ to } n \text{ terms}}{\sum\limits_{k=1}^{n}\dfrac{(\Delta l)_k}{a_k{}^4}}$$

$$= \frac{\pi t}{8\eta}\cdot\frac{(p_1 - p_2)}{\sum\limits_{k=1}^{n}\dfrac{(\Delta l)_k}{a_k{}^4}}.$$

This explains why it is necessary to consider the expression $\sum \dfrac{(\Delta l)_k}{a_k{}^4}$ when the tube is not uniform in section.

To proceed with the actual calibration, the capillary tube is first cleaned by drawing chromic acid and then water through it with the aid of a filter pump. To prevent the pump from ceasing to work it is advisable to dip one end of the capillary tube in the cleaning solution and withdraw it almost immediately. This is

repeated rapidly so that every pellet of liquid passing through the tube is followed by a bubble of air. A glass tube, tightly packed with cotton wool, is then attached to the capillary tube and warm dry air is drawn through. The tube is then fastened in a horizontal position to a metre scale. Let the length l be divided into n equal parts. A pellet of mercury whose length is slightly less than $\frac{l}{n}$ is then introduced into the capillary tube and caused to move along the tube so that it occupies each of the n divisions in turn. The distance between the curved ends of the pellet is measured for each position. Let λ_k be this distance when the pellet is in the k-th section of the tube. Then, if the tube is fairly uniform, we may take the radius as being constant over any one of the n sections; let it be a_k for the k-th section. Then if ρ is the density of mercury at the temperature at which the calibration is carried out,

$$\pi \rho {a_k}^2 \lambda_k = \mathrm{C},$$

where C is a constant for the whole tube; it is nearly equal to the mass of the mercury pellet. Thus

$$\Sigma\,(a_k)^2 = \frac{\mathrm{C}}{\pi\rho}\,\Sigma\left(\frac{1}{\lambda_k}\right).$$

If M is the mass of mercury required to fill the length l of the tube

$$\mathrm{M} = \pi \rho l\,\overline{a^2},$$

where $\overline{a^2}$ is the mean value of the squares of the radii corresponding to the different sections. [It is usually more convenient to determine M_0, say, the mass of mercury filling a length l_0, where $l_0 \doteqdot l$, and to calculate M by simple proportion.] But

$$\overline{a^2} = \frac{1}{l}\left[\Sigma\,(a_k)^2\,\frac{l}{n}\right].$$

$$\therefore\ \mathrm{M} = \pi\rho\,\Sigma\,(a_k)^2\,\frac{l}{n} = \mathrm{C}\,\frac{l}{n}\,\Sigma\left(\frac{1}{\lambda_k}\right).$$

$$\therefore\ \frac{1}{\mathrm{C}} = \frac{l}{n\mathrm{M}}\,\Sigma\left(\frac{1}{\lambda_k}\right).$$

Let a_0 be the effective value for the radius of the capillary tube. Then

$$\frac{l}{{a_0}^4} = \Sigma\left(\frac{1}{{a_k}^4}\right)\frac{l}{n} = \frac{\pi^2\rho^2}{\mathrm{C}^2}\,\Sigma\,(\lambda_k)^2\left(\frac{l}{n}\right),$$

$$= \pi^2\rho^2\,\frac{l^3}{n^3\mathrm{M}^2}\left[\Sigma\left(\frac{1}{\lambda_k}\right)\right]^2\Sigma\,(\lambda_k)^2.$$

Experimental determination of the viscosity of tap water at room-temperature.—To determine the viscosity of tap water at room-temperature the apparatus shown in Fig. 11·05(*a*) may be

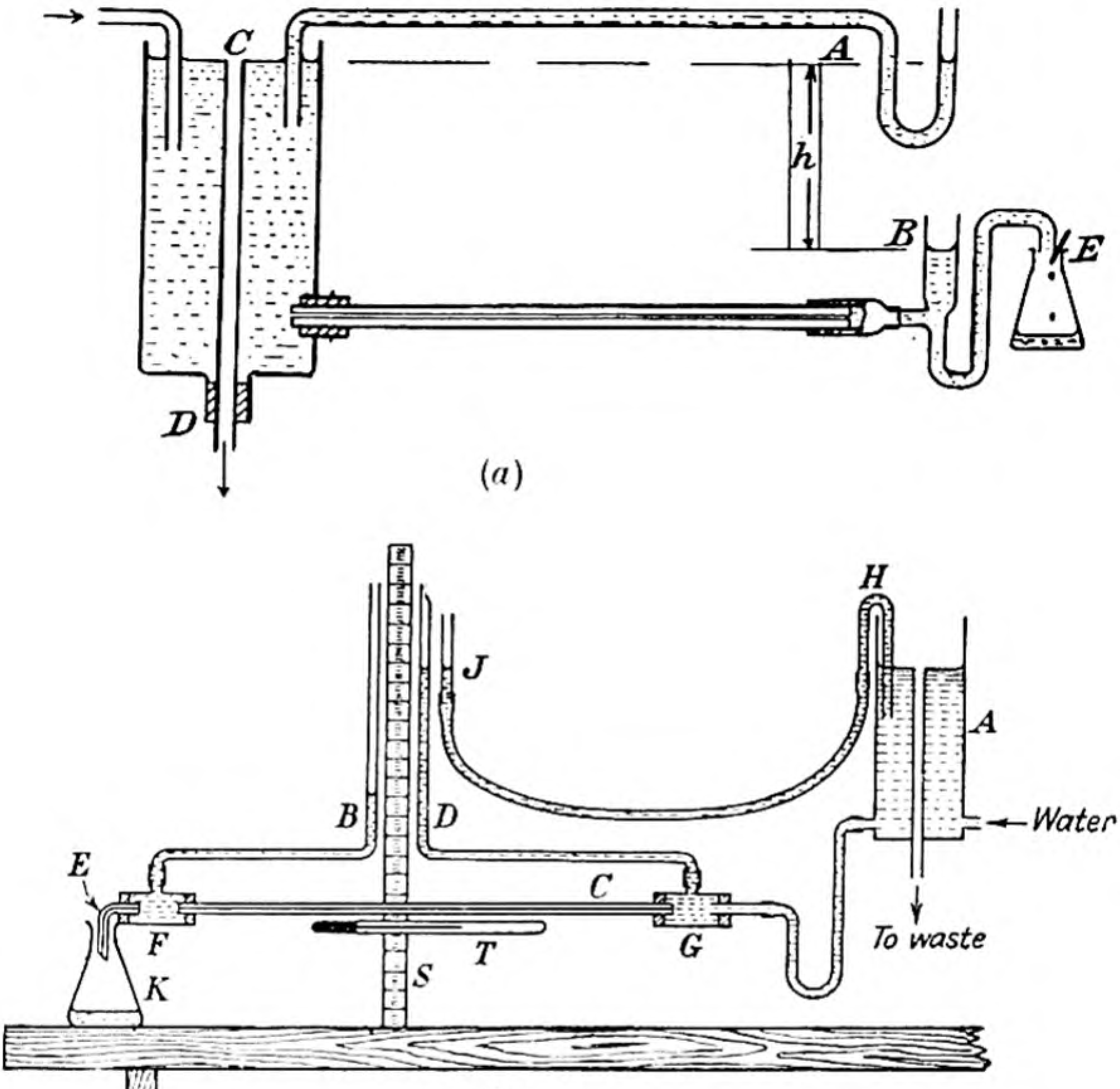


FIG. 11·05.—Experimental determination of the viscosity of tap-water.

used. It consists of a tall metal cylinder furnished with an overflow pipe CD. A capillary tube which has been freed from traces of grease in the usual way, of known length l and mean radius of cross-section a, is placed in a horizontal position and connected to the cylinder. Water enters this cylinder through the inlet tube shown and any excess is carried away along CD. Attached to the exit end of the capillary tube is a glass tube bent in the manner indicated. The pressure difference between the ends of the capillary tube is proportional to the vertical distance between the water-levels at B and C. This distance may be determined with the aid of a scale in mm. and a U-tube, filled with water and placed as shown, so that the levels at A and C are the same. When water flows

along the capillary tube, as each drop breaks away from E, the water-level at E changes, an effect due to the changes in pressure at E as the drops of water alter in shape. This disturbing factor may be avoided if a clean glass rod is placed in contact with the end E of the outlet-tube. The water then leaves the tube in a trickle and the level at B is constant.

An alternative form of apparatus is shown in Fig. 11·05(*b*). C is the capillary tube, about 0·1 cm. in diameter and one metre long. It is cleaned in the usual way and then fitted with a rubber bung at each end. F and G are two wide brass tubes one of which permits the capillary to be connected to the water tank while the other is fitted with a very narrow capillary tube E as shown. The end of E is chamfered so that the liquid escapes from it in definite drops. F and G are carried by two clamps and these are adjusted so that C is horizontal. To measure the drop in pressure along the tube C, when water is escaping steadily from the system, glass tubes B and D are attached as indicated. Unless the tube E is used no water will collect in B. The water-tank A, connected to the water supply, is fitted with an overflow tube so that a constant head of water may be maintained in the system; by raising or lowering the tank the magnitude of this head may be adjusted.

In order to test for the presence of air bubbles in the connecting tubes, which, if allowed to remain, would vitiate the steadiness of the flow, the tube E is closed and the water allowed to rise in B. A siphon tube, HJ, filled with water, has one end in the tank A, so that the level of the water in the tank is reproduced in the tube J and may be compared with the levels in B and D. Any difference in level shows that air bubbles are present and these must be removed by tilting the tubes before proceeding with the experiment.

To ensure that steady conditions exist in the viscometer the pressure difference between the ends of the capillary tube should be first adjusted to be at least 20 cm. of water and a determination of the mass of water escaping per unit time made by collecting and then weighing the water which exudes in ten minutes; this should be repeated and if the observations are consistent the viscometer will probably be free from air bubbles. To make quite sure of this the pressure head should be reduced to about 5 cm. of water and the mass of liquid escaping per unit time found. The two values of the rate of flow are then plotted against the pressure head and, provided that the flow of water through the capillary tube is so slow that the energy imparted to the water is negligible, if they lie on a straight line passing through the origin the system must be free from air bubbles. Then ascertain several more rates of flow for different pressure heads; represent these readings on the graph and from it deduce a mean value for the rate of flow

per unit head of pressure; hence obtain a value for the viscosity of tap water at the temperature of the viscometer.

If, in either method, the capillary tube has not been calibrated as on p. 539, it is necessary to determine the mean radius of the tube with as great an accuracy as possible as the fourth power of the radius is involved in the calculation. The tube should be dried and cleaned and sufficient clean mercury introduced almost to fill it. The length of the mercury thread is measured and then it is expelled into a weighing bottle of known mass. This is repeated several times and from the total length and total mass of all the mercury threads the mean radius of the tube is deduced.

Osborne Reynolds' method for investigating how the viscosity of a liquid varies with temperature or that of a solution with concentration.—The original apparatus was

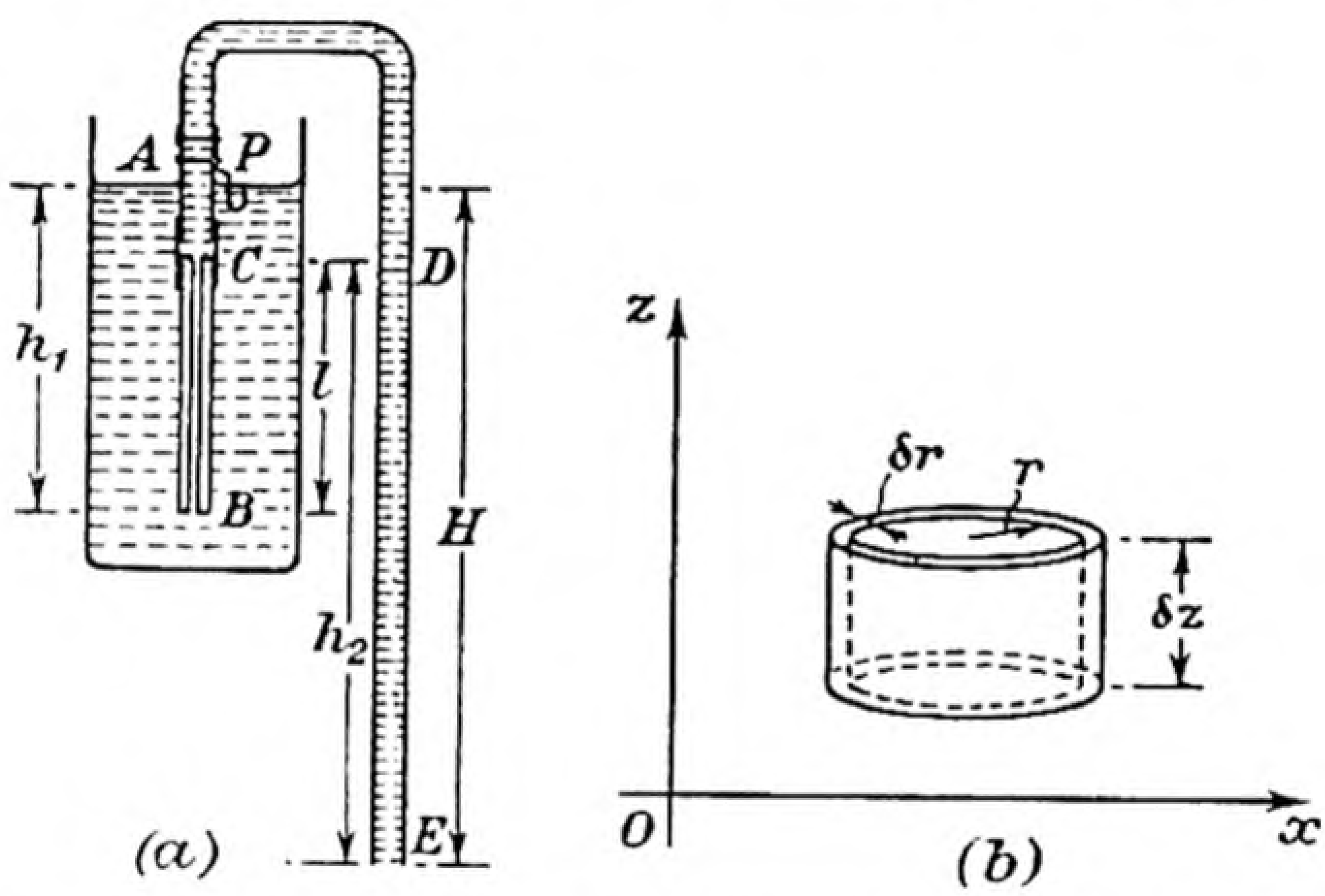


FIG. 11·06.—Theory of Osborne Reynolds' apparatus for the measurement of viscosity.

designed in 1886 and concerning it Reynolds writes:—'Since the value of η for water is known for all moderate temperatures, in order to obtain the value for oil it is only necessary to ascertain the relative times taken by the same volumes of oil and water to flow through the same channel, care being taken to make the channel such that there are no eddies and that the energy of the motion is small compared with the loss of head.' For this purpose, or to make an absolute determination of the viscosity of a liquid, the apparatus usually consists of a glass siphon connected by means of a short length of rubber tubing to a length of capillary tube, BC, Fig. 11·06(a). The capillary tube is completely immersed in the liquid under investigation and contained in a suitable vessel; the tube BC is vertical and the glass-rubber joints must be 'water-tight'. [If only comparative measurements are to be made the

glass tubes may be fused together; if this is done it is more difficult to determine the effective length of the capillary tube as would be required in an absolute determination of viscosity, but eddies are less likely to be formed.] P is a glass pin, bent as shown, and the level, A, of the liquid in the beaker is adjusted so that the point of the pin coincides with it. The advantage of using the pin in this way is that the point of the pin can readily be adjusted to coincide with its image formed by reflexion in the liquid surface. The siphon is filled with the liquid under investigation and the mass of liquid exuding in a given time ascertained. From this data and the dimensions of the apparatus, the value of η, the coefficient of viscosity of the liquid at the temperature of the experiment, may be calculated by means of the formula which we proceed to establish.

Let P_0 be the atmospheric pressure. Then the pressure at B is $P_0 + g\rho h_1$, where $h_1 = AB$, g is the intensity of gravity and ρ the density of the liquid at the temperature of the experiment. The pressure at $E = P_0$; that at $D =$ that at $C = P_0 - g\rho h_2$, where $h_2 = DE$.

$$\therefore\ P_B - P_C = g\rho(h_1 + h_2).$$

Let z be the distance of a point in the capillary tube from B, the positive direction of z being upwards. The pressure gradient along the tube is constant if the usual conditions of viscous flow are fulfilled. Hence

$$\frac{\partial p}{\partial z} = \alpha, \qquad \text{or} \qquad p = \alpha z + \beta,$$

where α and β are constants to be determined from the end conditions. Let l be the length of the capillary tube. Then at $z = 0$, $p = P_0 + g\rho h_1$, so that

$$\beta = P_0 + g\rho h_1,$$

and at $z = l$, $p = P_0 - g\rho h_2$, so that

$$\alpha = -\frac{g\rho(h_1 + h_2)}{l}.$$

$$\therefore\ \frac{\partial p}{\partial z} = -g\rho\frac{(h_1 + h_2)}{l}.$$

This equation fixes the sign and magnitude of the pressure gradient. To determine the rate of flow of liquid through the tube, consider a small cylindrical element with internal and external radii r and $(r + \delta r)$ respectively, and length δz—cf. Fig. 11·06(b). Let p be the pressure on the lower end of the element. Then the force due to the pressure of the liquid on the lower plane face of the element is

$$2\pi r\ \delta r \,.\, p,$$

and acts upwards. On the upper plane face the force is

$$2\pi r\,\delta r\left[p + \frac{\partial p}{\partial z}\,\delta z\right],$$

and this acts downwards. The resultant force in the direction of z increasing and due to pressure is therefore

$$-2\pi r\,\frac{\partial p}{\partial z}\,\delta r\,\delta z.$$

The weight of the cylindrical element is $2\pi r\rho g\,\delta r\,\delta z$, and this acts vertically downwards.

If u and $(u + \delta u)$ are the velocities of flow at points at distance r and $(r + \delta r)$ from the axis of the element, then from Newton's definition of viscosity we have

$$2\pi r\,\delta z\,\eta\,\frac{\partial u}{\partial r} = 2\pi\eta\,\delta z\; r\,\frac{\partial u}{\partial r},$$

as the force due to viscosity acting on the inner curved surface and this force acts downwards. [The important point to emphasize here is that since $\frac{\partial u}{\partial r}$ is negative the force on the inner curved surface will actually be directed upwards but no minus sign must be introduced into the above expression for the force if the recognized practice and methods of the calculus are not to be violated.] On the outer curved surface of the element the force due to viscosity is

$$2\pi\eta\,\delta z\left[r\,\frac{\partial u}{\partial r} + \frac{\partial}{\partial r}\left(r\,\frac{\partial u}{\partial r}\right)\delta r\right],$$

and this acts upwards. Since the liquid is moving without acceleration

$$2\pi r\,.\,\delta r\,.\,\delta z\,.\,\frac{\partial p}{\partial z} + 2\pi r\,.\,\delta r\,.\,\delta z\,.\,\rho g - 2\pi\eta\,.\,\delta z\,.\,\delta r\,\frac{\partial}{\partial r}\left(r\,\frac{\partial u}{\partial r}\right) = 0,$$

i.e.

$$\eta\,\frac{\partial}{\partial r}\left(r\,\frac{\partial u}{\partial r}\right) = r\rho g + r\,\frac{\partial p}{\partial z} = r\left(g\rho + \frac{\partial p}{\partial z}\right).$$

Since $\frac{\partial p}{\partial z}$ is independent of r, integrating we have

$$\eta\left[r\,\frac{\partial u}{\partial r}\right] = \frac{r^2}{2}\left[g\rho + \frac{\partial p}{\partial z}\right] + \mathrm{A},$$

where A is an integration constant.

$$\therefore\;\eta\,\frac{\partial u}{\partial r} = \frac{r}{2}\left[g\rho + \frac{\partial p}{\partial z}\right] + \frac{\mathrm{A}}{r}.$$

Integrating again we have

$$\eta u = \tfrac{1}{4}r^2\left(g\rho + \frac{\partial p}{\partial z}\right) + \mathrm{A}\ln r + \mathrm{B},$$

where B is another integration constant. Now $\mathrm{A} = 0$, since u is finite when $r = 0$, and at $r = a$, $u = 0$.

$$\therefore\ \mathrm{B} = -\frac{a^2}{4}\left(g\rho + \frac{\partial p}{\partial z}\right),$$

and

$$\therefore\ u = -\frac{1}{4\eta}\left(g\rho + \frac{\partial p}{\partial z}\right)(a^2 - r^2).$$

Let Q be the volume of liquid flowing per second. Then

$$\mathrm{Q} = -\int_0^a 2\pi r.\frac{1}{4\eta}\left(g\rho + \frac{\partial p}{\partial z}\right)(a^2 - r^2)\,dr = \frac{\pi}{2\eta}\left(g\rho + \frac{\partial p}{\partial z}\right)\left(-\frac{a^4}{4}\right)$$

$$= \frac{\pi a^4}{8\eta l}\,g\rho\mathrm{H},$$

where H is the depth of E below A.

$$\therefore\ \mathrm{Q} = \frac{\pi a^4 \mathrm{P}}{8\eta l}, \qquad \text{where } \mathrm{P} = g\rho\mathrm{H}.$$

As the liquid runs out from the siphon the capillary tube must be lowered so that the point of the pin is always in contact with the liquid surface. At best, this method only maintains P approximately constant.

If the apparatus is used to investigate how the viscosity of a liquid varies with temperature, then the temperature of the liquid in the siphon is not equal to that of the liquid as it flows through the capillary tube and it is difficult to estimate the correction with any degree of accuracy. When investigating how the viscosity at room-temperature of a solution varies with the concentration of the solute this difficulty does not arise.

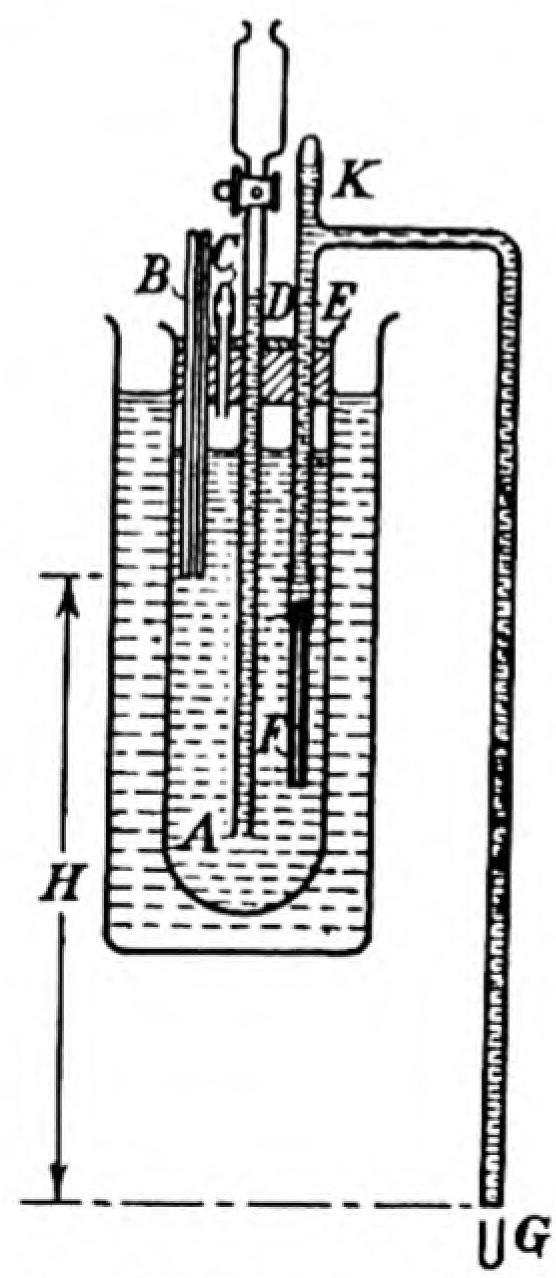


Fig. 11·07.—A modern form of Osborne Reynolds' viscometer.

A form of Osborne Reynolds' apparatus, as modified by the author, is shown in Fig. 11·07. It is true that the first difficulty just mentioned is not overcome but the pressure head can be maintained constant when experiments at room-temperature are performed.

A large tube A is fitted with a rubber bung through which pass four tubes. The tube B is a capillary tube about 1 mm. in diameter, its lower end reaching half-way down the tube A. C is a short tube which may be opened or closed with respect to the atmosphere by using the solid glass rod which is attached by means of rubber tubing to C. D is a wider tube attached to a small reservoir so that liquid may be introduced into the viscometer; by passing air through this tube the liquid in A can be well stirred before beginning the experiment. This is very necessary since the viscosities of most liquids change very rapidly with temperature and hence the temperature throughout A must be kept constant and uniform. E is a siphon to the end of which is attached, either by means of rubber tubing or by fusing the glass tubes together, the capillary tube F, through which the liquid flows. The capillary tube is about 10 cm. long and has an internal diameter of about 0·02 cm. The tube K attached to the siphon, enables the latter to be filled before commencing work; it is then closed and serves to collect any small bubbles that may appear in the viscometer during the progress of an experiment. If these bubbles are allowed to collect in the siphon the steadiness of the flow of liquid may be seriously affected.

When the temperature in A has become steady (the thermometer is not shown) the plug is inserted at C and the stop-cock in D closed. When the plug G at the lower end of the siphon is withdrawn, the liquid begins to exude from the apparatus; it will be replaced by air which enters through B and the head of liquid remains constant provided the level of the liquid in A never falls below the lower end of B. The coefficient of viscosity is then obtained from the formula already established.

It would of course be possible to arrange for the siphon to be almost entirely within A by providing this tube with a cork at its lower end, but the difficulties of maintaining a long column of liquid at a constant temperature are so well known that it is difficult to believe that this modification would be beneficial.

On the flow of an incompressible liquid downwards through a vertical capillary tube.—Let AB, Fig. 11·08(a), be the tube through which the liquid is flowing steadily. Consider the forces on the small cylindrical element whose ends are at distances z and $(z + \delta z)$ from A, while the radii of its curved surfaces are r and $(r + \delta r)$. Let u be the velocity in the liquid at a point on the inner curved surface of the above element. At a point on the outer curved surface the velocity will be $\left(u + \dfrac{\partial u}{\partial r}\,\delta r\right)$. The forces acting on the element are shown in Fig. 11·08(b), where it must be remembered that the forces due to the viscosity of the liquid are actually distributed over the curved surfaces. Likewise the forces

arising from the pressure in the liquid are distributed over the flat ends of the element. In the steady state the resultant force downwards is zero since the liquid is moving without acceleration.

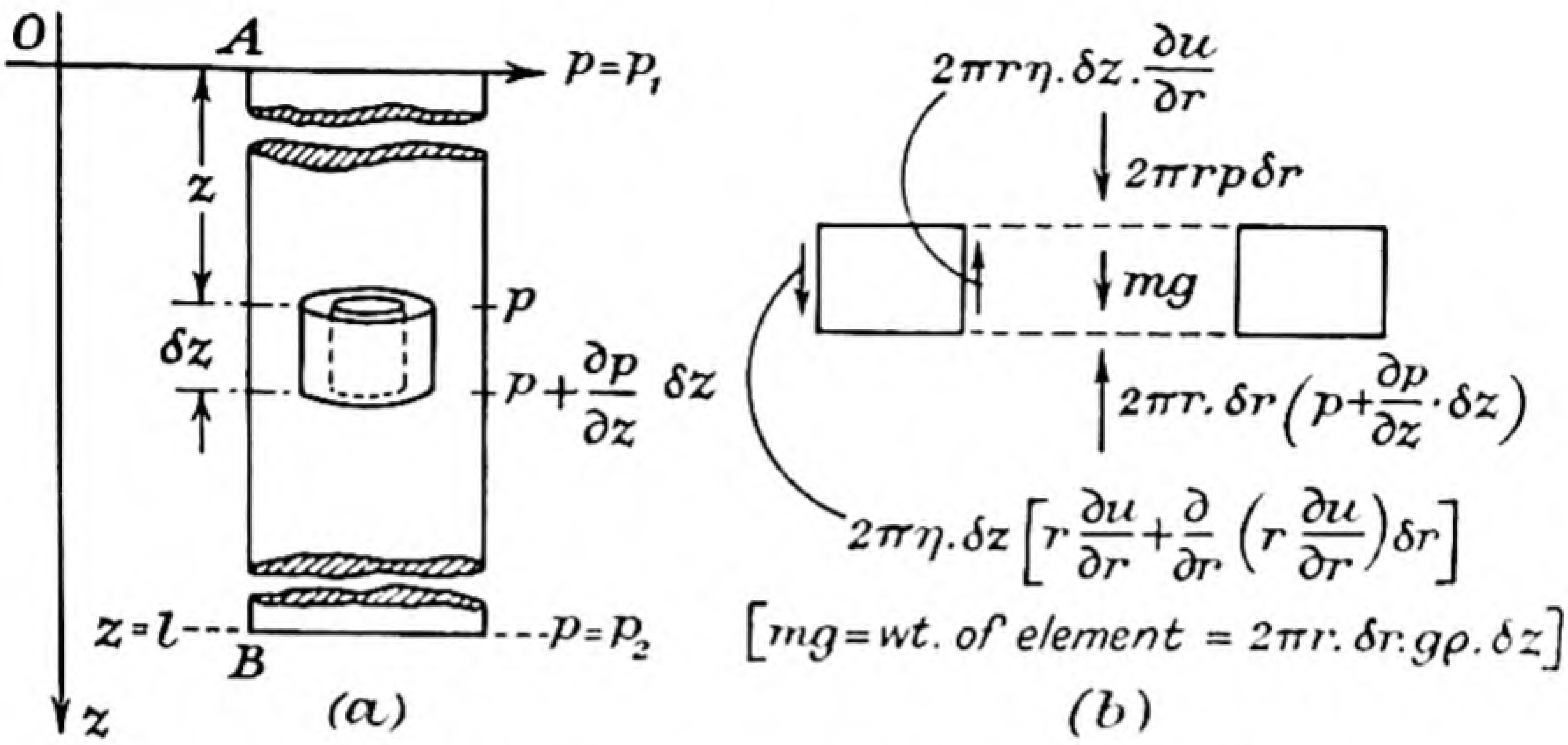


FIG. 11·08.—Theory of the steady flow of a liquid through a vertical capillary tube.

Hence

$$-2\pi r\eta\,.\,\delta z\,.\,\frac{\partial u}{\delta r} + 2\pi\eta\,.\,\delta z\left[r\frac{\partial u}{\partial r} + \frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)\delta r\right]$$

$$+ 2\pi r\,.\,\delta r\,.\,g\rho\,.\,\delta z + 2\pi r\,.\,p\,\delta r - 2\pi r\,.\,\delta r\left(p + \frac{\partial p}{\partial z}\delta z\right) = 0,$$

where g and ρ have their usual meanings.

$$\therefore\ g\rho r + \eta\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) - r\frac{\partial p}{\partial z} = 0.$$

Now $$\frac{\partial p}{\partial z} = -\left(\frac{p_1 - p_2}{l}\right),$$

so that $$\left(g\rho + \frac{p_1 - p_2}{l}\right)r = -\eta\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right).$$

$$\therefore\ \left(g\rho + \frac{p_1 - p_2}{l}\right)\frac{r}{2} + \frac{A}{r} = -\eta\frac{\partial u}{\partial r}, \qquad [A = \text{integ. const.}]$$

$$\therefore\ \left(g\rho + \frac{p_1 - p_2}{l}\right)\frac{r^2}{4} + A\ln r + B = -\eta u. \qquad [B = \text{integ. const.}]$$

But A = 0, since $u \neq \infty$ when $r \to 0$, and at $r = a$, $u = 0$.

$$\therefore \ \mathrm{B} = -\left[g\rho + \left(\frac{p_1 - p_2}{l}\right)\right]\frac{a^2}{4}.$$

$$\therefore \ u = \left[g\rho + \left(\frac{p_1 - p_2}{l}\right)\right]\left(\frac{a^2 - r^2}{4\eta}\right).$$

If Q is the volume of liquid flowing per second through the tube, we have

$$\mathrm{Q} = \int_0^a u.2\pi r.dr = \frac{\pi a^4}{8\eta}\left[\frac{p_1 - p_2}{l} + g\rho\right]$$

after the integration has been performed.

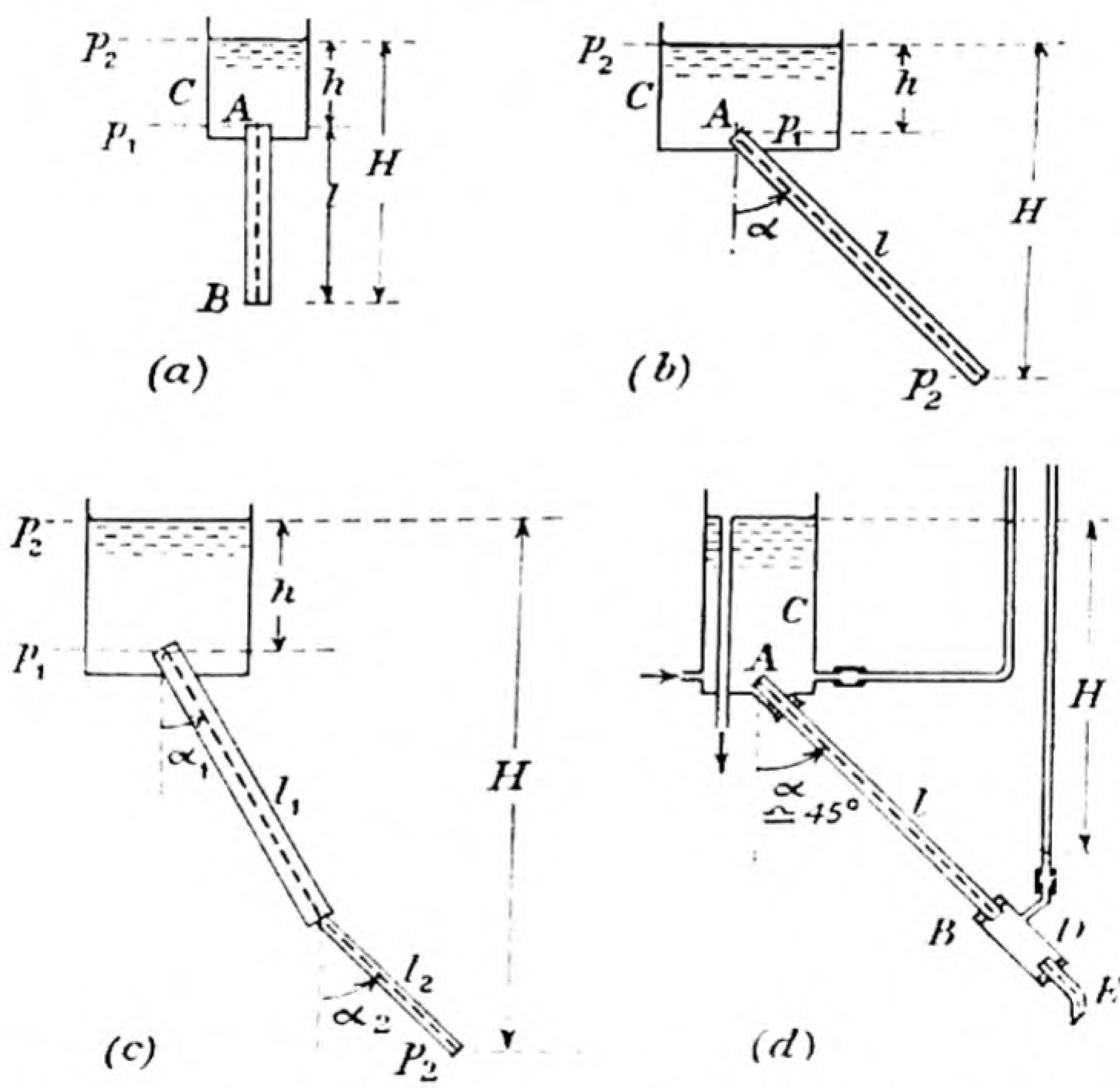


FIG. 11·09—Flow of an incompressible liquid through tubes which are inclined to the vertical.

To appreciate the full significance of this equation let us consider the 'ideal' vertical viscometer shown in Fig. 11·09(a). If H is the height of the liquid surface in the container C above the exit B and h its height above the entrance, A, to the capillary tube, we have

$$\mathrm{Q} = \frac{\pi a^4}{8\eta}.\frac{g\rho h + g\rho l}{l} = \frac{\pi a^4}{8\eta l}.g\rho\mathrm{H}.$$

When the capillary tube is inclined at an angle α to the vertical, as in Fig. 11·09(b), the component of the intensity of gravity along the axis of the tube is $g \cos \alpha$, so that the formula on p. 550 becomes

$$Q = \frac{\pi a^4}{8\eta}\left[\frac{p_1 - p_2}{l} + g \cos \alpha . \rho\right]$$

$$= \frac{\pi a^4}{8\eta}\left[\frac{g\rho h + g\rho l \cos \alpha}{l}\right]$$

$$= \frac{\pi a^4}{8\eta l} . H,$$

so that it is the height of liquid H which is effective in determining the rate of flow of a given liquid through a given capillary tube.

If the tube through which the liquid flows consists of two capillaries in series, with constants as indicated in Fig. 11·09(c), we have, if p is the pressure at the junction of the capillaries,

$$Q . \frac{8\eta l_1}{\pi a_1{}^4} = p_1 - p + g\rho l_1 \cos \alpha_1,$$

and

$$Q . \frac{8\eta l_2}{\pi a_2{}^4} = p - p_2 + g\rho l_2 \cos \alpha_2,$$

since Q is the volume of liquid flowing per second through each tube.

$$\therefore\ Q . \frac{8\eta}{\pi}\left[\frac{l_1}{a_1{}^4} + \frac{l_2}{a_2{}^4}\right] = g\rho[h + l_1 \cos \alpha_1 + l_2 \cos \alpha_2]$$

$$= g\rho H.$$

The viscosity of tap-water: flow through an inclined capillary tube.—An apparatus designed by the author for determining the viscosity of tap-water at room temperature is shown in Fig. 11·09(d). The capillary tube AB is fitted to the base of a metal container C which serves as a constant-head device of the usual type. In order to keep the pressure p_2 at the lower end of AB constant, a small brass tube D with a fine capillary tube E is attached to B. Manometer tubes arranged as shown enable a value for H to be determined directly without making any assumption regarding the values of p_1 and p_2; e.g. p_2 is not equal to atmospheric pressure; when D is not used p_2 varies quite considerably as the drops break away. By using a fine tube for E the changes in pressure at the exit end of D are not 'reflected' to affect the water level in the r.h.s. tube of the manometer. The viscosity is calculated by using the formula established in the previous paragraph.

A vertical viscometer for liquids.—A viscometer with its capillary tube in a vertical position is shown in Fig. 11·10. It is designed for students' use primarily and the minor changes necessary to make it a precision instrument will be apparent. A wide glass tube G contains the liquid under investigation and the capillary tube AB is carried in a vertical position by a rubber bung fitted to the lower part of G. A second rubber bung fitted into the upper end of the tube G carries the glass tubes C, D and E, together with the thermometer T_1. Liquid is introduced, when necessary, through E, the rubber cap on D being removed so that the displaced air may escape from the apparatus. The tube E also permits air to be blown through the liquid in order that it shall be thoroughly stirred before observations are made. With D closed, as liquid flows through the capillary tube AB air enters the apparatus through C so that the pressure at the lower end of C is almost constant and the effect of this slight variation may be eliminated by the method adopted below.

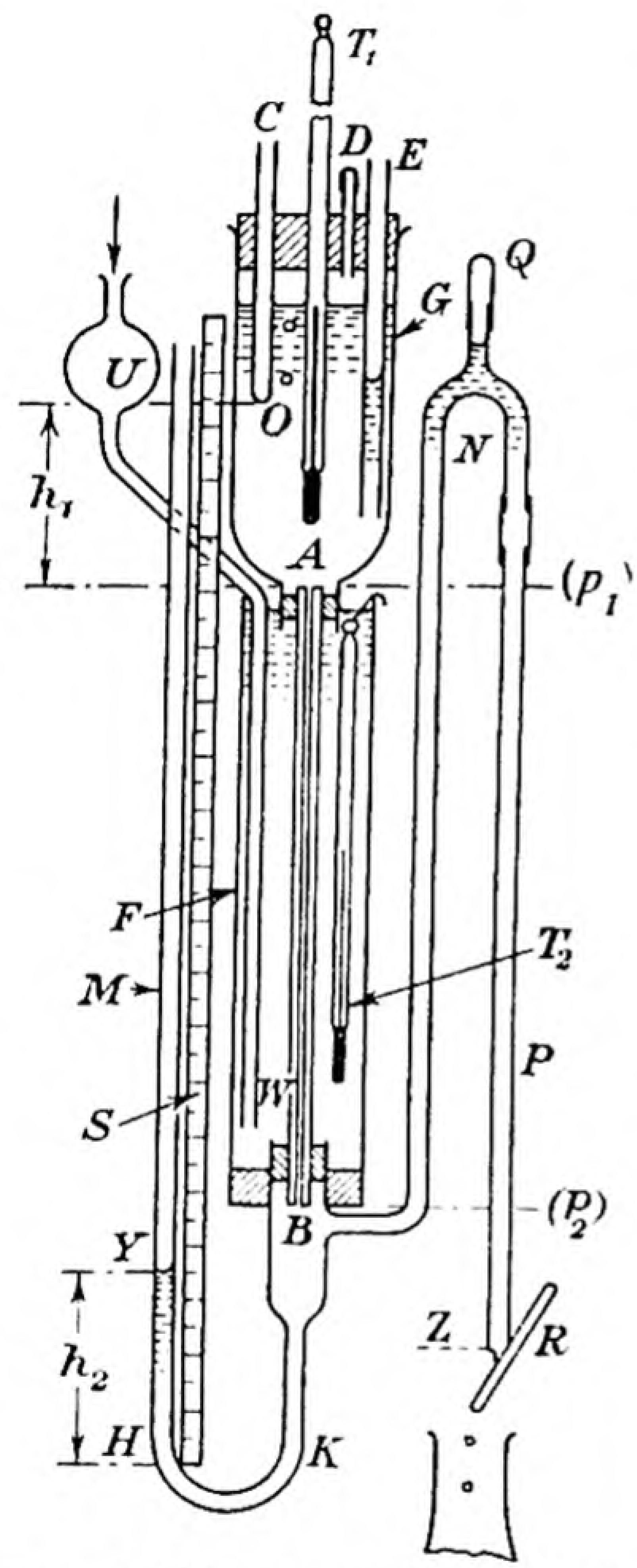


FIG. 11·10.—A vertical viscometer for liquids.

Now if drops of liquid are allowed to fall from B directly into the air the pressure at B within the liquid is not constant. To overcome this difficulty the lower end of the capillary tube AB is fitted into a glass tube of the shape indicated; this is connected to a tube M which serves as a manometer while the inverted U-tube BNZ to which it is connected provides a means of escape for the liquid. The tube Q attached to the top of the U-tube permits air bubbles to escape during the process of filling the viscometer; when this is in use Q is closed with a rubber cap.

In order to ensure steady conditions at the end of the tube Z, a short glass rod R is held rigidly against Z and the drops falling from R are collected in a weighed flask.

Let P be the atmospheric pressure. Then in the liquid at the

lower end O of the tube C the pressure is $(\mathrm{P} - \alpha)$, where α represents the reduction in pressure due to surface tension and a correction due to the fact that the surface of the bubble is not fixed in position. Thus α is variable, strictly, but probably assumes a constant mean value over the time required for a conveniently measurable quantity of liquid to escape. Hence the pressure p_1, at the upper end A of the capillary tube, is given by

$$p_1 = \mathrm{P} - \alpha + g\rho h_1,$$

where h_1 is the depth of A below O, and g and ρ have their usual meanings. Now at B the pressure is p_2, where

$$p_2 = \mathrm{P} + g\rho h_2 + \beta,$$

where β is a constant correction term due to the fact that the zero from which h_2 is measured is not at B—it is convenient to have a scale in cm., etc., S, resting on the lower portion of the tube HK—and also necessary in part because the pressure in the liquid at Y is not quite atmospheric.

The formula established on p. 550, therefore, may be written

$$\begin{aligned} \mathrm{Q} &= [(\pi a^4)/(8\eta l)][\mathrm{P} - \alpha + g\rho h_1 - \mathrm{P} - g\rho h_2 - \beta + g\rho l] \\ &= [(\pi a^4)/(8\eta l)][\mathrm{C} - g\rho h_2], \end{aligned}$$

where C is a constant for a given apparatus. If we make a series of observations using different lengths of the tube NZ, we may construct the straight-line graph $x = h_2$, $y = \mathrm{Q}$. The slope of this line is $-[(\pi a^4)/(8\eta l)]g\rho$, so that η may be found. If the tube AB is not quite uniform, it must be calibrated as on p. 539; then the slope of the graph will be

$$-\left[(\pi g\rho)\Big/\left\{8\eta\, \Sigma\left(\frac{\Delta l}{a^4}\right)\right\}\right].$$

Since the apparatus cannot conveniently be set up in a thermostat it is essential to work in a room where the temperature is fairly constant. It is also necessary to surround the capillary tube by a wide glass tube F containing water. This is stirred by forcing air through the glass tube UW, the bulb at U preventing water from entering the apparatus in which air under pressure is produced. A mercury thermometer T_2 gives the temperature of the capillary tube.

In collecting the drops it is advisable to place the flask in position immediately after a drop has fallen from the end of R and to remove it after a time interval when another drop has just fallen. The time between the fallings of these two drops must be measured with an accurate stop-watch.

The following result was obtained for distilled (air-free) water at 23·0° C. It will be seen that h_2 was varied from 18·55 cm. to

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52·70 cm., i.e. the position of Y varied from one near H to one near A. The temperature changed from 22·9° C. to 23·1° C. during the experiment.

Mass of water (gm.)	Time of flow (sec.)	Rate of flow (gm.sec.$^{-1}$)	h_2 (cm.)
13·723	604·1	0·0227	18·55
13·651	604·6	0·0226	18·55
3·600	610·5	0·0059	52·70
3·475	603·6	0·0058	52·70
12·240	605·0	0·0202	23·80
10·682	607·0	0·0176	29·20
8·568	602·1	0·0142	34·95
6·730	603·0	0·0112	41·00

These give $\eta = (9{\cdot}43 \pm 0{\cdot}07) \times 10^{-3}$ gm.cm.$^{-1}$ sec.$^{-1}$ at 23·0°C.

Viscometers for relative measurements.—In the apparatus used for the determination of the viscosity of tap-water at room temperature the head of liquid remains constant during the determination of any one particular rate of flow, and so the pressure excess at one end of the capillary tube over that at the other end remains invariable. When other liquids are under investigation the supply of liquid is restricted and the head of liquid may not remain constant, as in Osborne Reynolds' viscometer as generally used. The elimination of this variable pressure head is attained almost perfectly when relative viscosities are determined. One of the earliest types of viscometer for the comparison of viscosities is due to OSTWALD, and this viscometer is in general use today. Many slight changes in the design of the instrument have been introduced by subsequent workers. Some workers in this branch of physics maintain that the errors in relative determinations of viscosity are much less than in an absolute determination of this coefficient. This is undoubtedly true, and the

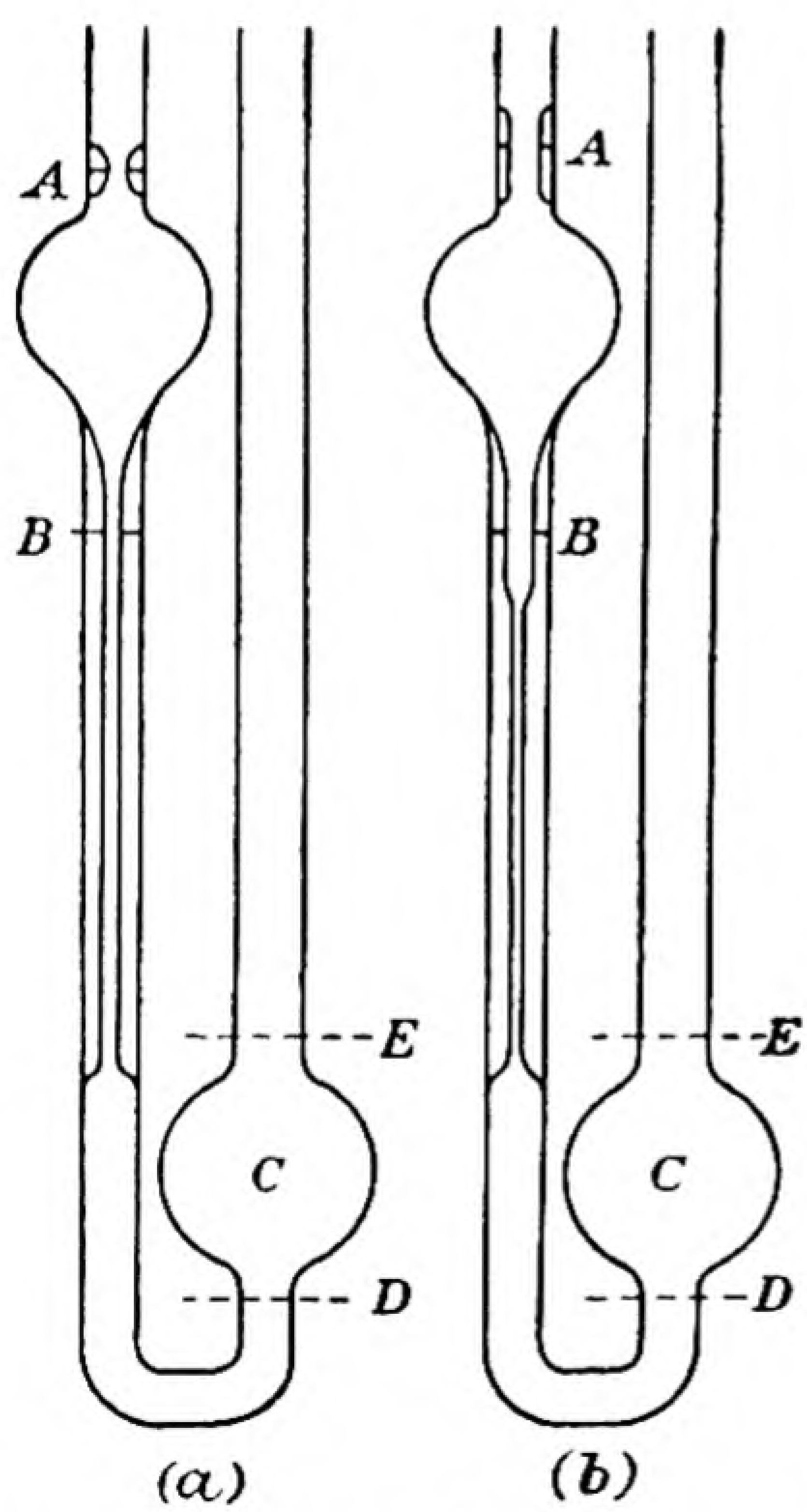


FIG. 11·11.—Ostwald viscometers.

greater precision obtained when relative measurements are made is due to the avoidance of determinations of errors where linear measurements are made and also to the fact that the apparatus is less complicated so that the control of temperature becomes more easy.

Elementary forms of this viscometer are shown in Fig. 11·11(a) and (b). A constant volume of liquid is introduced by means of a pipette into the right-hand limb of the apparatus and then caused to fill that part of the viscometer from just above A to just above the mark D. The liquid is then permitted to flow back and the time taken for it to fall from A to B is determined. To discover how, if this time of flow is observed for two liquids in the viscometer, it becomes possible to obtain a value for their relative viscosity the following analysis is necessary.

Theory of an Ostwald viscometer.—Fig. 11·12(a) is a diagram of an Ostwald viscometer in which, for the sake of generality, it is

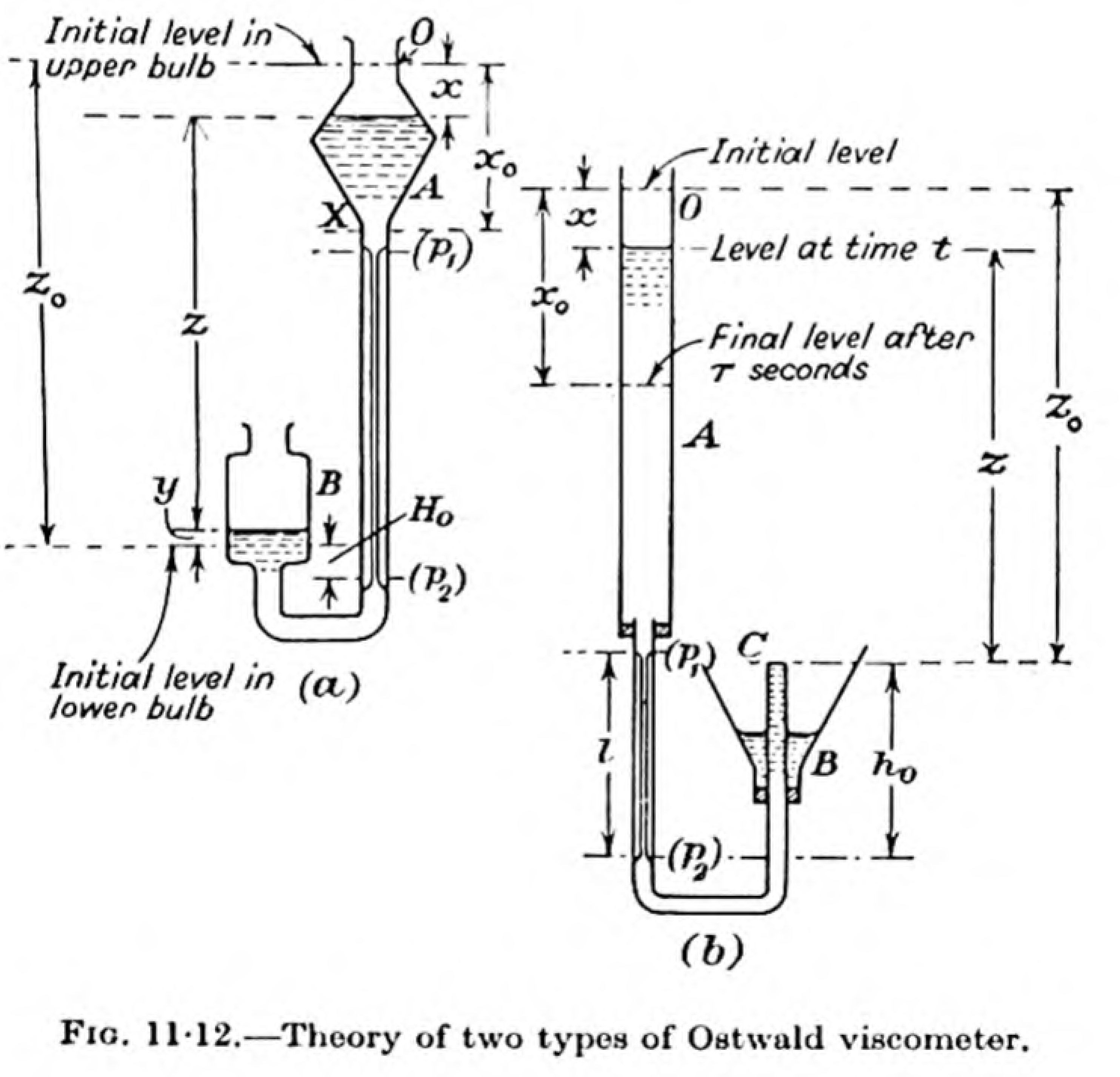


FIG. 11·12.—Theory of two types of Ostwald viscometer.

assumed that the upper and lower bulbs are not identical; the cross-section A of the upper bulb A is a function of x only, i.e. $A = \phi(x)$, while for B we may similarly write $B = \psi(y)$, where x is the vertical distance through which the meniscus in A falls in time t, and y is the distance through which the meniscus in B moves upward in the

same time. The volume of liquid which passes per unit time through a vertical capillary tube is given by

$$dV = \frac{\pi a^4 g \rho z}{8\eta l}\,dt, \qquad \text{[cf. p. 550].}$$

where z is the vertical distance between the liquid levels in A and B at the instant considered and the other symbols have their usual significance.

But if dx is the fall in A in time dt, and dy the corresponding rise in B, $A\,dx = dV = B\,dy$. If z_0 is the vertical distance between the liquid levels at time $t = 0$,

$$\begin{aligned} A\,dx &= \left[\frac{(\pi a^4 g\rho)}{(8\eta l)}\right][z_0 - x - y]\,dt \\ &= k\left(\frac{\rho}{\eta}\right)f(x)\,dt, \end{aligned}$$

where $f(x)$ may be written for $(z_0 - x - y)$, which is a function of x only since

$$dV = \phi(x)\,dx = \psi(y)\,dy.$$

$$\therefore\ V = \int_{t=0}^{t} dV = \int_0^x \phi(x)\,dx = \int_0^y \psi(y)\,dy.$$

Thus y is a function of x only.

Hence
$$\frac{A\,dx}{f(x)} = k\left(\frac{\rho}{\eta}\right)dt.$$

$\therefore$ If τ is the time required for the level of the liquid in A to fall from its initial position to a level $x = x_0$, say [shown by a mark X on the stem of the instrument],

$$\int_0^{x_0} \frac{A\,dx}{f(x)} = k\left(\frac{\rho}{\eta}\right)\tau.$$

Now for a given viscometer at constant temperature $\int_0^{x_0} \frac{A\,dx}{f(x)}$ is constant and therefore under such conditions $\rho\tau\eta^{-1}$ is invariable, so that if two liquids are used in turn in the viscometer

$$(\eta_1/\eta_2) = (\rho_1\tau_1)/(\rho_2\tau_2)$$

Another viscometer of this type was used by Ostwald and Auerbach† in their work on colloidal solutions. The required formula may be derived as follows. Fig. 11·12(*b*) shows in section an Ostwald and Auerbach viscometer. It will be assumed that the

† Ostwald and Auerbach, *Koll. Zeitsch.*, **41**, 56, 1927.

wide tube A is uniform in cross-section; let x be the distance through which the liquid meniscus in it falls in time t. Then if all corrections for surface tension effects are neglected,

$$dV = \frac{\pi a^4 g\rho z\, dt}{8\eta l},$$

and since $dV = A_0\, dx$, if A_0 is the constant cross-sectional area of the tube A, we have

$$A_0\, dx = \frac{\pi a^4 g\rho(z_0 - x)}{8\eta l}\, dt,$$

where z_0 is the vertical distance OC. Hence if τ is the time required for the liquid level in A to fall a distance x_0,

$$A_0 \int_0^{x_0} \frac{dx}{z_0 - x} = \frac{\pi a^4}{8\eta l} g\rho \int_0^{\tau} dt,$$

$$\therefore\ A_0 \ln \Big[(z_0 - x)\Big]_{x_0}^{0} = \frac{\pi a^4}{8\eta l} g\rho\tau,$$

i.e.

$$A_0 \ln \left(\frac{z_0}{z_0 - x_0}\right) = \frac{\pi a^4}{8\eta l} g\rho\tau,$$

so that a value for η may be obtained.

Example.—(i) An elliptically shaped vessel is formed by the revolution of an ellipse about its minor semi-axis, which is vertical. There is a small aperture at the topmost point of the vessel and a horizontal capillary tube, length l and radius r, is attached to the lowest point of the vessel. If this is filled initially with a liquid, density ρ and viscosity η, calculate the time which elapses before the vessel is empty. [Assume that a and b are the major and minor semi-axes of the ellipse.]

Let O be the centre of the ellipse, so that if ϕ is the eccentric angle, then $(a \cos \phi, b \sin \phi)$ is a point P on its circumference. If $\phi + \delta\phi$ defines a point Q close to P, the volume of liquid contained between the two horizontal planes through P and Q will be given by

$$\delta V = \pi x^2 \delta z,$$

where $x = a \cos \phi$ and z and $z + \delta z$ are the heights of P and Q above the horizontal plane through the lowest point on the ellipse. Thus

$$z = b + b \sin \phi;\quad \delta z = b \cos \phi\, .\, \delta\phi.$$

$$\therefore\ \delta V = (\pi a^2 \cos^2 \phi) b \cos \phi\, .\, \delta\phi.$$

But $-\delta V$ is the volume of liquid flowing through the capillary tube in time δt, say.

$$\therefore\ -\delta V = \frac{\pi r^4 (g\rho z)\, \delta t}{8\eta l}. \qquad [\because\ p = g\rho z]$$

$$\therefore\ -\pi a^2 b \cos^3 \phi\, .\, \delta\phi = kb(1 + \sin \phi)\delta t,$$

where $k = \pi r^4 g\rho \div 8\eta l$.

Since $\cos^3\phi = (1 - \sin^2\phi)\cos\phi$, we have

$$k\,.\,\delta t = -\frac{\pi a^2(1 - \sin^2\phi)\,d(\sin\phi)}{1 + \sin\phi}.$$

Hence t, the time required for the vessel to empty, is given by

$$kt = -\pi a^2 \int_{\phi=\frac{1}{2}\pi}^{\phi=-\frac{1}{2}\pi} (1 - \sin\phi)\,d(\sin\phi) = 2\pi a^2.$$

$$\therefore\ t = \frac{16a^2\eta l}{r^4 g\rho},$$

and this is independent of the length of the minor semi-axis.

Example.—(ii) From first principles obtain an expression for the rate at which an incompressible fluid flows with stream-line motion through a long horizontal capillary tube of diameter $2b$, when a straight wire of diameter $2a (a < b)$ is placed co-axially along the length of the tube.

If in the formula deduced, a is made zero, does the new formula give Poiseuille's equation for the flow of an incompressible fluid through a tube?

We have, cf. p. 539,

$$u = -\frac{p_1 - p_2}{4\eta l}r^2 + \mathrm{A}\ln r + \mathrm{B} = -\frac{\mathrm{P}}{4\eta l}r^2 + \mathrm{A}\ln r + \mathrm{B},\ \text{say}.$$

Now $u = 0$ when $r = a$ and when $r = b$, so that

$$\mathrm{A} = \frac{\mathrm{P}(b^2 - a^2)}{4\eta l}\left[\frac{1}{\ln\left(\frac{b}{a}\right)}\right]; \quad \mathrm{B} = \frac{\mathrm{P}a^2}{4\eta l} - \frac{\mathrm{P}(b^2 - a^2)}{4\eta l}\left[\frac{\ln a}{\ln\left(\frac{b}{a}\right)}\right].$$

$$\therefore u = \frac{\mathrm{P}}{4\eta l}\left[(a^2 - r^2) + \frac{b^2 - a^2}{\ln\left(\frac{b}{a}\right)}\ln\left(\frac{r}{a}\right)\right].$$

$\therefore$ Volume of liquid escaping per second

$$= \int_a^b u\,.\,2\pi r dr = \frac{\pi}{8\eta}\frac{(p_1 - p_2)}{l}\left[b^4 - a^4 - \frac{(b^2 - a^2)^2}{\ln\left(\frac{b}{a}\right)}\right].$$

For an answer to the last part of this question cf. ANDRADE, *Trans. Far. Soc.*, **27**, 201, 1931.

Corrections to Poiseuilles' formula.—Poiseuilles' equation $\mathrm{Q} = \dfrac{\pi a^4(p_1 - p_2)}{8\eta l}$ for the rate of steady flow of a liquid through a long capillary tube must be considered as an approximation since two important factors have been omitted. In the first place the pressure difference $p_1 - p_2 = p$, say, is utilized partly in giving kinetic energy to the liquid, and in the second place it is not correct to assume that there are no accelerations along the axis of the tube for near the inlet the accelerations do not become zero until a short

length of the capillary tube has been traversed. To correct for this it is usual to add a length α to that of the tube, where $\alpha = 1{\cdot}64a$.

To evaluate the so-called kinetic energy correction, WILBERFORCE† proceeded as follows. The work expended in driving into the capillary tube a volume Q δt from the reservoir and near to the entrance of the tube, is $pQ\,\delta t$, where p is the pressure excess at the entrance to the capillary. Now the kinetic energy imparted to the liquid per second on entering the capillary tube from a very large reservoir, and retained by the liquid until it leaves the tube, is the sum of the kinetic energies calculated for the elements of the liquid which pass any cross-section of the tube per second. Since through any small annulus of radius r and width δr, at a given cross-section there flows a volume $u.2\pi r.\delta r$ per second, the kinetic energy of this liquid is

$$\tfrac{1}{2}\rho u.2\pi r\,\delta r.u^2,$$

where, cf. p. 539,

$$u = \frac{\bar{p}}{4\eta l}(a^2 - r^2),$$

$\bar{p}$ being the effective pressure difference producing the forces which overcome the viscous resistance of the liquid.

The kinetic energy of the liquid flowing through the whole of the cross-section per second is therefore

$$\int_0^a \pi\rho u^3 r\,dr = \pi\rho\left(\frac{\bar{p}}{4\eta l}\right)^3\int_0^a (a^2 - r^2)^3 r\,dr$$
$$= \pi\rho\left(\frac{\bar{p}}{4\eta l}\right)^3.\frac{a^8}{8}.$$

But $Q = \dfrac{\pi a^4 \bar{p}}{8\eta l}$. Hence the kinetic energy of the liquid emerging per second from the tube is $\rho\dfrac{Q^3}{\pi^2 a^4}$.

Now the total work expended per second must be equal to the work done against the viscous forces, which is the effective pressure times the volume emerging per second, plus the kinetic energy retained by the liquid when it leaves the capillary tube. Thus

$$pQ = \bar{p}Q + \frac{\rho Q^3}{\pi^2 a^4},$$

or

$$\bar{p} = p - \frac{\rho Q^2}{\pi^2 a^4}.$$

Poiseuille's equation therefore becomes

$$Q = \frac{\pi a^4}{8\eta(l + \alpha)}.\left(p - \frac{\rho Q^2}{\pi^2 a^4}\right).$$

† *Phil Mag.*, **31**, 407, 1891.

Couette's method for determining the viscosity of a liquid; the elimination of some 'corrections'.—An examination of the formula for the effective pressure difference along a capillary tube through which liquid is flowing steadily, viz.,

$$\bar{p} = p - \frac{\rho Q^2}{\pi^2 a^4}$$

shows that the correction is independent of the length of the tube. Thus $\bar{p} = p - \beta$, where β is a constant for tubes of radius a. To eliminate the two corrections discussed in the previous paragraph

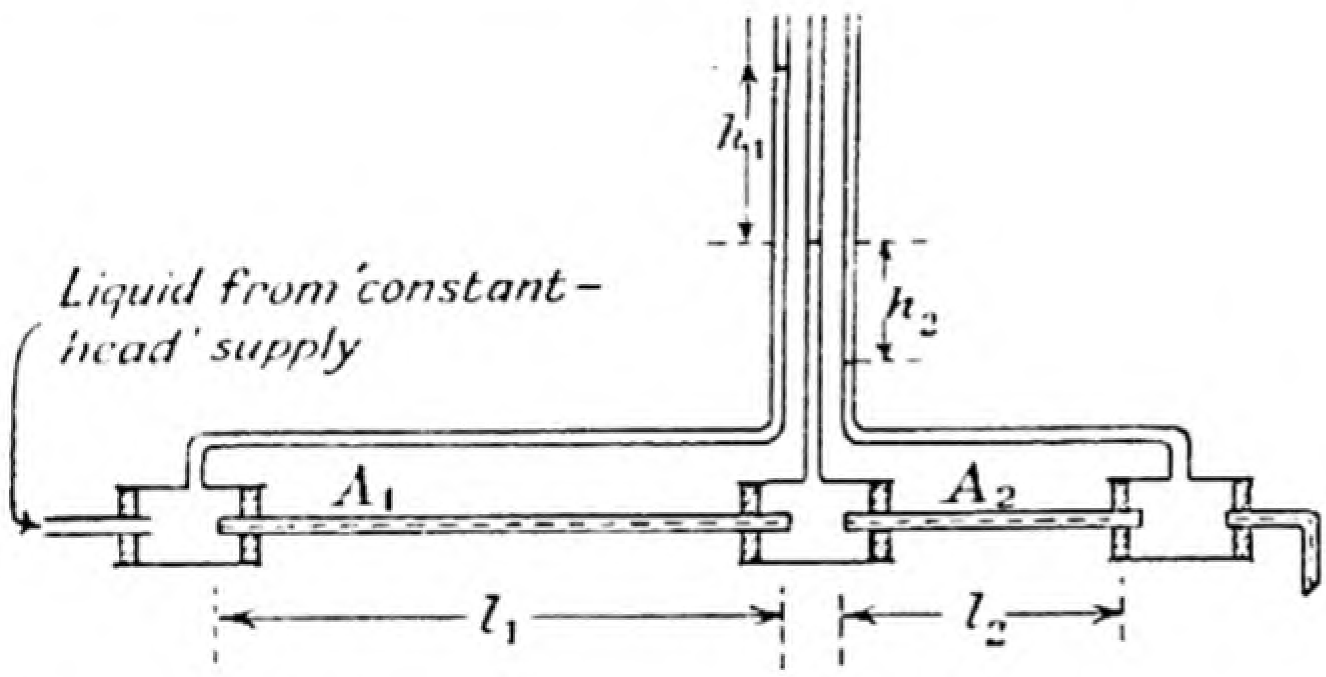


FIG. 11·13.—Couette's method for determining the viscosity of a liquid.

COUETTE devised the following experiment. A_1 and A_2 are two capillary tubes of the same diameter but different lengths, l_1 and l_2 respectively, arranged as indicated in Fig. 11·13. The pressure differences between the ends of the tubes are obtained by observing the heights h_1 and h_2 respectively. If these pressure differences are P_1 and P_2, then since the same volume of liquid passes through each tube per second, we have

$$Q = \frac{\pi a^4}{8\eta(l_1 + \alpha)}(P_1 - \beta) = \frac{\pi a^4}{8\eta(l_2 + \alpha)}(P_2 - \beta).$$

But
$$Q(l_1 + \alpha) = \frac{\pi a^4}{8\eta}(P_1 - \beta),$$

and
$$Q(l_2 + \alpha) = \frac{\pi a^4}{8\eta}(P_2 - \beta).$$

$$\therefore\ Q(l_1 - l_2) = \frac{\pi a^4}{8\eta}(P_1 - P_2) = \frac{\pi a^4}{8\eta}g\rho(h_1 - h_2).$$

The main difficulty in this experiment is to obtain tubes with equal and constant cross-sectional areas.

The comparison of viscosities by an Ostwald type viscometer.—The elementary theory of an Ostwald viscometer given on p. 555, is slightly in error since no correction has been made for the kinetic energy of the liquid or the effective length of the tube. Now by using the corrected form of Poiseuille's equation given on p. 559 it is easily shown that if V is the volume of liquid emerging in time t, then

$$\eta = \frac{\pi a^4 pt}{8\mathrm{V}(l+\alpha)} - \frac{\rho \mathrm{V}}{8\pi(l+\alpha)t}.$$

Since p is of the form $g\rho h$, the above equation may be written

$$\eta = \rho\left(\mathrm{A}t - \frac{\mathrm{B}}{t}\right),$$

where A and B are constants.

To determine values for these constants the viscometer is filled in turn with two liquids whose kinematic viscosities ν_1 and ν_2 are known. If t_1 and t_2 are the times required for a volume V of each liquid to be discharged from the capillary, we have

$$\mathrm{A} = \frac{\nu_2 t_2 - \nu_1 t_1}{t_2^2 - t_1^2}, \quad \text{and} \quad \mathrm{B} = \frac{t_1 t_2}{t_2^2 - t_1^2}(\nu_2 t_1 - \nu_1 t_2)$$

The viscometer may then be used to determine the viscosity of another liquid provided its density is known.

THE TRANSPIRATION OF GASES AND VAPOURS THROUGH CAPILLARY TUBES

On the steady flow of a compressible fluid at ordinary pressures through a capillary tube—Meyer's formula.—In the investigations above concerning the flow of a liquid through a capillary tube it is assumed that the liquid is incompressible and its density constant. This is clearly inadmissible when the transpiring substance is a gas. In the case of liquids it is found that the velocity u is constant at all points equidistant from the axis of the tube; in dealing with gases the starting point in our discussion is the fact that equal masses of gas pass each cross-section in a given time. This requires ρu to be constant at all points at a fixed distance from the axis of the tube, where ρ is the density of the gas at the point considered. Hence since ρ decreases with decrease in pressure the velocity of flow will increase as the exit end of the tube is approached. Since $\rho \propto p$, where p is the pressure pu must be independent of x, where x is the distance of the point from the entrance of the tube. In addition we shall assume that p is independent of r.

(*a*) ***Proof from First Principles:*** The equation to the motion is, as for a liquid, cf. p. 539,

$$\frac{\partial p}{\partial x} = \frac{\eta}{r}\cdot\frac{\partial}{\partial r}\left(r\cdot\frac{\partial u}{\partial r}\right).$$

$$\therefore p\cdot\frac{\partial p}{\partial x} = \frac{\eta p}{r}\cdot\frac{\partial}{\partial r}\left(r\cdot\frac{\partial u}{\partial r}\right) = \frac{\eta}{r}\cdot\frac{\partial}{\partial r}\left[r\cdot\frac{\partial(pu)}{\partial r}\right].$$

Integrating with respect to x,

$$\frac{p_2^2 - p_1^2}{2} = \frac{\eta}{r}\cdot\frac{\partial}{\partial r}\left[r\cdot\frac{\partial(pu)}{\partial r}\right](x_2 - x_1),$$

or

$$\frac{p_1^2 - p_2^2}{2l} = -\frac{\eta}{r}\cdot\frac{\partial}{\partial r}\left[r\cdot\frac{\partial(pu)}{\partial r}\right],$$

where $l = (x_2 - x_1)$.

Calling the left-hand side of the above equation A and integrating with respect to r, we have

$$A\int\frac{r}{\eta}.dr = -\int d\left[r.\frac{\partial(pu)}{\partial r}\right].$$

$$\therefore \frac{A}{\eta}.\frac{r^2}{2} = -r.\frac{\partial(pu)}{\partial r} + \kappa_1,$$

where κ_1 is an integration constant.

$$\therefore \frac{A}{2\eta}.r\,dr = -d(pu) + \frac{\kappa_1}{r}.dr.$$

$$\therefore \frac{A}{4\eta}.r^2 = -pu + \kappa_1 \ln r + \kappa_2,$$

where κ_2 is an integration constant.

Now, from the 'end' conditions, $\kappa_1 = 0$, and $\kappa_2 = \dfrac{A}{4\eta}a^2$.

$$\therefore pu = \frac{A}{4\eta}(a^2 - r^2) = \frac{p_1^2 - p_2^2}{8\eta l}.(a^2 - r^2).$$

If Ω_2 is the volume emerging per second at pressure p_2, then

$$p_2\Omega_2 = p\Omega,$$

where Ω is the volume, measured at pressure p, which passes per second across that section where the pressure is p. But

$$\Omega = 2\pi \int_0^a ru\,dr.$$

$$\therefore\ p_2\Omega_2 = \frac{\pi a^4}{16\eta l}(p_1^2 - p_2^2) = p_1\Omega_1 = \mu\mathscr{R}\mathrm{T},$$

where μ is the mass flowing per second, $\mathscr{R} = \frac{\mathrm{R}}{\mathrm{M}}$ [R is the universal gas constant, M is the molecular weight], and T is the absolute temperature.

(*b*) ***Proof of Meyer's formula for a gas assuming the formula for a liquid:*** It has already been stated that the assumption made in establishing theoretically Poiseuille's equation for a liquid cannot hold for gases. Since gases are easily compressible it would be expected that the velocity would increase as the gas passed through the tube, i.e. from the region of high to that of low pressure. Actually, the parabolic formula for the distribution of velocities over any cross-section of the tube, must be modified to some extent by the presence of a radial component but, with Meyer, it may be assumed as a first approximation, that if p and $p + \delta p$ are the pressures at cross-sections of the tube defined by x and $x + \delta x$, then Ω, the volume of gas, measured at pressure p, which passes per second through the element of tube considered, is given by

$$\Omega = -\frac{\pi a^4}{8\eta}\cdot\frac{dp}{dx},$$

i.e. $-\frac{dp}{dx}$ is written for $\frac{p}{l}$.

If μ is the mass of gas flowing per second, $\mu = \rho\Omega$, where ρ is the density of the gas at pressure p (and absolute temperature T).

$$\therefore\ \mu = -\frac{\pi a^4}{8\eta}\cdot\rho\cdot\frac{dp}{dx} = -\frac{\pi a^4}{8\eta}\cdot\frac{p}{\mathscr{R}\mathrm{T}}\cdot\frac{dp}{dx}.$$

$$\therefore\ \mu\int_0^l dx = -\frac{\pi a^4}{8\eta}\cdot\frac{1}{\mathscr{R}\mathrm{T}}\int_{p_1}^{p_2} p\,.dp$$

$$\therefore\ \mu\mathscr{R}\mathrm{T} = \frac{\pi a^4}{8\eta l}\left(\frac{p_1^2 - p_2^2}{2}\right) = p_1\Omega_1 = p_2\Omega_2 \text{ as before.}$$

If, in general, V is the volume of gas flowing through the capillary tube in time t, we have

$$p_1\mathrm{V}_1 = \frac{\pi a^4}{8\eta l}\left(\frac{p_1^2 - p_2^2}{2}\right)t = p_2\mathrm{V}_2.$$

Correction for slip.—In the above proofs for Meyer's formula it has been assumed that the fluid in contact with the walls of the tube remains at rest when the fluid flows through the tube. Experiments on liquids support this assumption but for gases there is evidence to show that slipping occurs, i.e. the gas flows as if the radius of the tube were increased by λ, where λ is the mean-free-path. This means that on account of 'slip' the flow of a gas through a tube of radius a is the same as that through a tube of radius $a + \lambda$, in which the slip were zero. Hence Meyer's equation for the transpiration of a gas through a capillary tube becomes

$$\mu \mathscr{R}\mathrm{T} = \frac{\pi a^4}{16\eta l}(p_1^2 - p_2^2)\left(1 + \frac{4\lambda}{a}\right), \left[\because\ (a + \lambda)^4 \simeq a^4\left(1 + 4\frac{\lambda}{a}\right)\right].$$

Experimental determination of the viscosity of hydrogen (oxygen) at room temperature.—The apparatus, designed by LEHFELDT, and shown in Fig. 11·14, consists of a water voltameter W

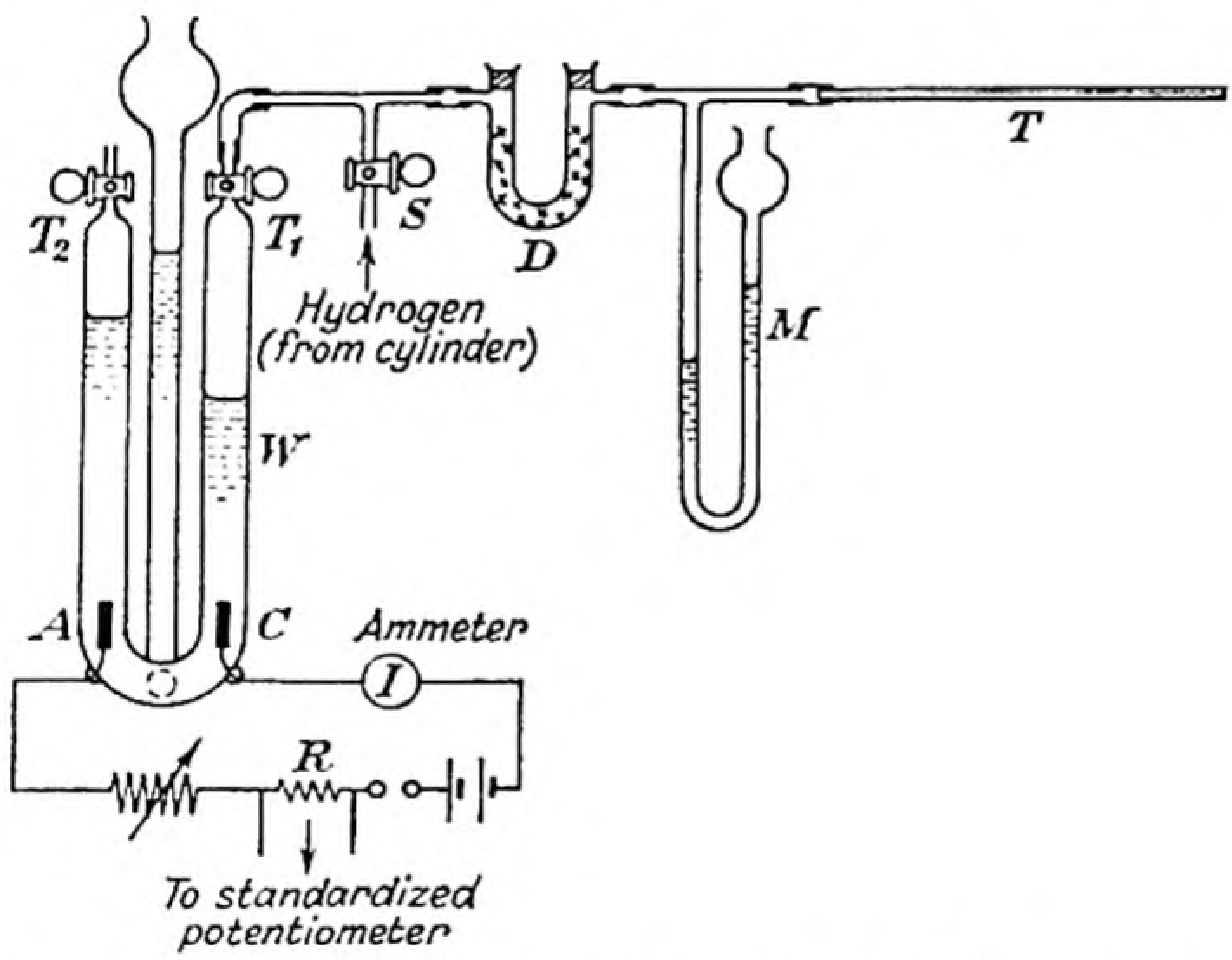


FIG. 11·14.—Lehfeldt's apparatus for determinnig the viscosity of hydrogen.

through which a steady current is passed. Hydrogen is liberated at the cathode C and leaves the voltameter through the stop-cock T_1. It then passes through the drying tube D, containing calcium chloride, and finally escapes through the graduated capillary tube T. The pressure of the hydrogen in excess of that of the atmosphere when it arrives at the entrance to T is measured by the xylol manometer M. The hydrogen escapes from T at atmospheric pressure.

The air within the apparatus is first removed by passing hydrogen from a cylinder of that gas through it; the side tube S permits

a connexion to be made to the cylinder. During this part of the experiment the acidulated water above C should reach T_1 which should be closed; the xylol should also be made to fill the left-hand side of the manometer limb completely and T should be removed so that the stream of hydrogen may be rapid. The apparatus must then be tested for leaks. To do this, close the open end of the capillary tube by means of rubber tubing and a glass rod and generate hydrogen until the pressure of the gas is about 10 cm. of xylol above atmospheric. Then read the positions of the liquid surfaces in the manometer and repeat these readings at intervals for about ten minutes. If no significant variations in level are observed, the apparatus is leak tight for the pressures required.

The steady current through the voltameter is adjusted to lie in the range 0·2–0·25 amp. [for a capillary tube for which $l = 50$ cm. and its mean radius $= 0{\cdot}014$ cm.]. In a fairly short time the manometer readings should become constant, showing that the rate at which hydrogen is being generated in the voltameter is equal to the rate at which it is escaping through the capillary tube. An ammeter, G, may be used to measure the current approximately but the current should finally be measured by means of a potentiometer.

The mean room-temperature and the mean barometric height at the time of the experiment should be recorded.

If the steady current used is I amp., the mass of hydrogen generated per second is Iz, where z is the electrochemical equivalent of hydrogen. This is equal to μ, the mass of hydrogen escaping through the tube per second, which is also given by the equation

$$\mu \mathscr{R} \mathrm{T} = \frac{\pi a^4 (p_1^2 - p_2^2)}{16 \eta l},$$

where $\mathscr{R}$ is the gas constant for one gramme of hydrogen,
T is the absolute temperature of the gas (the room),
a is the effective radius and l the length of the capillary tube,
η is the viscosity of hydrogen at temperature T,
and p_1 and p_2 are the pressures of the gas on entering and leaving the capillary tube respectively. Hence,

$$\mathrm{I}z = \frac{(p_1 - p_2)(p_1 + p_2)}{16 \eta \mathscr{R} \mathrm{T}} \cdot \frac{\pi a^4}{l}.$$

Thus a value for the viscosity of hydrogen, at room temperature, may be found.

The viscosity of oxygen, at room temperature, may be determined by connecting D and T to T_2. During the course of each experiment both stop-cocks T_1 and T_2 must be kept open.

Anderson's method for determining the viscosity of air at room temperature.—The following simple method for determining

the viscosity of air is due to ANDERSON,† and the theory is as follows. S, Fig. 11·15(a), is a glass or metal container of volume V to which is attached a capillary tube AB, of length l, and mean radius a. Let p_0 be the constant pressure outside the apparatus. Suppose

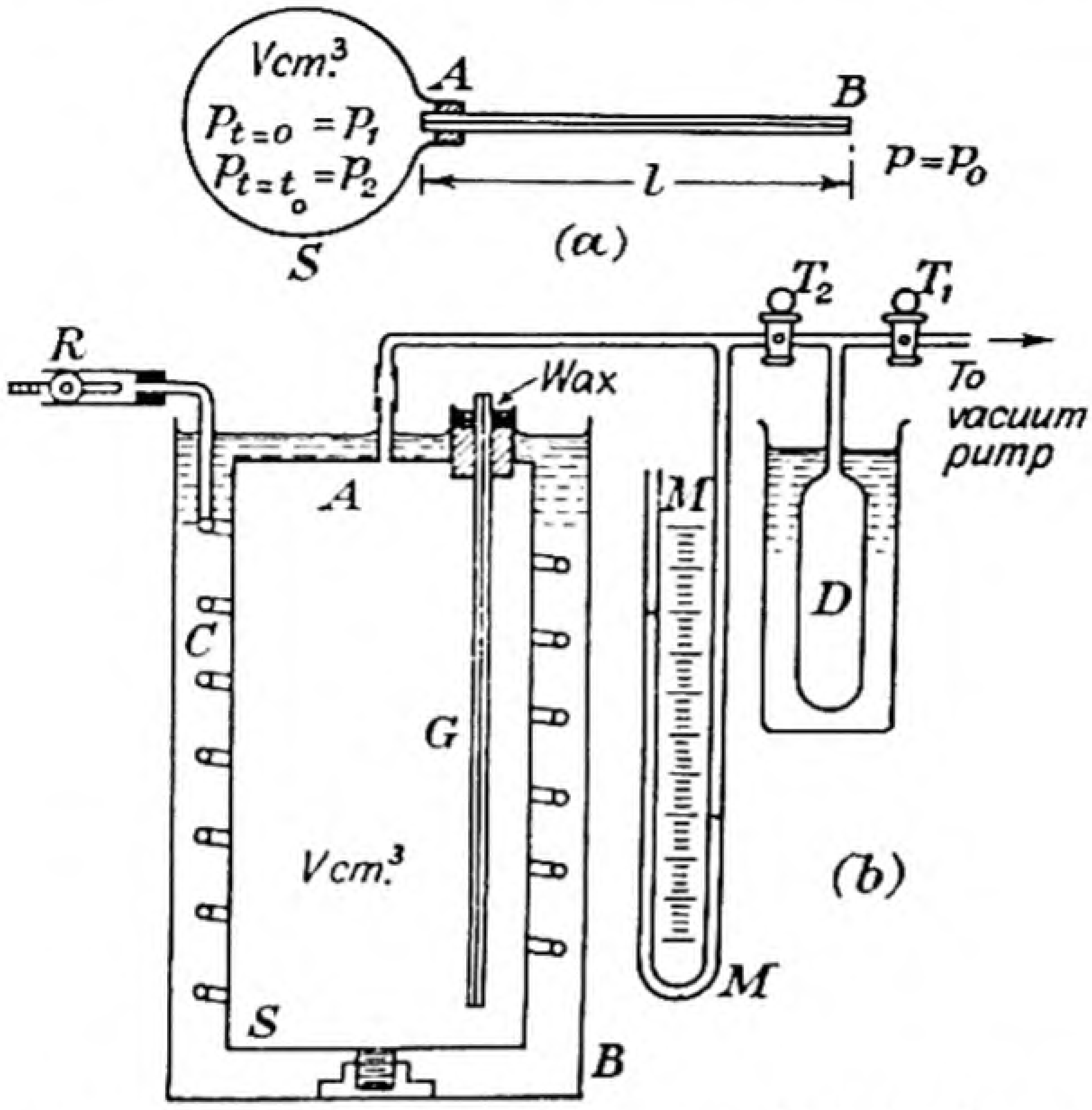


FIG. 11·15.—Anderson's method, as modified by the author, for finding experimentally the viscosity of air.

that in t_0 seconds the pressure of the air in S falls from p_1 to p_2. Let p be the pressure in S at time t. Then $pV = m\mathscr{R}T$, where $\mathscr{R}$ is the gas constant for 1 gm. of air, T is the absolute temperature and m the mass of the air in S. At time $t + \delta t$ let the pressure in S be $p + \delta p$, and $m + \delta m$ the mass of gas. Then $V\,\delta p = \mathscr{R}T.\delta m$.

$\therefore$ Mass of gas escaping per second

$$= \mu = -\frac{dm}{dt} = -\frac{V}{\mathscr{R}T}.\frac{dp}{dt}$$

$$= \frac{(p^2 - p_0{}^2)}{16\eta l}.\frac{\pi a^4}{\mathscr{R}T}.$$

$$\therefore\ -V.\frac{dp}{dt} = \frac{\pi a^4}{16\eta l}(p^2 - p_0{}^2).$$

$$\therefore\ \frac{1}{2p_0}\ln\frac{p + p_0}{p - p_0} = \frac{\pi a^4 t}{16\eta l V} + C,$$

where C is an integration constant determined by the fact that $p = p_1$, when $t = 0$. Also, since when $t = t_0$, $p = p_2$, we have

† *Phil Mag.*, **92**, 1022, 1921.

$$\ln \frac{(p_2 + p_0)}{(p_2 - p_0)} \cdot \frac{(p_1 - p_0)}{(p_1 + p_0)} = \frac{t_0}{\eta} . \kappa,$$

where $\kappa = \dfrac{\pi a^4 p_0}{8lV}$.

In carrying out this experiment it is difficult to keep V constant on account of the changes in effective volume caused by the displacement of the manometric liquid, used in a tube attached to S to measure changes in pressure, but if V is large and the limbs of the manometer not too wide, then the error on this account is negligible. A convenient set-up, designed by the author, is shown in Fig. 11·15 (*b*). S is a hollow brass cylinder of about 5 to 10 litres volume. A brass lid A is soft-soldered over its mouth and tubes from A lead to a manometer M, a cycle-tyre valve R and the necessary connecting tubes. A similar lid, but with no apertures, forms the base of the viscometer. To keep the temperature reasonably constant the cylinder S is surrounded by a wider brass container B and this is filled with water. To overcome the effects of the fluid thrust on S, this is screwed to the base of B as shown.

The air which is forced into the apparatus is passed through a drying tube and, after leaving the valve R, the air passes through a long copper-tube spiral C so that it enters S at a temperature equal to that of the water in B.

Well-greased rubber connexions enable the apparatus to be assembled.

To determine the volume of V a glass bulb D, of known volume, and stop-cocks T_1 and T_2 are fixed to the apparatus. With T_1 open and T_2 closed, the bulb D is exhausted; when T_1 is closed and T_2 is opened the changes of the levels of the manometer fluid permit the volume of S to be found if it is justifiable to assume Boyle's law to be valid for air.

A calibrated capillary tube, G, passes through a hole in the lid of A and is held in position with wax or Chatterton's compound. The end of G open to the atmosphere is closed with rubber tubing and a glass rod while the volume of S is found.

When the glass rod is removed air escapes from the viscometer and by measuring the changes in pressure which occur during a known time interval, the viscosity of air at the temperature of the experiment can be found from the formula already established.

Now the kinetic theory shows that the viscosity of a gas varies with temperature (T° K) according to the formula

$$\eta = \frac{\kappa T^{\frac{1}{2}}}{1 + \dfrac{S}{T}},$$

where κ and S are constants, the latter being termed Sutherland's constant, cf. p. 605.

With the viscometer just described, it is possible to carry out determinations of the viscosity of air (or coal gas) at 0° C. and at 50° C. and hence to determine Sutherland's constant for the gas under examination; for

$$\frac{\eta_{50}}{\eta_0} = \left(\frac{323}{273}\right)^{\frac{3}{2}} \left(\frac{S + 273}{S + 323}\right),$$

so that S may be calculated when η_0 and η_{50} are known.

Rankine's method for determining the viscosity of a gas (available in small quantities).—This apparatus was devised by Rankine† for the purpose of measuring the viscosity at room temperature of each of the rare gases, neon, argon, krypton and xenon.

Consider a closed glass vessel consisting of two connected limbs, one a fine capillary tube and the other of much greater cross-sectional area, yet sufficiently narrow for a pellet of mercury to remain intact in it, cf. Fig. 11·16(*a*). Let V be the volume unoccupied by mercury (the volume of the capillary being considered negligible). Let P be the steady pressure of the gas in the apparatus when the latter is held horizontally, and let p be the difference in pressure caused by the mercury pellet when the apparatus is vertical. Then if p_1 is the pressure and v_1 the volume at any time below the mercury, p_2 and v_2 the corresponding quantities above the mercury,

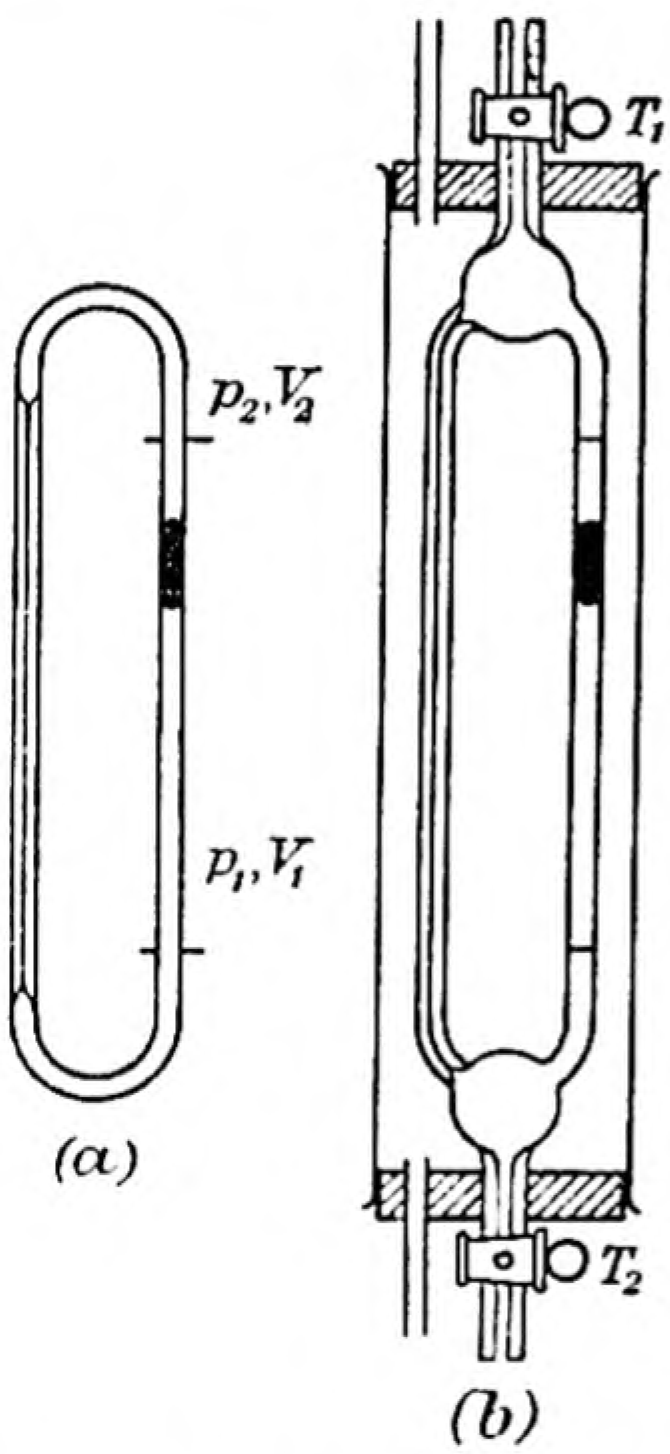


FIG. 11·16.—Rankine's viscometer for gases available in small quantities.

$$V = v_1 + v_2 \qquad \text{and} \qquad p_2 = p_1 - p.$$

Further, if we assume that the temperature remains constant,

$$PV = p_1v_1 + p_2v_2,$$

always.

Let m be the mass of gas in the apparatus above the mercury pellet at any instant. Then $m\mathscr{R}T = p_2v_2$, where $\mathscr{R}$ is the gas constant per unit mass of gas. The mass of gas, δm,

† *Proc. Roy. Soc.*, A, **83**, 265, 1910.

which transpires in time δt is given by

$$\mathscr{R}\mathrm{T}\,\delta m = \frac{\pi a^4}{16\eta l}(p_1 - p_2)(p_1 + p_2)\,\delta t,$$

$$= \frac{\pi a^4}{8\eta l}(p_1 - p_2)\left(\frac{p_1 + p_2}{2}\right)\delta t,$$

$$= \kappa(p_1 - p_2)\left(\frac{p_1 + p_2}{2}\right)\delta t, \qquad \text{where } \kappa = \frac{\pi a^4}{8\eta l}.$$

But $\quad \mathscr{R}\mathrm{T}\,\delta m = \delta(p_2 v_2).$

$$\therefore\ p_2\,dv_2 + v_2\,dp_2 = \kappa p\left(\frac{2p_2 + p}{2}\right)dt.$$

Now from the isothermal condition, we have,

$$\mathrm{PV} = p_1 v_1 + p_2 v_2 = (p_2 + p)(\mathrm{V} - v_2) + p_2 v_2$$

$$= p_2\mathrm{V} + p\mathrm{V} - pv_2.$$

$$\therefore\ p_2 = \mathrm{P} - p + \frac{p}{\mathrm{V}}.v_2.$$

Hence p_2 is a linear function of v_2, and also p_1 is a linear function of v_1.

$$\therefore\ \left(\mathrm{P} - p + \frac{pv_2}{\mathrm{V}}\right)dv_2 + v_2\frac{p}{\mathrm{V}}.dv_2 = \kappa p\left[\frac{2\mathrm{P} - 2p + 2\frac{p}{\mathrm{V}}.v_2 + p}{2}\right]dt$$

$$\therefore\ \left(\mathrm{P} + 2v_2.\frac{p}{\mathrm{V}} - p\right)dv_2 = \kappa p\left[\frac{2\mathrm{P} - p + \frac{2p}{\mathrm{V}}.v_2}{2}\right]dt.$$

Let $2\mathrm{P} - p + \frac{2pv_2}{\mathrm{V}} = \xi$. Then $d\xi = \frac{2p}{\mathrm{V}}.dv_2$.

$$\therefore\ \frac{\mathrm{V}}{2p}(\xi - \mathrm{P})\,d\xi = \tfrac{1}{2}\kappa p\xi\,dt.$$

or

$$\frac{\mathrm{V}}{p}\left(1 - \frac{\mathrm{P}}{\xi}\right)d\xi = \kappa p\,dt.$$

In the interval of time from 0 to t, let ξ change from ξ_0 to ξ_t and v_2 from $(v_2)_0$ to $(v_2)_t$, then

$$\frac{V}{p}\int_{\xi_0}^{\xi_t}\left(1 - \frac{P}{\xi}\right)d\xi = \kappa \int_0^t p\, dt.$$

$$\therefore\ \frac{V}{p}\Big[\xi - P \ln \xi\Big]_{\xi_0}^{\xi_t} = \kappa pt.$$

$$\therefore\ \frac{V}{p}\left[\xi_t - \xi_0 - P \ln \frac{\xi_t}{\xi_0}\right] = \kappa pt.$$

$$\therefore\ \frac{V}{p}\left[\frac{2p}{V}\{(v_2)_t - (v_2)_0\} - P \ln\left\{\frac{2P - p + \dfrac{2p}{V}.(v_2)_t}{2P - p + \dfrac{2p}{V}.(v_2)_0}\right\}\right] = \kappa pt.$$

Suppose now that the mercury describes a symmetrical displacement. Such a displacement is indicated by the two marks on the fall tube; let v be the volume between them. Then

$$V = (v_2)_0 + (v_2)_t \quad \text{and} \quad (v_2)_t - (v_2)_0 = v.$$

$$\therefore\ (2v_2)_t = V + v, \quad \text{and} \quad (2v_2)_0 = V - v.$$

The left-hand side of the above equation then becomes

$$2v - \frac{PV}{p}.\ln\left\{\frac{1 + \dfrac{pv}{2PV}}{1 - \dfrac{pv}{2PV}}\right\}.$$

As $\dfrac{pv}{2PV}$ is less than unity, we may expand the logarithmic term. Calling $\dfrac{p}{PV} = \alpha$, and retaining only terms to the third power, the expression under consideration reduces to

$$2v - \frac{2}{\alpha}\left[\frac{\alpha v}{2} + \frac{\alpha^3 v^3}{24}\right] = v\left[1 - \frac{\alpha^2 v^2}{12}\right] = v,$$

since $\dfrac{\alpha^2 v^2}{12} \rightarrow 0$, for all mercury pellets likely to be used in practice.

$$\therefore\ \frac{\pi a^4 pt}{8\eta l} = v.$$

Rankine's viscometer as modified for use at room and at steam temperatures.—This viscometer was designed in 1910 by RANKINE to investigate how the viscosities of the inert gases varied

with temperature. The viscometer is shown in Fig. 11·17, and consists of a complete circuit formed by two glass tubes, one a capillary and the other a wider tube containing a pellet of mercury which forces the gas in the apparatus through the capillary when the viscometer is in a vertical position. The capillary tube, A, about a metre long, is bent twice as shown to reduce the overall length of the apparatus without seriously altering the time occupied by the mercury pellet, P, in descending. Two small bulbs, B_1 and B_2, each about 1 cm.^{3} in volume, serve to contain the mercury pellet during the process of exhausting or of filling the apparatus with gas.

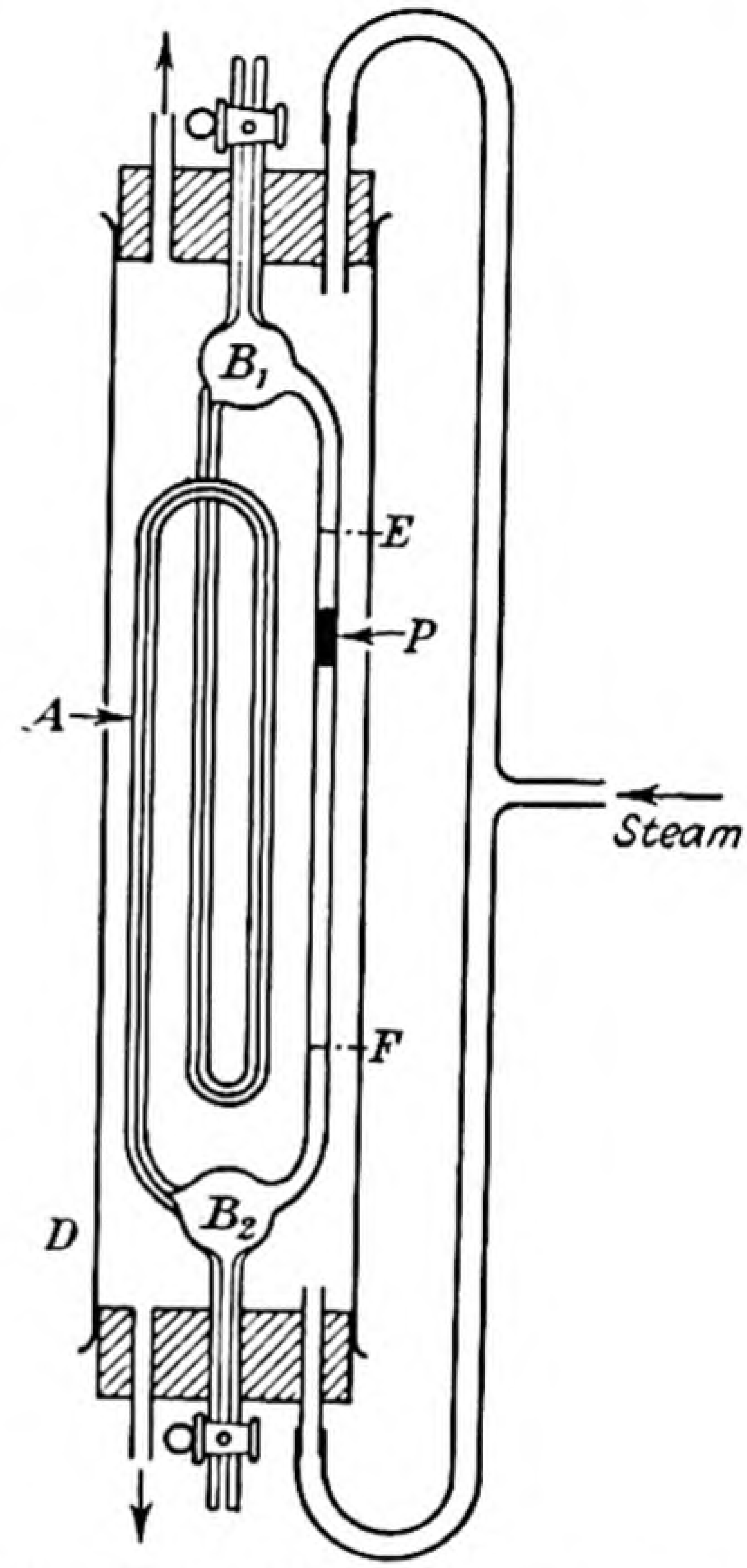


FIG. 11·17.—Rankine's viscometer for investigating how the viscosity of a gas varies with temperature.

The apparatus is surrounded by a steam jacket, D; a thermometer is inserted to measure room temperature only, the steam temperature being calculated from the barometric height and the thermometer never being allowed to remain in position while the steam is passing.

Apart from some small corrections the ratio of the viscosities of a gas at two different temperatures is approximately equal to the ratio of the times of fall at those temperatures of the mercury pellet between the fiducial marks E and F on the 'fall-tube'.

In 1921 work by Rankine and the author showed that an important modification in the method of making observations is necessary. Quite noticeable changes in the curvatures of the ends of the mercury pellets are observed, not only when the gas within the viscometer is changed but also when the temperature of the same gas alters. Originally it had been assumed that the effect of capillarity in reducing the full hydrostatic pressure due to the weight of the pellet was constant. The variations in driving pressure are probably due to changes in the angle of contact between mercury and glass rather

than to a variation in the surface tension of mercury. The effect is eliminated by making observations in every instance both with the pellet intact and broken into two segments approximately equal in length.

If it is assumed that the capillary effect is doubled when the pellet is divided into two segments and trebled when the segments are three in number [as indeed proves to be so experimentally], we may denote it as a certain fraction α of the full hydrostatic pressure due to the weight of the mercury pellet. If t_1, t_2 and t_3 are the observed times of descent of the pellet, when in one, two or three segments, respectively, it drives equal volumes of gas through the capillary tube, we have

$$(1-\alpha)t_1 = (1-2\alpha)t_2 = (1-3\alpha)t_3,$$

so that

$$\alpha = \frac{t_2 - t_1}{2t_2 - t_1} = \frac{t_3 - t_1}{3t_3 - t_1},$$

and it is the equality of these two fractions which indicates, in practice, that the capillary effect is additive. If t is the time of fall which we may expect in the absence of all capillary effects, i.e. when the full hydrostatic pressure of the mercury is operative, then

$$t = t_1(1-\alpha).$$

This time t is then strictly proportional to the viscosity of the gas in the apparatus [except for a small correction for 'slip', which can be applied independently].

Either bulb, B_1 or B_2, may be used to divide the pellet into the required number of segments.

Rankine's method for determining the viscosity of a vapour and for investigating its variation with temperature.—This method was devised by RANKINE† to measure the viscosity of the vapour of a volatile liquid (e.g. bromine); in it the superheated vapour transpired through a capillary tube. The special features of the method lay in the devices which were adopted for measuring the pressures on the two sides of the capillary tube and in estimating the mass of vapour which transpired in a given time. In its simplest form Rankine's viscometer would consist of a capillary tube connecting two bulbs, A and B, Fig. 11·18, which contain only the liquid and its vapour. These two bulbs are immersed in baths maintained at different temperatures, and the capillary has a temperature which is greater than that of either bulb. In this way a suitable difference of pressure is established between the two ends of the capillary tube, and hence the vapour will pass through the capillary from the hotter to the cooler side—by evaporation in the

† *Proc. Roy. Soc.*, A, **88**, 575, 1913.

former and condensation in the latter. When the steady state has become established the mass of vapour which has transpired in a given time can be determined by observing the volume of liquid which disappears from the hotter side A, provided we know the density of the liquid at the temperature of A.

It is obvious that if we wish to determine the mass of liquid which has disappeared from A with any degree of precision, A should be

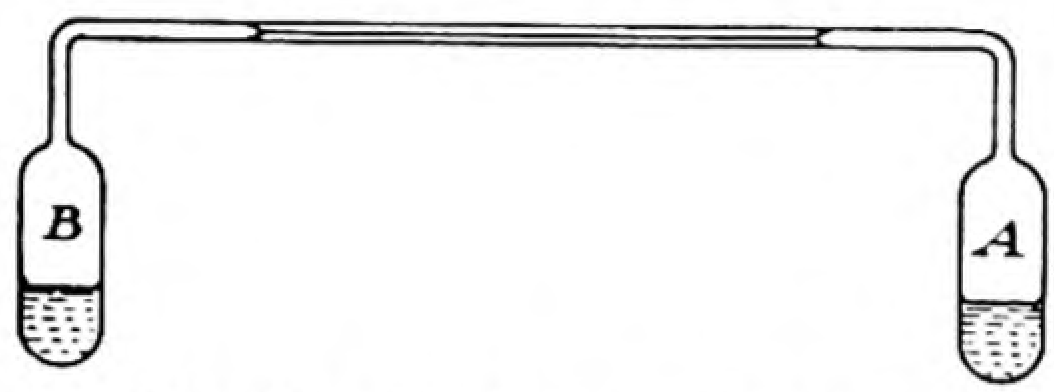


FIG. 11·18.—Principle of Rankine's viscometer for superheated vapours.

in the form of a uniform tube of narrow bore. In such an arrangement, on account of the loss of heat by evaporation, and the failure of conduction and convection in so narrow a tube to establish equality of temperature, the temperature of the liquid surface becomes lower than that of the surrounding bath. If, however, this decrement can be measured, the actual driving pressure forcing the vapour through the capillary tube can be found, the saturation pressure itself being known.

In practice, however, the experiment is scarcely so simple as this, for on account of the smallness of the volume of liquid corresponding to a large volume of vapour, it is desirable that A and B should take the form of tubes, narrow in bore, although not comparable in this respect with the capillary itself. In this case the pressure at the two ends of the capillary are no longer equal to the saturation vapour pressures of the liquid at temperatures respectively equal to those of the baths in which A and B are immersed. This must be attributed to the fact that when the rate of distillation through the capillary is large, heat can neither enter A nor leave B rapidly enough to secure equilibrium between the liquid and its vapour. The consequence is that the pressure in A is less than, and that in B greater than the corresponding saturation pressures, and in order to find the true pressures means have to be adopted for estimating the above differences. A consideration of the diagram of the apparatus actually used, cf. Fig. 11·19, will show how this was achieved. It will be seen, that the vessels A and B are U-tubes, sealed at the ends remote from the capillary. They are enclosed in water baths, the temperatures of which differ by several degrees. Let us suppose that that containing A is at the higher temperature. The temperature of the bath containing A must be lower than that of any other

part of the apparatus except B, and, with this restriction, the temperature of the bath C containing the capillary tube may be maintained at the value at which it is desired to determine the viscosity of the vapour. The vapour which evaporates from A passes through the glass spiral S_1 in order that it may be raised to the desired temperature before entering the capillary. After emerging from the other end it is eventually condensed to liquid in B.

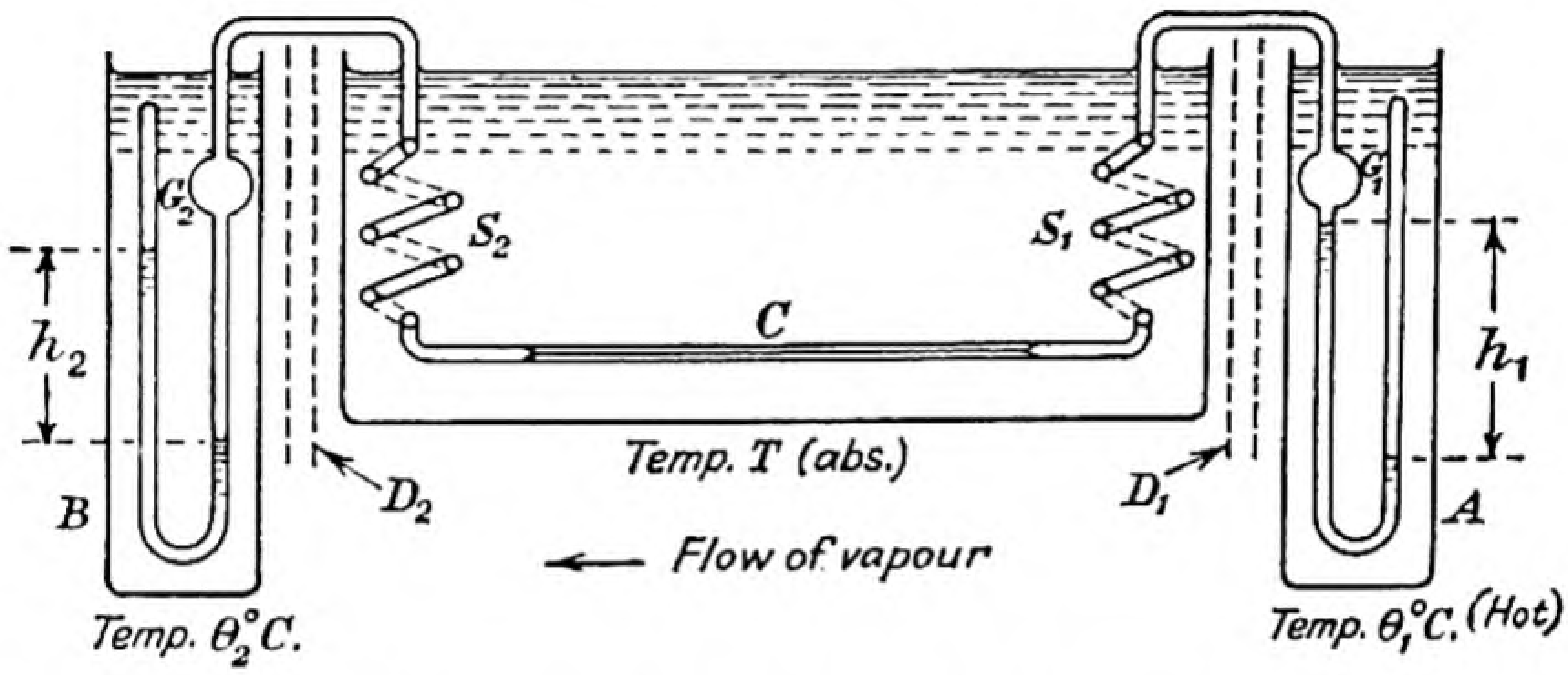


FIG. 11·19.—Rankine's viscometer for use with the vapours of volatile liquids. [The large bulbs G_1 and G_2 facilitate condensation on the cooler side of the viscometer; the stirrers and thermometers are *not* shown].

Now let us consider what happens in the closed limbs of the vessels A and B. The variations in level to which the liquid is subject in these limbs is not such as to involve appreciable evaporation or condensation there; consequently the saturation pressures are maintained. In the limbs adjacent to the capillary, however, owing to rapid evaporation and condensation the pressures differ from the saturation values, but by amounts which can be easily estimated by observing the differences of level in the two limbs of the U-tubes. Thus suppose θ_1 and θ_2, $\theta_1 > \theta_2$, are the temperatures of the baths containing A and B respectively, ρ_1 and ρ_2 the densities of the liquid at these temperatures, p_1 and p_2, the pressures at the two ends of the capillary, and h_1 and h_2, the differences of levels of the liquid in the respective U-tubes, then, if P_1 and P_2 are the saturation vapour pressures of the liquid at temperatures θ_1 and θ_2 respectively,

$$p_1 = P_1 - g\rho_1 h_1,$$

and

$$p_2 = P_2 + g\rho_2 h_2.$$

Thus, provided that the saturation vapour pressures and the liquid densities over a small temperature range are known, the pressures controlling the transpiration can be found without using anything in the nature of an ordinary pressure gauge. Further, a knowledge of the specific volumes of vapour and liquid provides

means of determining the rate of flow through the capillary, from observations of the volume evaporation and condensation of the liquid.

Meyer's transpiration formula, which is the basis of this method for measuring the viscosity of a vapour, is

$$\eta = \frac{\pi a^4}{16l} \cdot \frac{(p_1^2 - p_2^2)t}{p_1 V_1},$$

where t is the time in which a volume of vapour, V_1 (at pressure p_1), transpires. This formula is developed on the assumption that the substance is an ideal gas. In these experiments the vapour was superheated and under a low pressure so that the deviations will almost certainly be small. In 1924† the author used this method, with some slight modifications, to study the viscous properties of water vapour over the range 100° C.–260° C., and carried out some experiments in which the difference in pressure between the ends of the capillary were halved; the value for η was not affected beyond the limits of error inherent to the method.

As the above formula stands, it is not best suited for the purposes of calculation; it requires to be put in terms of m, the mass of substance transpiring in time t. Since $p_1 V_1 = m\mathscr{R}T_1 = m \cdot \frac{R}{M} \cdot T_1$, where R is the universal gas constant, $\mathscr{R}$ the gas constant for the liquid per gm., M its molecular weight and T_1 the temperature, on the absolute scale, of the vapour, as it passed through the capillary tube, we have

$$\eta = \frac{\pi a^4}{16l} \cdot \frac{(p_1^2 - p_2^2)t}{m} \cdot \frac{1}{\mathscr{R}T_1} = \frac{\pi a^4}{16l} \cdot \frac{(p_1^2 - p_2^2)t}{m} \cdot \frac{M}{RT_1}.$$

Several small corrections have to be applied, but for an account of these, the original papers must be consulted.

Experimental determination of the viscosity of steam at atmospheric pressure.—The principle of this method of measuring the viscosity of steam at about 100° C. is due to GREGORY, but the apparatus described below was designed by the author and possesses several marked advantages over the original apparatus in which an inverted Dewar flask was used. Fig. 11·20(a) shows the principle of the method. Water is boiled in a metal container, A, from which all air has been expelled. The steam escapes through a calibrated capillary tube, C, and may then be condensed in a U-tube, partially immersed in cold water. A sensitive mercury-in-glass thermometer, T, gives the temperature of the vapour, but in

† *Proc. Roy. Soc.*, A, **106**, 83, 1924.

order that the bulb of the thermometer shall not be deformed mechanically by the pressure of the steam, the thermometer is placed in a tube closed at its lower end and containing an oil

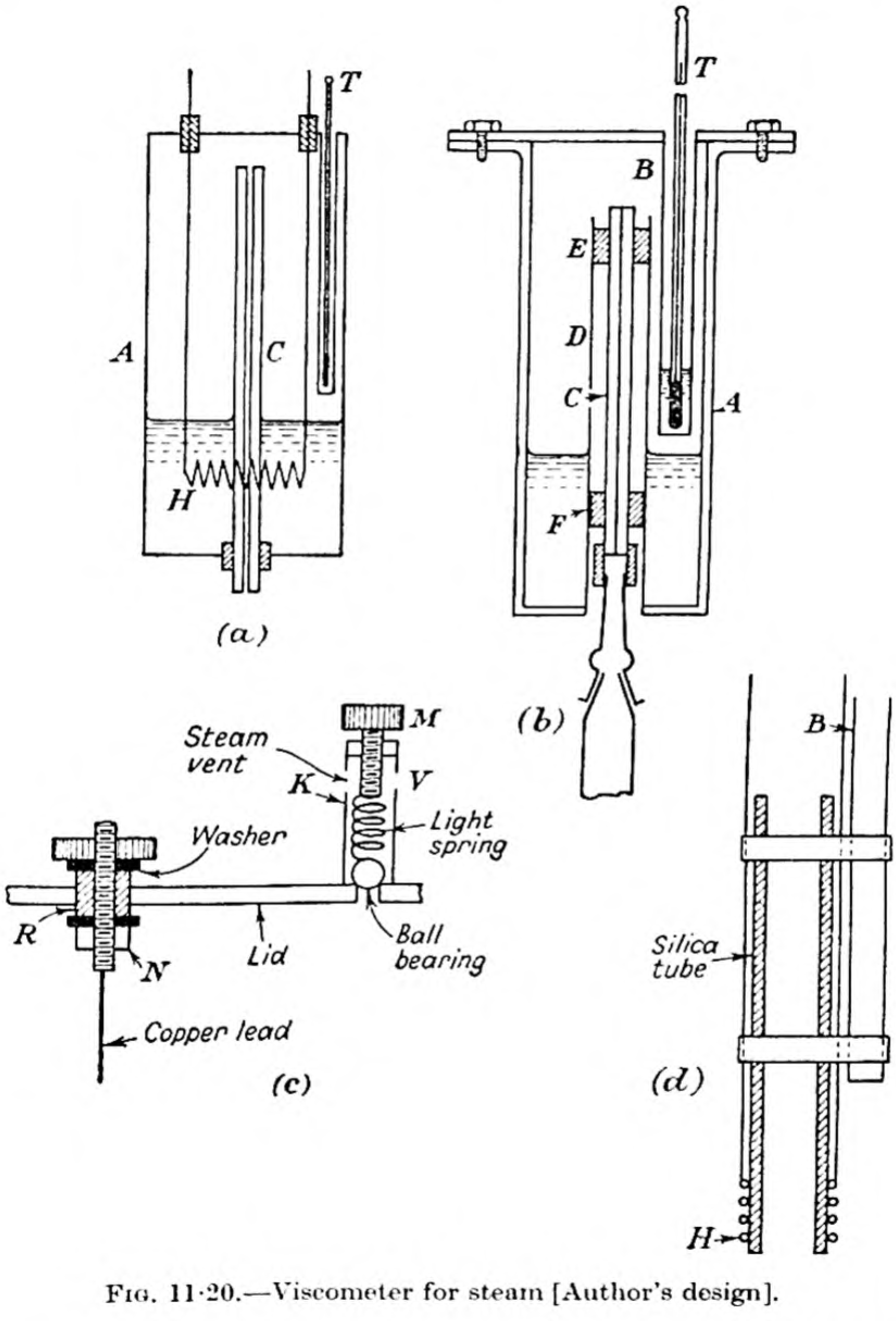


FIG. 11·20.—Viscometer for steam [Author's design].

[b.p. > 100° C.] to give good thermal contact between the bulb and the heated tube. When this temperature is known the pressure of the vapour can be deduced from tables; the viscosity of the vapour is then calculated with the aid of Meyer's formula as given on p. 563.

Details of the actual viscometer are shown in Fig. 11·20(*b*). A is a brass vessel, flanged at the top so that a brass lid, with an intervening rubber gasket, may be fixed in position with screws. This lid carries the tube B, in which the thermometer, T, is placed. The lid also carries the insulated leads to the heating coil, H, and a safety-valve, V, which will be described later. Into the base of the tube A there is soldered a brass tube D. A rubber bung, E, fits into D and supports the capillary tube, C. F is a loosely fitting cork which prevents C from wobbling about. By means of a short length of rubber tubing a U-tube may be attached to the exit end of C and this, when partly immersed in cold water, serves to catch the steam which escapes in a measured time interval.

Fig. 11·20(*c*) gives details of the safety-valve and of one of the insulated leads to the heating coil, H. A brass terminal is passed through a piece of thick-walled rubber tubing, R, which just fits into a hole in the lid of the viscometer. Brass washers are placed as shown and when the nut, N, is screwed into position the rubber is expanded and the joint is 'gas-tight'. The valve, V, consists of a short piece of brass tubing, K, at the base of which is a steel sphere resting on a small hole. A light spring holds the sphere down and the thrust on it is controlled by means of the screw, M.

Details of the electric heater are shown in Fig. 11·20(*d*). The coil consists of a few turns of constantan wire, hard soldered to thick copper leads. The coil is wound on a silica tube of such size that it slips easily over the brass tube D. The silica tube is held in position by copper bands fixed to the tube B. Thus the apparatus is readily assembled and may be dismantled to clean the capillary, etc.

To commence the experiment the water is boiled vigorously, the valve V being open so that air may be expelled from the apparatus. The thrust on the spring is then increased slightly so that the pressure of the saturated water vapour in the viscometer increases. This pressure is deduced from the steady reading of the thermometer. Steam is then escaping through the valve and also through the capillary tube. The viscosity of steam is then found as indicated earlier. The experiment is then repeated with temperatures not exceeding about 102·5° C. [$p = 90$ cm. of mercury].

ROTATIONAL AND OSCILLATION VISCOMETERS

Introductory.—The viscosity of such a substance as Lyle's 'Golden Syrup', or glycerol, is so great at room temperatures that the capillary tube viscometer becomes a very inconvenient instrument to use. In such cases a rotational viscometer may be used.

Such an instrument consists of a vertical cylinder, of known dimensions, which is caused to rotate uniformly, within a coaxial cylinder of known radius, the annular space between the cylinders containing the liquid under investigation. The depth to which the inner cylinder is immersed must be known and the experiment consists in determining the constant angular velocity of the inner cylinder when a steady couple is applied to it. In the elementary theory of the action of such a viscometer it is assumed that the motion of the liquid is the same as if the depth of immersion were infinite; a so-called 'end-correction' occurs in practice and the experiment is usually carried out so that this end-correction may be eliminated. In theory it is immaterial whether the outer cylinder is rotated and

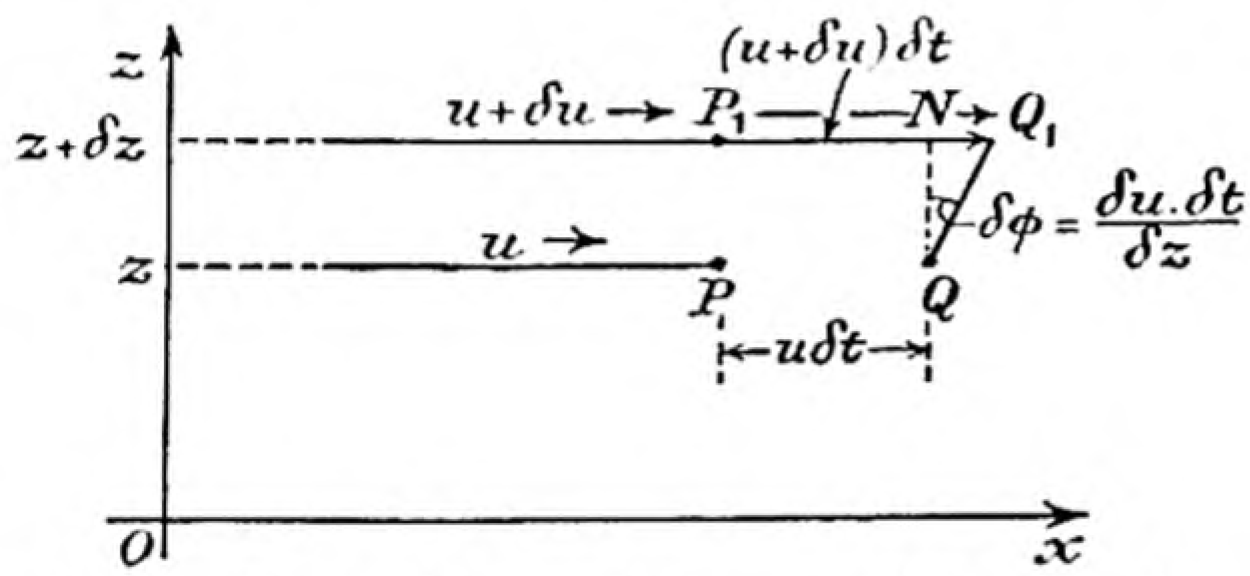


FIG. 11·21.—Newton's hypothesis—another point of view.

the inner one kept stationary or vice-versa, for the problem is one of relative rotation. In the case of liquids having a high viscosity the simplest construction is obtained by applying a known couple to the inner cylinder by means of a system of weights and pulleys and measuring the rate of rotation after a steady state has been reached. When the torque necessary to provide a suitable speed becomes so low that the friction of the pulleys becomes a disturbing factor, the apparatus is made so that the outer cylinder may be rotated at constant speed, the inner cylinder being suspended by a torsion wire. The deflexion is a measure of the couple on this cylinder.

Before attempting to show how this couple may be related to the viscosity of the fluid between the cylinders it is necessary to reconsider Newton's hypothesis of viscous flow. This hypothesis is epitomized by the equation, cf. p. 536, $\frac{\mathrm{F}}{\mathrm{A}} = \eta \frac{du}{dz}$. It is sometimes more instructive to write this equation in a different form. Thus, if two layers at a distance δz apart in a liquid are considered, where the velocities are u and $u + \delta u$, respectively, then in time δt a particle in the faster moving layer will advance a distance $\delta u\ \delta t$ further than a particle in the other layer cf. Fig. 11·21. If $\delta\phi$ is

the angle through which the straight line connecting the positions of the particles rotates in time δt, i.e. PP_1 rotates to the position QQ_1, then

$$\delta\phi = \frac{\delta u \, \delta t}{\delta z}, \quad \text{or} \quad \frac{d\phi}{dt} = \frac{du}{dz}.$$

Thus

$$\frac{F}{A} = \eta \frac{d\phi}{dt},$$

or the stress is directly proportional to the rate of shear if the viscosity, η, is constant.

On the torque acting on a cylinder placed in a rotating fluid.—Let the space betmeen two coaxial cylinders be filled with a viscous fluid. If the outer cylinder is made to rotate with constant angular velocity the couple exerted on the inner cylinder may be calculated; hence, if this couple is measured we have a means of finding the viscosity of the fluid. Let a and b, Fig. 11·22(a), be the

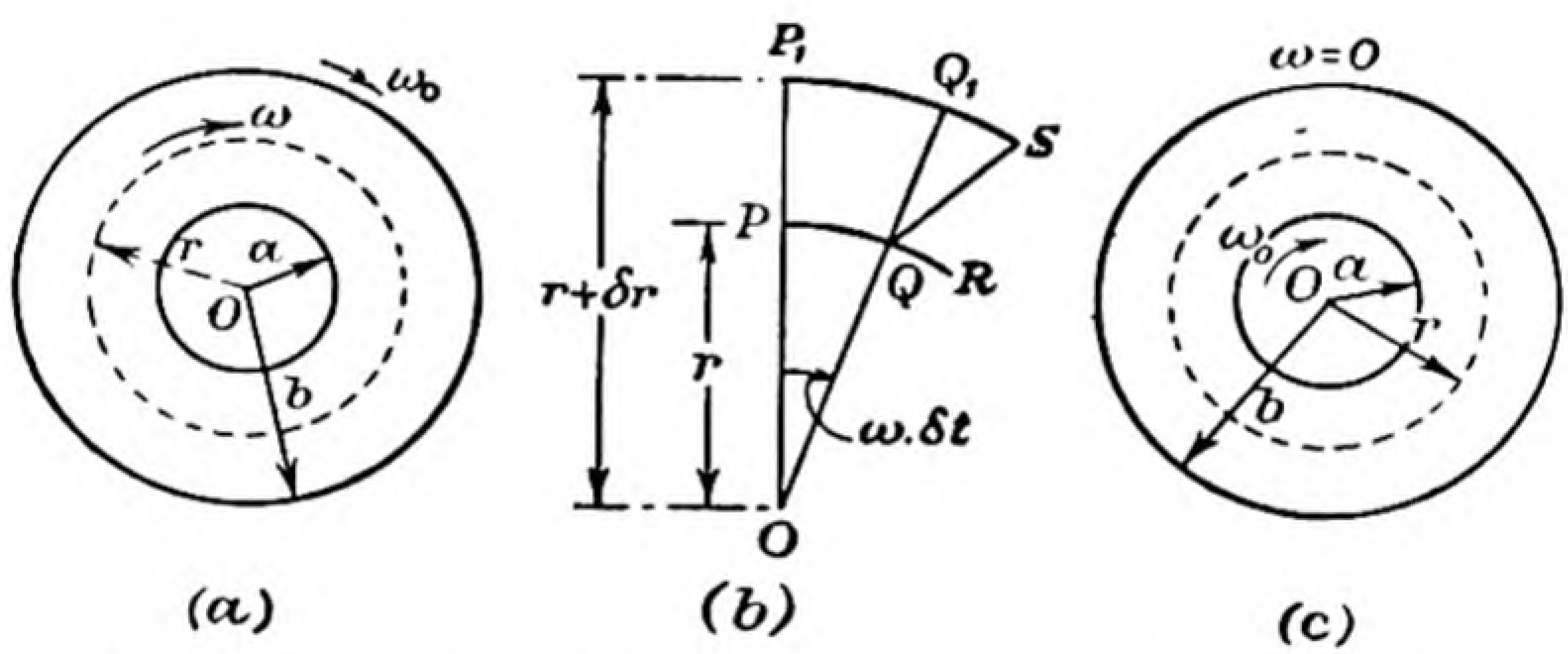


FIG. 11·22.—On the torque on a cylinder placed in a steadily rotating fluid.

radii of the inner and outer cylinders and consider a length l of the cylinders; end effects are neglected for the moment. The inner cylinder is at rest, the couple exerted on it being balanced against the couple due to the torsion in its suspension. Let ω_0 be the angular velocity of the outer cylinder; this velocity is taken to be clockwise and the innermost layer of liquid will be at rest.

Let r be the radius of a cylindrical surface within the fluid. If ω is the angular velocity of a point on this surface, let $\omega + \delta\omega$ be the velocity at points distance $r + \delta r$ from the axis. Let PQR and P_1Q_1S, Fig. 11·21(b), be arcs of circles of radii r and $r + \delta r$. Let $\widehat{POQ} = \omega . \delta t$. Then the particles which were at P and P_1 at time t will be at Q and S at time $t + \delta t$.

Now $PQ = r.\omega.\delta t$, and $P_1S = (r + \delta r)(\omega + \delta\omega)\delta t$.

$$\therefore\ Q_1S = r.\delta\omega.\delta t, \qquad \text{if } \delta\omega\ \delta r \text{ is neglected.}$$

$$\therefore\ \widehat{Q_1QS} = r\frac{\partial\omega}{\partial r}\delta t$$

$$\therefore\ \text{Rate of shearing} = r\frac{\partial\omega}{\partial r}.$$

$\therefore$ Tangential stress at the surface of the cylinder, of radius r, is

$$\eta r\frac{\partial\omega}{\partial r}, \qquad \text{[cf. p. 579]}$$

where η is the viscosity of the fluid.

Hence the fluid outside the cylinder of radius r and length l exerts on a strip of this cylinder of width δs, where $\oint ds = 2\pi r$, a force $l\eta.r\frac{\partial\omega}{\partial r}\ \delta s$, and the moment of this force about the axis of the cylinder is

$$\eta l r^2\frac{\partial\omega}{\partial r}.\delta s.$$

Hence the total torque on the cylinder considered is

$$\eta l r^2\frac{\partial\omega}{\partial r}\oint ds = 2\pi\eta l r^3\frac{\partial\omega}{\partial r}.$$

Let Γ be the anticlockwise couple which must be applied externally to prevent the inner cylinder from rotating. When conditions are steady there is no angular acceleration of the liquid, i.e. the total couple on the liquid between the cylindrical surfaces defined by $r = r$ and $r = a$ is zero.

$$\therefore\ 2\pi\eta l r^3\frac{\partial\omega}{\partial r} - \Gamma = 0, \quad \text{or} \quad 2\pi\eta l\, d\omega = \Gamma\frac{dr}{r^3}.$$

$$\therefore\ 2\pi\eta l\omega = -\frac{\Gamma}{2}.\frac{1}{r^2} + A,$$

where A is an integration constant whose value, given by the boundary condition that when $r = a$, $\omega = 0$, is $A = \frac{1}{2}.\frac{\Gamma}{a^2}$.

Since when $r = b$, $\omega = \omega_0$,

$$2\pi\eta l\omega_0 = \tfrac{1}{2}\Gamma\left[\frac{1}{a^2} - \frac{1}{b^2}\right].$$

$$\therefore\ \eta = \frac{\Gamma}{4\pi l\omega_0}\left[\frac{1}{a^2} - \frac{1}{b^2}\right].$$

When the inner cylinder rotates, cf. Fig. 11·22(c), the liquid outside the cylindrical surface of radius r again exerts a torque $2\pi\eta l r^3 \dfrac{\partial\omega}{\partial r}$ on that surface and in a clockwise direction. If Γ is the clockwise couple causing the rotation of the inner cylinder, then since the fluid between the surfaces $r = a$ and $r = r$ has zero angular acceleration, we have

$$\Gamma + 2\pi\eta l r^3 \frac{\partial\omega}{\partial r} = 0.$$

Integrating this equation and using the boundary conditions $r = a$, $\omega = \omega_0$; $r = b$, $\omega = 0$, we find

$$\eta = \frac{\Gamma}{4\pi l\omega_0}\left[\frac{1}{a^2} - \frac{1}{b^2}\right].$$

Alternative proof: It has just been shown that the rotating fluid outside a cylinder of radius r and length l exerts on it a torque $2\pi\eta l r^3 \dfrac{\partial\omega}{\partial r}$. Now consider the fluid within two cylindrical surfaces r and $r + \delta r$. The torque on the inner surface must be equal and opposite to the torque on the outer surface since the liquid is in equilibrium, i.e.

$$\left[2\pi\eta l r^3 \frac{\partial\omega}{\partial r}\right]_{r=r} = \left[2\pi\eta l r^3 \frac{\partial\omega}{\partial r}\right]_{r=r+\delta r}$$

$$= [\ldots]_{r=r} + \frac{\partial}{\partial r}\left\{2\pi\eta l r^3 \frac{\partial\omega}{\partial r}\right\}\delta r.$$

$$\therefore\ 3\frac{\partial\omega}{\partial r} + r\frac{\partial^2\omega}{\partial r^2} = 0.$$

To solve this equation we have

$$\frac{\dfrac{\partial^2\omega}{\partial r^2}}{\dfrac{\partial\omega}{\partial r}} = -\frac{3}{r}.$$

Integrating we get

$$\ln\frac{\partial\omega}{\partial r} = \text{const} - 3\ln r,$$

i.e.

$$\frac{\partial\omega}{\partial r} = \mathrm{A}r^{-3},$$

where A is a constant. Hence $\omega = -\frac{1}{2}Ar^{-2} + B$, where B is a constant. The boundary conditions are $r = a$, $\omega = 0$, and $r = b$, $\omega = \omega_0$,

so that
$$A = 2\omega_0\left[\frac{1}{a^2} - \frac{1}{b^2}\right]^{-1}.$$

Now the inner cylinder of the apparatus is in equilibrium under the action of the anticlockwise couple Γ and the clockwise torque exerted by the liquid, viz.

$$2\pi\eta l\left[r^3\frac{\partial\omega}{\partial r}\right]_{r=a}.$$

$$\therefore\ \Gamma = 2\pi\eta l.A,$$

or, as before,
$$\eta = \frac{\Gamma}{4\pi l\omega_0}\left[\frac{1}{a^2} - \frac{1}{b^2}\right].$$

A cylindrical viscometer in practice.—In the theoretical analysis of the torque exerted by a moving viscous fluid on a cylinder the torques exerted on the ends of the cylinder have been neglected; by using in turn two cylinders of lengths l_1 and l_2 this unknown torque may be eliminated, provided no other change is made, for we have

$$\Gamma_1 = 4\pi\eta\omega_0\cdot\frac{a^2b^2}{(b^2 - a^2)}l_1 + \Gamma(B),$$

and
$$\Gamma_2 = 4\pi\eta\omega_0\cdot\frac{a^2b^2}{(b^2 - a^2)}l_2 + \Gamma(B),$$

where $\Gamma(B)$ denotes the torque on the ends. Hence

$$\eta = \frac{\Gamma_1 - \Gamma_2}{4\pi(l_1 - l_2)\omega_0}\left[\frac{1}{a^2} - \frac{1}{b^2}\right].$$

In 1890 Couette† designed the first really successful cylindrical viscometer so that such instruments are known by his name. Here we shall only discuss one or two modifications of his original instrument.

Millikan's method for an absolute determination of the viscosity of a gas at room temperature.—These experiments were made by Gilchrist‡ and later the apparatus was improved by Harrington,§ both of whom worked under the direction of Millikan, who had found that at that time the accuracy with which

† *Ann. Chim. Phys.*, **21**, 433, 1890.
‡ *Phys. Rev.*, **1**, 124, 1913.
§ *Phys. Rev.*, **8**, 738, 1916.

the elementary charge could be determined was limited mainly by possible inaccuracies in the value adopted for the coefficient of viscosity of air. It was decided to work with a constant deflexion method since such a method was exceedingly simple and direct, and appeared to be more suitable for absolute determinations of η than are either the capillary tube or oscillating body methods. [Zemplen† used this method for a direct determination of η for air but he used concentric spheres and his results were high.]

Gilchrist decided to use coaxial cylinders for two reasons:—

(*a*) it appeared to be the simplest method and the one in which the errors could be readily detected and their effect minimized,

(*b*) the theory is simple and the formula derived is exact. It has already been shown that Γ, the couple acting on the inner cylinder, provided the 'end effect' can be eliminated, is given by

$$\Gamma = \frac{4\pi\omega_0\eta l a^2 b^2}{(b^2 - a^2)},$$

and this is equal to $b_0\phi$, where ϕ is the steady angular deflexion of the inner cylinder and b_0 is the couple per unit twist in the suspension. If T is the period of oscillation of the inner cylinder about its axis of suspension, I being the corresponding moment of inertia [the outer cylinder being removed], then

$$\mathrm{T} = 2\pi\sqrt{\frac{\mathrm{I}}{b_0}}.$$

$$\therefore\ \eta = \frac{\pi\phi\mathrm{I}(b^2 - a^2)}{a^2b^2\mathrm{T}^2\omega_0 l}.$$

The inner cylinder was made of brass, the walls being as thin as possible to reduce the mass. The wall was supported by sheets of aluminium perpendicular to the axis. The wall of the cylinder was made parallel to the line of suspension.

The principle of the guard rings, used to eliminate 'end effects', is indicated in the diagram, Fig. 11·23(*a*), which also shows the principal dimensions.

To find I the inner cylinder was suspended by a steel piano wire 0·36 mm. in diameter and the period found. An annular ring of aluminium, with a cross bar, was placed symmetrically on the cylinder and the period again determined. Then,

$$\mathrm{T}_1 = 2\pi\sqrt{\frac{\mathrm{I}_1}{b_0}} \quad \text{and} \quad \mathrm{T}_2 = 2\pi\sqrt{\frac{\mathrm{I}_1 + \mathrm{I}_2}{b_0}}.$$

† *Ann. der Phys.*, **29**, 869, 1909; **38**, 71, 1912.

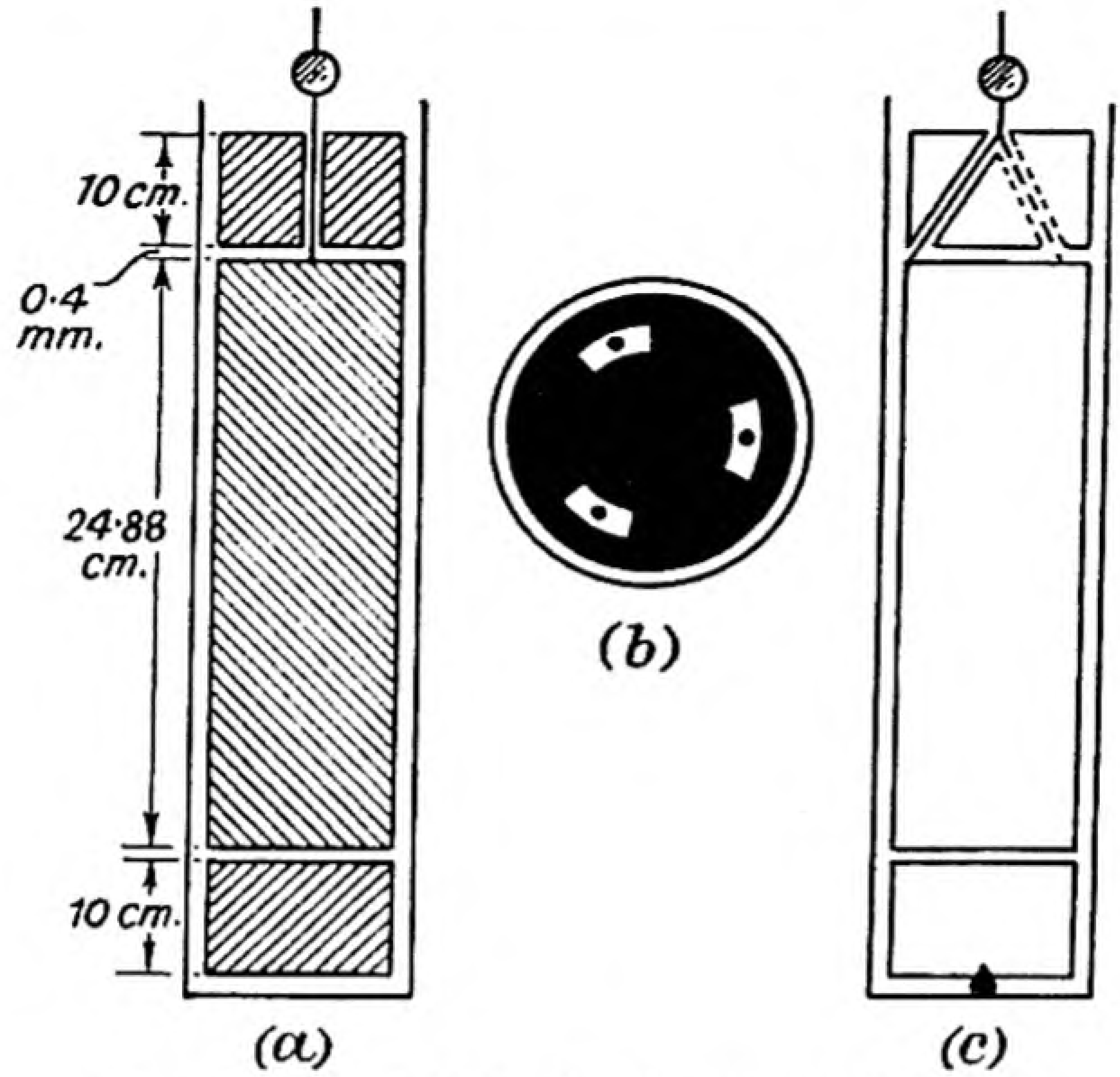


FIG. 11·23.—Millikan's viscometer for gases.

The following observations are typical:—

$$T_1 = 19{\cdot}7183 \text{ sec.}, \qquad T_2 = 28{\cdot}239 \text{ sec.}$$

$$I_2' = M' . \left[\frac{c^2 + d^2}{3}\right] = 22{\cdot}157 . \left[\frac{0{\cdot}376^2 + 9{\cdot}2265^2}{3}\right] \text{gm.cm.}^2,$$

c being the half-width and d the half-length of the bar.

$$I_2'' = M'' . \left[\frac{e^2 + f^2}{2}\right] = 97{\cdot}3771 . \left[\frac{9{\cdot}2075^2 + 8{\cdot}1608^2}{2}\right] \text{gm.cm.}^2,$$

where e and f are respectively the outer and inner radii of the ring.

$\therefore I_2 = I_2' + I_2'' = 8002{\cdot}4$ gm.cm.^{2} and hence $I_1 = 7615{\cdot}8$ gm.cm.^{2}.

The method of driving the outer cylinder and the actual manner of suspending the inner cylinder are shown in Fig. 11·23(b) and (c).

Harrington found for air,

$$\eta_{23^\circ \text{C.}} = 1{\cdot}8226 \times 10^{-4} \text{ gm.cm.}^{-1}\text{sec.}^{-1}.$$

A precision determination of the viscosity of air at 20° C.— The major part of the discrepancy between Millikan's oil-drop method value for the electronic charge and that obtained by X-rays, cf. Vol. VI, is attributable to an error in the measured viscosity of air. The only methods available for making accurate measurements of the viscosity of a gas depend upon the rotation of a cylinder or the flow of the gas through a capillary tube. This latter method is experimentally the more easy to carry out but the numerous

corrections which must be made render the method unsuitable for precision measurements. The rotating cylinder method possesses the distinct advantages of precision in the mechanical construction of the apparatus and the freedom from theoretical objections.

In 1939 BEARDEN made a precision determination of the viscosity of air at 20·00° C.; he used a rotating cylinder method which differed mainly in one respect from that used by Millikan. The rotating cylinder A, Fig. 11·24, was the inner one because the external diameter of a long cylinder may be made and measured more accurately than its inner diameter. The deflexion cylinder B was short so that its inner surface could be machined accurately; it was protected by guard rings G and its internal radius was found by determining the mass of water required to fill the cylinder completely when it rested on a flat base. A was rotated by means of an electromagnetic drive; this enabled it to be rotated in a vacuum and the apparatus could then be filled with dry air without upsetting the driving mechanism. All measurements were made at 20·00° C. and a Beckmann thermometer inside the apparatus showed that the temperature therein did not vary by more than 0·01 deg. C. in a day. The steel base of the apparatus was made horizontal and the suspended cylinder arranged coaxially with the guard rings and the rotating cylinder.

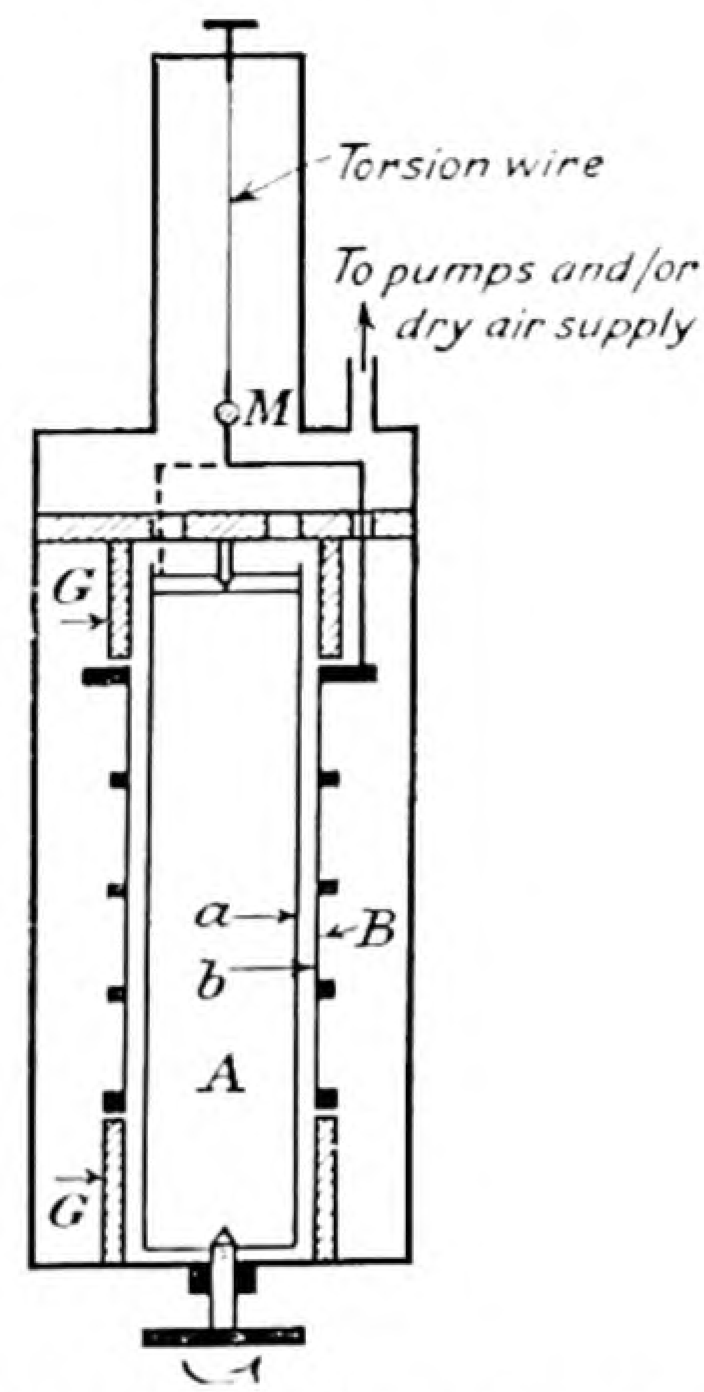


FIG. 11·24.—A precision viscometer for use with gases.

The optical system for measuring deflexions of the outer cylinder was a telescope and comparator. This was set parallel to the mirror in its zero position and the screws of the cross-hairs and comparator were calibrated for errors. The distance from the vertical wire in front of the light source to the mirror M was measured with the aid of a standard bar. The distance was corrected for the refractive index of the glass of the window in front of the mirror.

Because the cylinder A could not easily be brought to rest completely, either in its rest position or when deflected, observations were made by setting the cylinder oscillating enough to displace the image by about 0·005 cm. The extreme positions of several oscillations were measured with the vernier cross-hairs and the mean

† *Phys. Rev.*, **56**, 1023, 1939.

added to the comparator reading, the comparator being set on almost the steady position of the image. The drive for the cylinder B was very constant, the motor ($\frac{1}{2}$ H.P.) being synchronized with a precision electric tuning fork, made of elinvar, using a thyraton circuit. The torsion constant of the suspension wire was found by measuring the period of oscillation of suspended objects whose moments of inertia could be calculated. These measurements were made in a separate bell-jar, which was evacuated to about 10 mm. of mercury so that damping could be neglected. The torsion wire was specially made of tungsten treated at high temperature in dry hydrogen and always kept slightly under tension. This was found to prevent 'zero drift'. It was shown by experiment that there was no difference between the 'static' and 'dynamic' torsion constant. Now it is possible that an error may be made in determining the torsional constant of the suspension, for whereas the wire is used statically in the viscosity measurements it is used dynamically in finding the period of oscillation. If the two constants are not the same one might expect the torsional constant to be a function of the period of oscillation. Six inertia objects with different moments of inertia about the axis of rotation were used but the wire finally used showed no variation.

It was also verified by experiment that the effective length of the deflected cylinder was the mean of the actual length and the distance between the guard rings. In a typical determination, the pressure inside the apparatus was reduced to 10 mm. of mercury and a check for leaks made. Dry air was admitted to a pressure of 74·5 mm. of mercury. The apparatus was left for several hours to become steady. The zero position was measured, then deflexions with the cylinder rotating in one and then the other direction were measured.

The mean value for the viscosity of air, calculated by the formula proved on page 583, was found to be

$$(1819{\cdot}20 \pm 0{\cdot}06) \times 10^{-7}\ \text{gm.cm.}^{-1}\ \text{sec.}^{-1}\ \text{at}\ 20{\cdot}00°\text{C.}$$

If Millikan's oil-drop data are used with the above value for the viscosity of air, one finds

$$e = -4{\cdot}815 \times 10^{-10}\ \text{e.s.u.},$$

whereas the X-ray value, as calculated by Dunnington, is

$$e = -4{\cdot}8025 \times 10^{-10}\ \text{e.s.u.}$$

Searle's viscometer for liquids.—A sectional diagram of this viscometer is shown in Fig. 11·25. The inner cylinder, A, is carried by an axle working in fixed bearings, the lower bearing being at the top of the pillar, C, which supports the outer cylinder, B. The cylinder, B, may be raised or lowered and clamped in any desired position by means of the screw, S. The axle supporting the inner

cylinder carries a drum, D, so that A may be caused to rotate by means of two equal masses, M, supported in pans, of known mass, attached to a string wound round the drum D and passing over two pulleys, P_1 and P_2. The top of the pillar to which the outer cylinder is clamped carries a perforated baffle plate, Q, loosely fitting the outer cylinder. The liquid under investigation fills the space below the baffle plate and also part of the space between the two cylinders. By moving the outer cylinder upwards the length of the inner cylinder surrounded by the liquid may be varied. To facilitate varying the depth of immersion of the inner cylinder the outer cylinder is clamped to the pillar C by means of a screw-clamp, S, acting on a tube projecting from the base of the cylinder. This projecting tube is partially slotted parallel to its axis so that by means of the screw S a liquid-tight joint is made with the pillar. The baffle plate Q prevents the rotation of the inner cylinder from causing any appreciable motion of the liquid in the region under the plate; thus for a given value of ω_0, the angular velocity of the inner cylinder, there is a definite 'end-effect', α, i.e. the distribution of velocity near the lower end of A is practically independent of the position of B on its support.

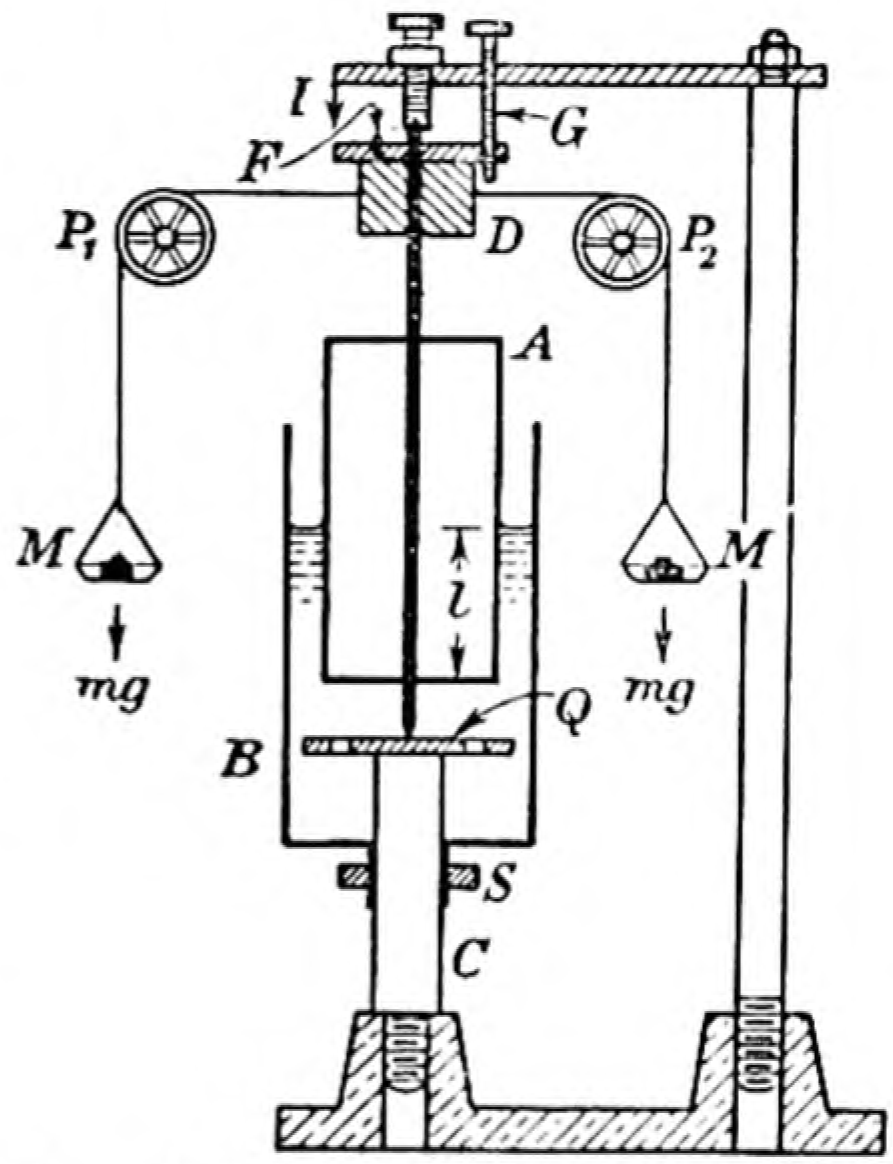


FIG. 11·25.—Searle's rotational viscometer for very viscous liquids.

A scale in millimetres is engraved upon the inner cylinder and this may be observed through a vertical slot in the wall of the outer cylinder. This slot is covered with a thin glass plate cemented to B and secured in position by a brass frame. In this way the depth of immersion of the inner cylinder in the liquid may be determined. Above the drum the axle carries a brass plate with a fiducial mark F on its edge; when the locking pin G is in position the inner cylinder is fixed and F coincides with a fixed mark, I, on the instrument. These two marks are used in determining the time of revolution of the inner cylinder.

The internal diameter, $2b$, of the cylinder B, and the external diameter, $2a$, of A must be measured at several places with the aid of calipers and mean values found.

The liquid is then placed in the viscometer, and after all air bubbles have disappeared the outer cylinder is adjusted so that only a short length of the inner cylinder is immersed—if the inner cylinder were raised out of the liquid it would be necessary to wait some considerable time for the liquid to drain away from it. Observations should then be made of the period of rotation, T, for different loads and a constant depth of immersion of the inner cylinder. If friction at the bearings and pulleys is constant and small, the product mT where m is the mass of each pan and its load, should be proportional to the effective depth of immersion $(l + \alpha)$. The intercept of the graph formed by plotting $m\mathrm{T} = x, l = y$, gives the 'end-correction' α.

It remains to measure the diameter of the drum and of the string; the effective diameter so far as the couple applied is concerned is equal to the sum of these two diameters, say δ, i.e. $\Gamma = gm\delta$, where g is the intensity of gravity. Since

$$\eta = \frac{g\delta(b + a)(b - a)}{8\pi^2 a^2 b^2} \times \frac{m\mathrm{T}}{(l + \alpha)},$$

the viscosity, η, may be deduced from the slope of the linear graph by plotting $x = l$, $y = m\mathrm{T}$. The observations must be made as quickly as possible in order that large variations in temperature shall not occur.

To determine T accurately the pin G should be removed and the inner cylinder rotated by hand through about one quarter of a revolution in a direction opposite to that which it will travel when the masses M are allowed to fall. A stop-watch is started as the fiducial mark F passes the index I and the time of each subsequent 'transit' noted. A mean value for the period of rotation is then found in the usual way by subtracting (say) the fourth reading from the first, the fifth from the second, etc.

The viscosity of gases at low temperatures.—The viscometer to be described is typical of a type known as oscillation viscometers; in each such viscometer a suspended body oscillates about a vertical axis of symmetry.

In certain circumstances the simple design of an oscillation viscometer is an advantage which outweighs the theoretical disadvantages inherent thereto. This is particularly so when the viscosity has to be determined at low (or high) temperatures. It was for reasons such as these that Vogel† avoided transpiration methods. In general, only comparative measurements can be made; in the experiments by Vogel the viscosity of air at 0° C. was taken as $1{\cdot}724 \times 10^{-4}$ gm.cm.$^{-1}$sec.$^{-1}$. This is a mean value, which happens to be identical with that found by Rankine.

† *Ann. der Phys.*, **43**, 1235, 1914.

The principal parts of the apparatus used by Vogel are shown in Fig. 11·26. Two glass plates, A, A, were held in position inside the bulb portion of a glass apparatus, G, by means of two springs, S, S. Metal blocks, B, B, kept these plates at a fixed and uniform distance apart. D was a disc, made of glass or silver, which was suspended so that it could oscillate in a horizontal plane between A, A. H was a torsion head which carried a fine platinum wire P (0·05 mm. diameter and 20 cm. long), to which was attached at its lower end a nickel wire N (1 mm. thick and 22 cm. long). A plane mirror was rigidly attached to N; an astatic pair of magnets enabled the system to be set in motion by bringing up a magnet which was afterwards removed.

The glass vessel G carried the whole of the above system and it could be evacuated or filled with the gas under investigation. Only the lower portion of the apparatus was placed in the cryostat, so that the oscillation could always be observed through the window W. The wire P was sufficiently far removed from the colder portion of the apparatus for its rigidity to remain practically unaffected by temperature variations.

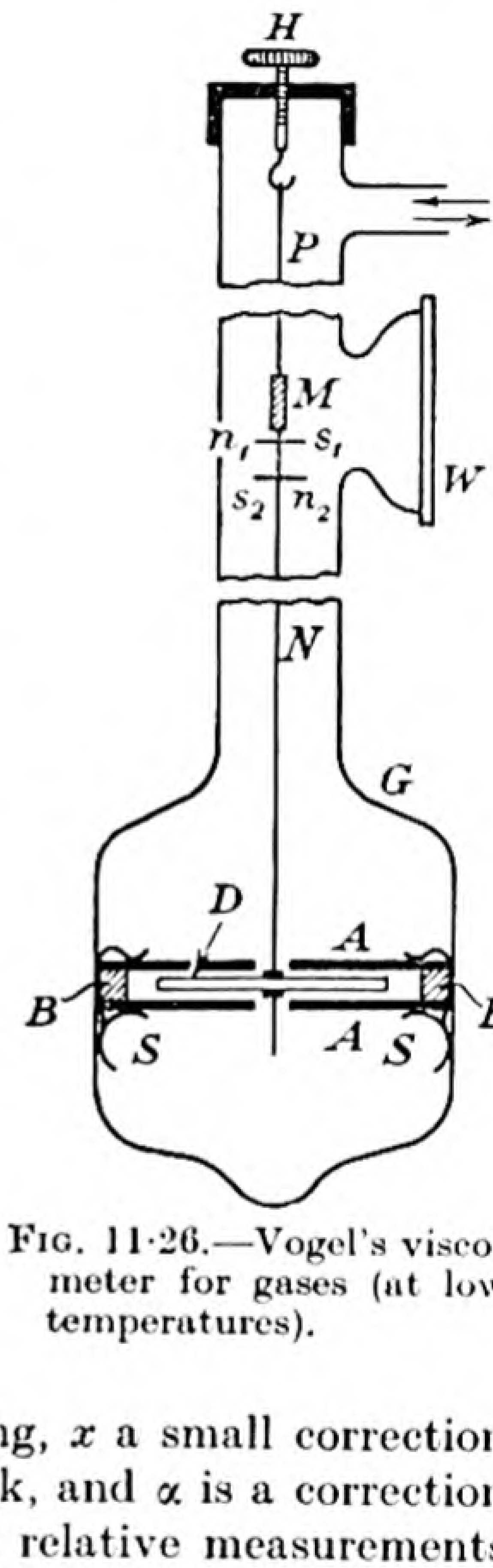


FIG. 11·26.—Vogel's viscometer for gases (at low temperatures).

According to Maxwell† the viscosity, η, of the gas is related to λ, the logarithmic decrement per half-cycle, by the equation

$$\eta = \frac{\lambda - \alpha}{\tau C(1 + x)},$$

where C is a constant, τ the period of swing, x a small correction term which was practically zero in this work, and α is a correction for accessory frictional effects. Since only relative measurements are made it is not necessary to know the value of C. The quantity α is the most important correction and earlier workers have often taken it to be zero. In Vogel's experiments where the dimensions of the apparatus were smaller than usual and especially at low temperatures when η itself is small, it was very necessary to determine α. To do this Vogel used two viscometers which were identical except that in one the suspended disc was made of glass, in the other of silver, yet both discs had the same moment of inertia, about the

† *Phil. Trans.*, **156**, 249, 1866.

axis of rotation, i.e. the surface areas of the discs alone were different. Experiments were made with each instrument in a particular gas at two different temperatures. Experiments were also made with the apparatus exhausted; α was then calculated. In general

$$\alpha = \alpha_1 + \alpha_2,$$

where α_1 is due to extraneous frictional effects due to the motion of the mirror etc. in the gas, and α_2 is due to 'elastic hysteresis' in the suspension wire. It was found that α_1 was proportional to $\eta^{\frac{1}{2}}$ approximately.

Temperatures were measured by means of a platinum resistance thermometer.

The following method was used to find λ. Let a_1 be the first swing to the right, and b_1 the first to the left. Then $\lambda = \ln \frac{a_1}{b_1} = \ln k$ (say). Hence, with a notation that is self-explanatory,

$$\frac{a_n}{a_{n+15}} = k^{30} \quad \text{and} \quad \frac{b_n}{b_{n+15}} = k^{30}.$$

$$\therefore \quad \frac{a_n + b_n}{a_{n+15} + b_{n+15}} = k^{30},$$

so that by taking logarithms, λ can be found. This manner of taking the readings is most convenient, since it is not necessary to know the zero position.

Finally, the viscosity of any gas, or of air at any temperature other than at 0° C., was found from the equation

$$\frac{\eta}{\eta_{\text{air}}} = \frac{\lambda - \alpha}{\lambda_{\text{air}} - \alpha_{\text{air}}},$$

where the viscosity of air is taken to be that at the appropriate temperature.

OTHER METHODS FOR DETERMINING VISCOSITY

Determination of the viscosity of a fluid by the revolving disc method.—In this method, which is applicable both to liquids and gases, a thin circular disc of large diameter is mounted so that it may be rotated with constant angular velocity about a vertical axis through its centre and normal to its plane—cf. Fig. 11·27. At a little distance above there is supported by means of a torsion wire another disc, the planes of the discs being horizontal. The torsion wire coincides with the axis of rotation of the lower disc. When the lower disc is in motion a couple is exerted on the upper

disc so that this moves through a certain angle, θ, until the couple due to the viscous drag is equal to the restoring couple due to the resulting twist in the torsion wire.

In obtaining an expression for this couple it will be assumed that the angular velocity of any layer of the liquid is a linear function of the distance of that layer from the upper disc. For this to be

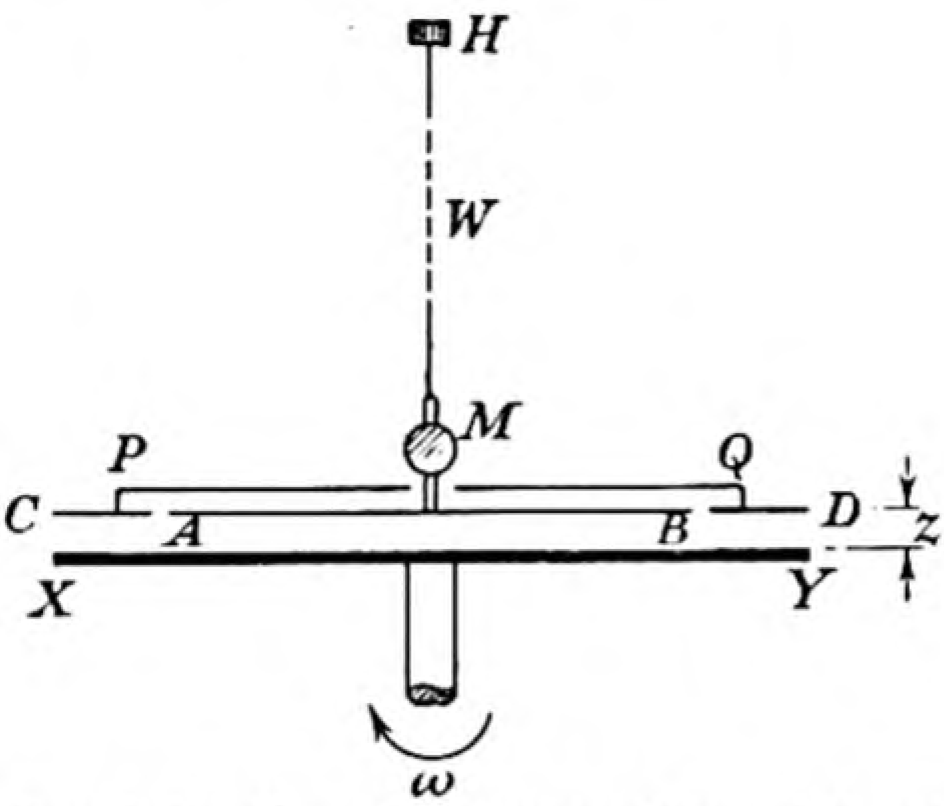


FIG. 11·27.—Revolving disc apparatus for determining the viscosity of a fluid.

fulfilled the disc would have to be infinite in diameter and the rotation infinitely slow. If z is the distance between the discs, ω the angular velocity of the rotating disc, the vertical velocity gradient at a distance r from the axis of rotation will be, under the above conditions, constant and equal to $\frac{r\omega}{z}$. Hence the elementary torque on that portion of the upper disc defined by adjacent rings of radii r and $r + \delta r$, is, cf. p. 580, given by

$$\delta\Gamma = \left[\oint \eta(\delta r\,.\,ds)\,\frac{rw}{z}\right].r = 2\pi r\,\delta r\,.\,\eta\,.\frac{r\omega}{z}\,.r = 2\pi r^3\,.\frac{\eta\omega}{z}\,.\,\delta r.$$

$$\therefore\ \text{Total couple on disc} = \Gamma = 2\pi\eta\,\frac{\omega}{z}\int_0^a r^3\,dr = \tfrac{1}{2}\pi\eta\,.\frac{\omega}{z}\,.a^4,$$

where a is the radius of the disc [Maxwell†].

In deriving the expression for the couple on the stationary disc in a revolving disc viscometer, it has been tacitly assumed that the velocity gradient is always given by $r\omega z^{-1}$, i.e. even up to the edge of the disc and that there is no drag arising from viscous forces on the upper surface of the stationary disc. To fulfil these requirements an apparatus is assembled as in Fig. 11·27. XY is the rotating disc; it is about 40 cm. in diameter and, for use with

† Collected Papers or *Phil. Trans.*, **156**, 249, 1866.

gases, makes one or two complete revolutions per second. About 3 mm. above this is the upper disc AB on which the torque, due to viscosity, is exerted. This disc nearly fills a circular aperture in a larger disc CD and thus the flow of fluid is practically the same as if AB and CD were one disc. In this way edge-effects are eliminated and the theory developed above applies. A shallow cap PQ rests on CD and protects the upper surface of AB from viscous drag. The upper disc is suspended, in the usual way from a torsion head H, by means of a fine wire W, and a mirror M, rigidly attached to the upper disc, permits an accurate evaluation of the angular deflexion of the disc to be made.

If b is the couple per unit twist in the suspension, then when the disc is in equilibrium.

$$\Gamma = b\theta,$$

or

$$\eta = \frac{2zb\theta}{\pi\omega a^4}.$$

Brillouin showed that the radius of the upper disc should be taken as

$$a + \gamma\left[1 - \frac{\pi}{8}.\frac{\gamma}{z}\right],$$

where γ, the clearance between the ring and suspended disc, is small compared with z.

Stokes' law.—NEWTON† established the fact that when a body is acted upon by a constant force and also experiences a resistance to its motion proportional to any power of its velocity, a stage is ultimately reached when that body attains a constant or terminal velocity; the force resisting the motion is then equal and opposite to the constant force applied to the body. STOKES first showed that when a sphere is moving slowly with uniform velocity through an infinite extent of viscous fluid, the resistance to the motion is $6\pi r\eta u$, where r is the radius of the sphere and u its terminal velocity. The equation is established mathematically in Lamb's Hydrodynamics; here we can only use the method of dimensional analysis to find the type of expression which will represent the force acting on the falling sphere. Thus, we write

$$\mathrm{F} = \mathrm{A}r^{\alpha}\eta^{\beta}u^{\gamma},$$

where A, α, β and γ are constants; it is only α, β and γ which are determinable by the method now used. Since

$$[\mathrm{F}] = [\mathrm{M}][\mathrm{L}][\mathrm{T}^{-2}], \quad [r] = [\mathrm{L}],$$
$$[\eta] = [\mathrm{M}][\mathrm{L}^{-1}][\mathrm{T}^{-1}], \quad \text{and} \quad [u] = [\mathrm{L}][\mathrm{T}^{-1}],$$

† *Principia*, Lib. II, Prop. viii, Cor. 2.

we have

$$MLT^{-2} = L^{\alpha}.M^{\beta}L^{-\beta}T^{-\beta}L^{\gamma}T^{-\gamma}.$$

Comparing indices, we and,

$$1 = \beta,$$

$$1 = \alpha - \beta + \gamma,$$

$$-2 = -\beta - \gamma.$$

$$\therefore \alpha = 1 \quad \text{and} \quad \gamma = 1.$$

Hence $$F = Ar\eta u,$$

and Stokes showed that when the extent of the fluid is limitless,

$$A = 6\pi.$$

$$\therefore F = 6\pi r\eta u.$$

The falling sphere.—When a sphere, falling through a viscous medium of unlimited extent, has reached its terminal velocity, u, the resistance to its motion has just been shown to be $6\pi r\eta u$, and this force must be equal and opposite to the effective weight of the sphere, i.e.

$$\tfrac{4}{3}\pi r^3(\rho - \sigma)g = 6\pi r\eta u,$$

or $$\eta = \frac{2}{9}.\frac{(\rho - \sigma)g}{u}.r^2,$$

where g is the intensity of gravity, ρ the density of the material of the sphere, and σ that of the fluid.

Perhaps the most important use to which this equation has been put was by MILLIKAN who applied it to find the size of the oil drops in his experimental determination of the electronic charge; a value for the viscosity of air at room temperature had therefore to be determined [cf. p. 582]. Our purpose at present, however, is to show how this equation may be used to determine the viscosity of a highly viscous fluid.

Experimental determination of the viscosity of very viscous oils at room temperature.—The expression

$$\eta = \frac{2}{9}.\frac{(\rho - \sigma)g}{u}.r^2,$$

shows that if the terminal velocity acquired by a sphere falling through a viscous medium can be measured, we have a means of determining the coefficient of viscosity of the medium, for all other quantities, except η, are easily ascertained.

Let us suppose that castor oil is the liquid whose viscosity is to be determined. This is placed in a glass cylinder, A, Fig. 11·28, about

70 cm. long and 10 cm. wide. Spheres of known diameter are dropped into the liquid. The terminal velocity is deduced from observations on the time required for the sphere to travel between two fiducial marks. Now the liquid is limited by the walls of the vessel and has a finite depth. The conditions stipulated by the above theory are therefore not fulfilled. It may be shown, however, that if the sphere falls between two fiducial marks B_1 and B_2 (10 cm. from the top and bottom of the liquid respectively), then the motion is uniform. Further, if the diameter of the spheres does not exceed 0·2 cm. and a vessel 10 cm. wide is used, no correction is necessary for the effect of the walls of the vessel. If λ is the distance between the fiducial marks, and t the time of transit,

$$\eta = \frac{2}{9} . r^2 g \left(\frac{\rho - \sigma}{\lambda} \right) t,$$

so that $r^2 t$ is constant for a given liquid at a constant temperature. If therefore r^2 is plotted against $\frac{1}{t}$ a straight line should be obtained if the conditions of the theory have been satisfactorily fulfilled. From the slope of the line η may be deduced.

For the experiment use steel spheres † of diameters 2, 3, 4, 5 thirty-seconds of an inch. Note the value of the diameter of each sphere in centimetres, using a micrometer screw gauge. Place three or four spheres of each size in a little oil in a watch-glass and roll them about until their surfaces are thoroughly wetted by the oil, and then transfer one on the point of a penknife to the top of a short glass tube, about 0·5 cm. in diameter, placed just above the oil, the axis of this tube coinciding with that of the vessel, B. In this way the sphere is caused to fall axially through the oil in the wide vessel. Set the chronometer going when the sphere reaches the first fiducial mark (a piece of black cotton wound once round the vessel), and observe the time of transit between B_1 and B_2. When this has been repeated for all the spheres available, a plot of r^2 against t^{-1} should give a straight line from the slope of which η for castor oil may be deduced.

As the viscosity of an oil changes very rapidly with temperature, the latter should be read to within 0·1 deg. C.; it is for this reason also that the diameters of all the balls are measured first and the fall experiments carried out one after the other as quickly as possible. A few experiments should also be made with very small spheres of Rose's metal [Tin, 1; lead, 1; bismuth, 2: M.P. 94·5° C.]; their diameters should be measured with the aid of a microscope having an eye-piece scale which must be calibrated. With these spheres, observations can be made of the time of fall through successive short distances (5 cm.) and the constancy of the limiting velocity verified.

† Such as are used in ball-bearings.

In practice two important corrections to the theory of the falling-sphere given above have to be considered. The viscous drag $6\pi r\eta u$ was calculated by Stokes for a sphere falling under gravity in an infinite ocean of liquid. The corrections are due to the

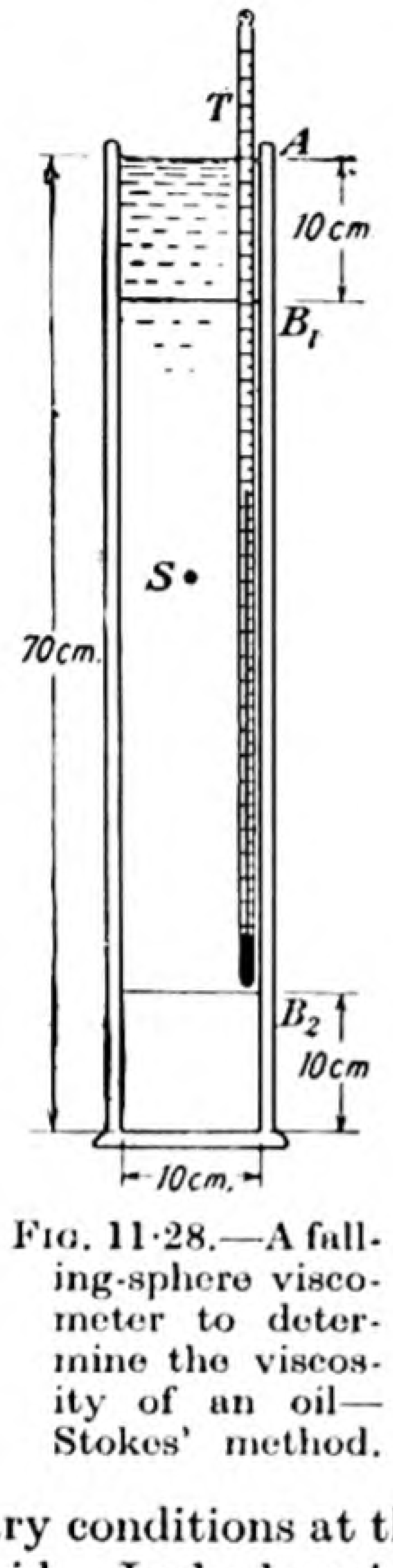


FIG. 11·28.—A falling-sphere viscometer to determine the viscosity of an oil—Stokes' method.

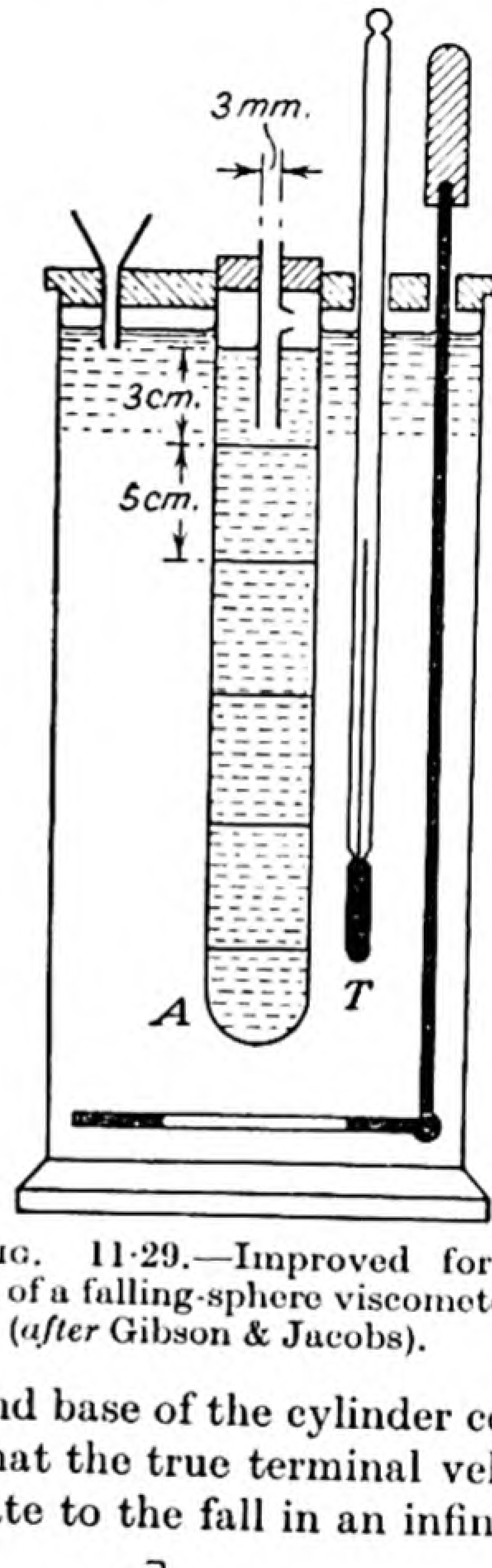


FIG. 11·29.—Improved form of a falling-sphere viscometer (*after* Gibson & Jacobs).

boundary conditions at the walls and base of the cylinder containing the liquid. Ladenburg† showed that the true terminal velocity u_∞, i.e. the terminal velocity appropriate to the fall in an infinite ocean of fluid was given by

$$u_\infty = u\left[1 + 2{\cdot}4\frac{r}{\mathrm{R}}\right],$$

where u is the observed velocity and R is the radius of cross-section of the cylindrical vessel containing the fluid. No satisfactory agreement appears to have been reached concerning the correction for the fact that the height of the cylinder is not infinite, but the correction is very small.

An improved form of falling-sphere viscometer due to Gibson and Jacobs is shown in Fig. 11·29. No description appears necessary.

† *Ann. der Phys.*, **22**, 287, 1907; 447, 1907.

Experimental method for investigating the Ladenburg correction.—This method is due to FLOOD† and is as follows. Neglecting all corrections, the equation for the steady motion under gravity of a sphere in a fluid is

$$u = \frac{2}{9} \cdot \frac{(\rho - \sigma)gr^2}{\eta}.$$

It therefore follows that if spheres of different radii are allowed to fall through the liquid and ρ, σ and η remain unchanged, r^2u^{-1} is constant. To test this a glass tube, about 75 cm. long and 4·5 cm. in diameter, was provided with a bung at its lower end, clamped in a vertical position, nearly filled with glycerine, (a Visco-static or silicone lubricating oil should be used to-day), and then left for all air bubbles to escape. Two horizontal marks, about 10 cm. from each end of the tube, were made and steel spheres ranging in diameter from $\frac{1}{16}$ in. to $\frac{1}{4}$ in. allowed to fall axially down the tube. The velocity of fall in each instance was determined. When r^2 was plotted against u, the graph exhibited a slight but systematic curvature. This indicated that a small correction term was necessary and Flood therefore suggested that

$$\frac{r^2}{u} = \frac{9\eta}{2(\rho - \sigma)g} \cdot f\left(\frac{r}{R}\right),$$

where $f\left(\frac{r}{R}\right)$ is an undetermined non-dimensional function. If

$$f\left(\frac{r}{R}\right) = a_0 + a_1\left(\frac{r}{R}\right) + a_2\left(\frac{r}{R}\right)^2 + \ldots,$$

then $a_0 = 1$, since when $\frac{r}{R} \to 0$, $f\left(\frac{r}{R}\right)$ must be unity. Let us assume $a_2 = a_3 = \ldots = 0$. Calling $\frac{9\eta}{2(\rho - \sigma)g} = \kappa$, we then have

$$\frac{r^2}{u} = \kappa + \kappa a\left(\frac{r}{R}\right),$$

where a is now written for a_1. If our assumptions are legitimate, the graph obtained by plotting $\frac{r^2}{u}$ against $\left(\frac{r}{R}\right)$ should be linear. This was verified and a was shown to be 2·33, which is in good agreement with Ladenburg's value 2·4.

An experimental study of the motion of steel spheres falling through a highly viscous medium contained in a vertical tube.—The object of this experiment is to investigate how the

† Flood, *Sch. Sci. Rev.*, No. 57, Oct. 1933.

terminal velocity of the falling sphere depends upon the diameter of the sphere when this varies from a very small value to one nearly equal to that of the cross-section of the fall-tube itself; also to obtain a value for the viscosity of the liquid. To ensure that the spheres fall with their centres along the axis of the tube it is desirable to release them from the same position by means of an electromagnet. When an attempt to do this is made the fact emerges that the spheres,

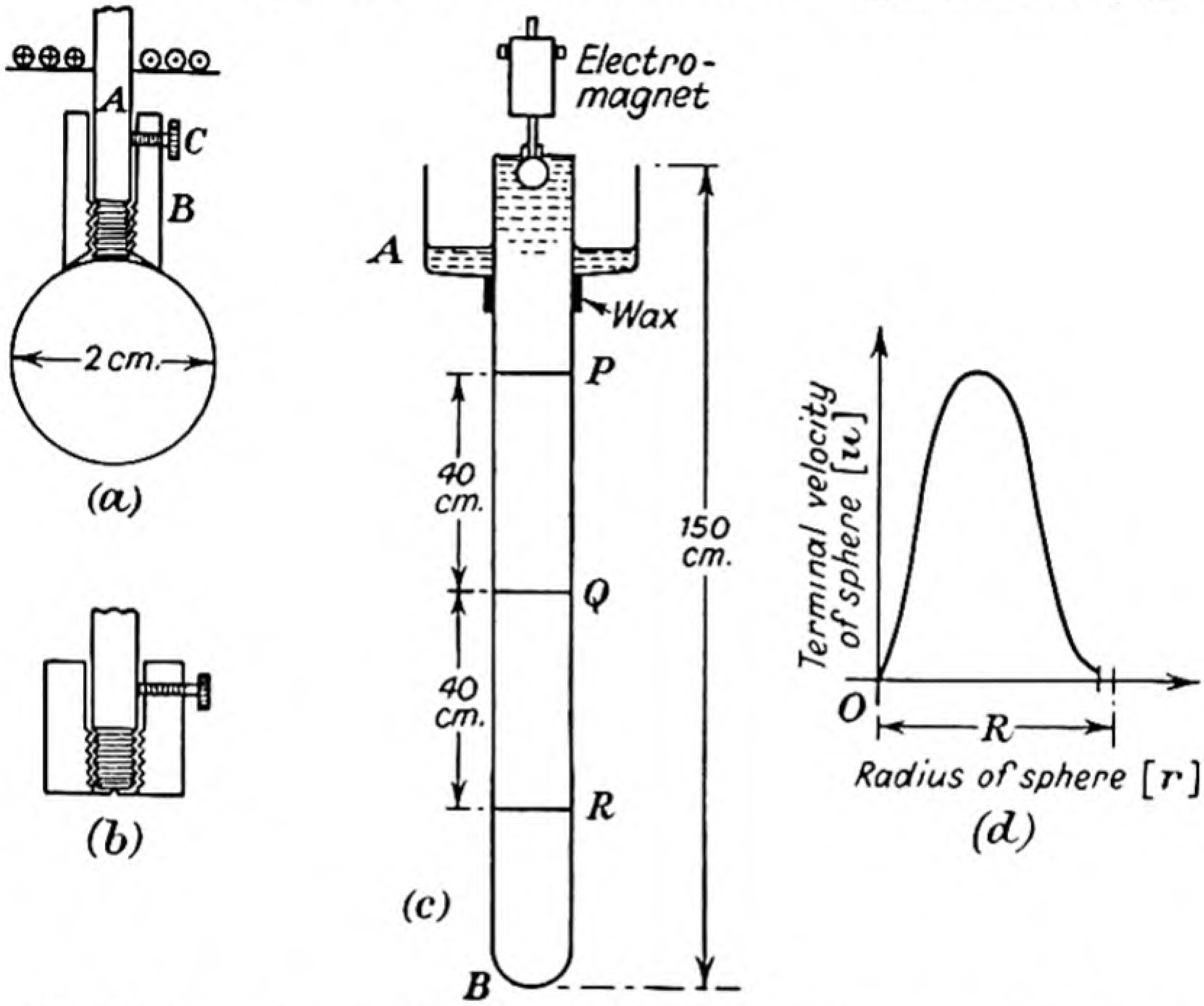


FIG. 11·30.—Experimental investigation of the motion of a sphere along the axis of a vertical tube filled with oil.

except the larger ones, remain attached to the electromagnet even when the exciting current is made zero. To overcome this it is necessary to provide a non-magnetic gap between the sphere and the core of the electromagnet and it is soon found that the gap must be variable. Fig. 11·30(a) shows the lower end of the electromagnet and its attachments. The end, A, of the core is screwed and on this moves a brass collar, B; C is a clamping screw. The lower end of the brass collar is made conical and for the larger spheres, say those greater than 0·3 mm. diameter, is very satisfactory. In general it is necessary to adjust the gap for each size of sphere used.

For the smaller spheres another brass collar is made and this is provided with a flat end to which a very thin piece of gun-metal sheet is soldered, cf. Fig. 11·30(b). The gun-metal is slightly depressed at its midpoint so that the spheres may be made to fall

axially along the vertical column of liquid, although for these spheres this criterion is not quite so essential as for the larger ones.

The glass tube used in this experiment is about 1·5 metres long and 2·5 cm. in internal diameter. It is shown in Fig. 11·30(c) and is provided with a brass collar, A. This is required since it is necessary to fill the fall-tube B completely with liquid (a Viscostatic or silicone oil). This is so because the spheres must be totally covered with liquid, or very nearly so, before they are released from the electromagnet. If this precaution is not taken a large air bubble often clings to the falling sphere when an interpretation of the motion becomes exceedingly difficult, if not impossible.

P, Q and R are three marks on the tube and are such that PQ = QR. These portions of the liquid column are the more central and by timing the motion of the spheres across these distances one is able to conclude with certainty whether or not the terminal velocity has been attained by each sphere used.

Fig. 11·30(d) shows a typical curve obtained by plotting the terminal velocity u against r, the radius of the sphere. If, as shown on p. 596, a plot of $\frac{r^2}{u}$ against r is made, a value for the Ladenburg correction is obtained.

A test should also be made to examine the validity of assuming whether or not $f\left(\frac{r}{R}\right)$ may not take the form exp $\left[\gamma\left(\frac{r}{R}\right)\right]$ where γ is a constant. Such a series may be applicable since

$$\exp \gamma x = 1 + \gamma x + \tfrac{1}{2}\gamma^2 x^2 + \text{higher terms}.$$

γ may be found by plotting $\log\left(\frac{r^2}{u}\right)$ against $\left(\frac{r}{R}\right)$.

If a value for κ has been found independently, another test may be made to find the constants α_1 and α_2 in the equation

$$\frac{r^2}{u} = \kappa\left[1 + \alpha_1\left(\frac{r}{R}\right) + \alpha_2\left(\frac{r}{R}\right)^2\right].$$

This may be written

$$\left(\frac{r^2}{u\kappa} - 1\right) = \alpha_1\left(\frac{r}{R}\right) + \alpha_2\left(\frac{r}{R}\right)^2,$$

or
$$\left(\frac{r^2}{u\kappa} - 1\right)\frac{R}{r} = \alpha_1 + \alpha_2\left(\frac{r}{R}\right).$$

A graph of $\left(\frac{r^2}{u\kappa} - 1\right)\frac{R}{r}$ against $\left(\frac{r}{R}\right)$ is then plotted; it is linear

over the range where this formula is applicable. It is found that this equation does actually represent the motion of the spheres over a larger range of diameters than is given by Ladenburg's formula.

Another formula†, which takes account of the finite width of the jar containing the liquid through which a sphere falls, is

$$u_\infty = u_0\left(1 - \frac{r}{R}\right)^{-2\cdot 4}.$$

This becomes

$$u_\infty = u\left(1 + 2\cdot 4\,\frac{r}{R}\right)$$

when $\frac{r}{R}$ is small and is found to fit in with experimental results very well for large values of $\frac{r}{R}$. An interesting feature of this equation is that the factor $\left(1 - \frac{r}{R}\right)^{-2\cdot 4}$ becomes infinite as $\frac{r}{R} \to 1$; this is what would be expected. To test the formula it may be written

$$u_\infty = u\left(1 - \frac{r}{R}\right)^{\beta},$$

where β is a constant, when a plot of log u against log $\left(1 - \frac{r}{R}\right)$ should be a straight line with a slope β.

Example.—Small uniform globules of a heavy liquid fall from rest at regular intervals of time from a fine jet placed just within the surface of a lighter viscous liquid at rest in a tall jar. How will the globules be distributed at a given instant?

If m is the effective mass of the falling drop, we have, with the usual notation

$$m\ddot{x} = mg - \mu\dot{x} \qquad \text{or} \qquad \ddot{x} + \alpha\dot{x} = g,$$

where $\alpha = \mu m^{-1}$.

For the complementary function

$$\lambda(\lambda + \alpha) = 0.$$

$$\therefore\ x = A + B\exp(-\alpha t),$$

where A and B are constants.

For the particular integral

$$x = \frac{g}{D^2 + \alpha D} = \frac{1}{\alpha}\left(\frac{1}{D} - \frac{1}{\alpha} + \frac{D}{\alpha^2} - \dots\right)g = \frac{gt}{\alpha} - \frac{g}{\alpha^2}.$$

$$\therefore\ x = A + B\Big[\exp(-\alpha t)\Big] + \frac{gt}{\alpha} - \frac{g}{\alpha^2}.$$

Now at $t = 0$, $x = 0$ and $\dot{x} = 0$, so that $A = 0$ and $B = \frac{g}{\alpha^2}$.

$$\therefore\ x = \frac{g}{\alpha^2}\Big[\{(\exp(-\alpha t)\} - 1 + \alpha t\Big].$$

† Suggested in a private communication from Dr. A. R. Stokes.

Let the first drop have fallen for t seconds; then if τ is the time interval between successive drops breaking away from the fine jet and x_n is the distance traversed by the nth drop,

$$x_n = \frac{g}{\alpha^2}\Big[\exp\{-\alpha(t - \overline{n-1}\tau)\} - 1 + \alpha(t - \overline{n-1}\tau)\Big].$$

Hence for drops which have been falling for a considerable time,

$$x_1 - x_2 = \frac{g}{\alpha^2}\alpha\tau = \frac{g\tau}{\alpha},$$

$$x_2 - x_3 = \qquad = \frac{g\tau}{\alpha}, \text{ etc.}$$

Thus such drops are equally spaced.

Rectilinear motion in a resisting medium.—If a mass m be supposed to move without rotation in a straight line in a resisting medium, the resistance is a function of the velocity of the body. If $\phi(v)$ is the resistance and F the external force acting along the line of motion, the equation of motion is

$$m\dot{v} = \mathrm{F} - \phi(v).$$

Now, with Newton, it is usual to assume that $\phi(v) = kv^2$, where k is a constant depending upon the density of the medium and on the area of the greatest section of the body taken normal to the direction of motion. Hence

$$m\dot{v} = \mathrm{F} - kv^2 = k(\mathrm{V}^2 - v^2),$$

if we assume F to be constant and equal to $k\mathrm{V}^2$. If the initial velocity is less than V, the velocity will continue to increase until it reaches a value V. It is for this reason that V is called the terminal velocity of the body.

Integrating the above equation we get

$$t = \frac{m\mathrm{V}}{2\mathrm{F}} \ln\left(\frac{\mathrm{V}+v}{\mathrm{V}-v}\right),$$

no integration constant being added if t is measured from the instant when the body is at rest.

An interesting and important instance of such a motion is that of a body projected vertically downwards from rest. Then $\mathrm{F} = mg$, and we have

$$t = \frac{\mathrm{V}}{2g} \ln\left(\frac{\mathrm{V}+v}{\mathrm{V}-v}\right),$$

which gives $$v = \mathrm{V}\tanh\frac{gt}{\mathrm{V}}.$$

Since $v = \dfrac{dx}{dt}$, we find $x = \dfrac{\mathrm{V}^2}{g}\ln\cosh\dfrac{gt}{\mathrm{V}}$.

For a body projected vertically upwards the equation of motion is

$$\frac{dv}{dt} = -g\left(1 + \frac{v^2}{V^2}\right).$$

If V_0 is the initial velocity, we find

$$t = \frac{V}{g}\left(\tan^{-1}\frac{V_0}{V} - \tan^{-1}\frac{v}{V}\right).$$

Also, since $v = 0$ at the highest point, the time of ascent to that point is given by

$$\frac{V}{g}\tan^{-1}\frac{V_0}{V}.$$

If z is the distance traversed, this being measured upwards from the point of projection, we have

$$v = \frac{dz}{dt} = \frac{dz}{dv}\cdot\frac{dv}{dt} = \frac{dz}{dv}\left[-g\left(\frac{v^2 + V^2}{V^2}\right)\right].$$

$$\therefore\ z = \frac{V^2}{2g}\ln\left(\frac{V_0{}^2 + V^2}{v^2 + V^2}\right).$$

If Z is the height of ascent, we get, since v is then zero,

$$Z = \frac{V^2}{2g}\ln\left(\frac{V_0{}^2 + V^2}{V^2}\right).$$

Example.—The resistance to the motion of a ship in still water may be assumed to be kv^3, where v is the speed and k a constant. Owing to small changes in thrust due to the propeller, the speed at time t is given by $v = a + b\cos\omega t$, where a, b and ω are constants. If m is the mass of the ship, find the thrust of the propellers at any instant and the power exerted. Compare the average power with that necessary when the speed of the ship is constant, i.e. $b \to 0$.

The thrust at any instant is given by

$$F = m\dot{v} + kv^3.$$

Thus

$$F = -m\omega b\sin\omega t + k(a + b\cos\omega t)^3,$$

so that the power supplied is given by

$$P = Fv = -m\omega b\sin\omega t(a + b\cos\omega t) + k(a + b\cos\omega t)^4.$$

If this is expanded we get terms in $\sin\omega t$, $\sin\omega t\cos\omega t$, $\cos\omega t$, $\cos^2\omega t$, $\cos^3\omega t$ and $\cos^4\omega t$. The mean values of $\sin\omega t$, $\sin\omega t\cos\omega t$, $\cos\omega t$ and $\cos^3\omega t$, taken over any number of complete cycles, are all zero, while

$$\overline{\cos^2\omega t} = \overline{\tfrac{1}{2} + \tfrac{1}{2}\cos 2\omega t} = \tfrac{1}{2},$$

and

$$\overline{\cos^4\omega t} = \overline{(\tfrac{1}{2} + \tfrac{1}{2}\cos 2\omega t)^2}$$
$$= \overline{\tfrac{1}{4} + \tfrac{1}{2}\cos 2\omega t + \tfrac{1}{4}\cos^2\omega t} = \tfrac{1}{4} + \tfrac{1}{8} = \tfrac{3}{8}.$$

Hence

$$\bar{P} = ka^4 + \tfrac{1}{2}(6ka^2b^2) + kb^4(\tfrac{3}{8})$$
$$= k(a^4 + 3a^2b^2 + \tfrac{3}{8}b^4) > ka^4.$$

The viscosity of pitch.—Experimental determinations of the viscosity of pitch, paraffin wax, and glass were made by TROUTON and ANDREWS.† In each experiment a constant torque was applied to one end of a uniform cylinder of the substance whose viscosity was to be determined, and the relative motion of the ends of the cylinder observed—actually, one end was fixed. From these

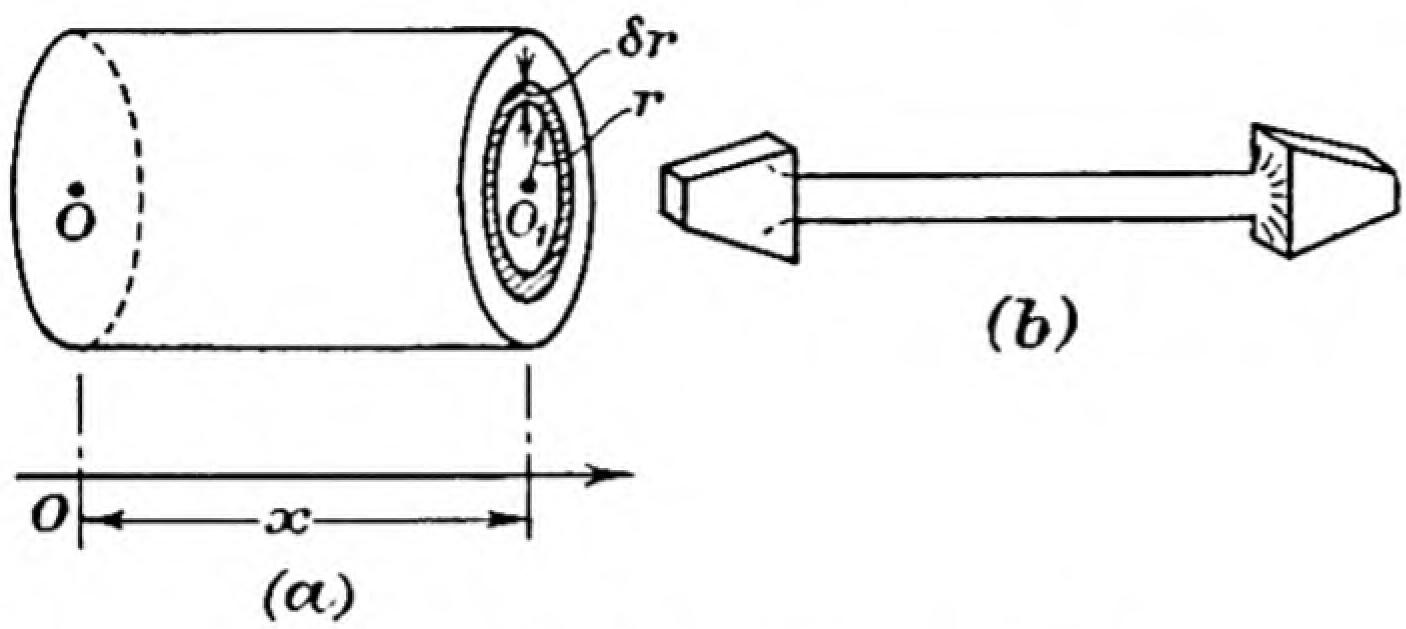


FIG. 11·31.—The viscosity of pitch.

observations, the diameter and length of the rod, and the magnitude of the applied torque, the viscosity, η, could be calculated from the equation we proceed to establish.

Consider a normal cross-section of the cylinder at a distance x from its fixed end—cf. Fig. 11·31(a). Let δF be the tangential force exerted on that portion of the cross-section defined by circles of radii r and $r + \delta r$. Then, from Newton's hypothesis,

$$\delta F = \eta\,(\text{area})(\text{velocity gradient}), = \eta(\delta r\,.\,\delta s)\,\frac{\partial u}{\partial x}. \qquad \text{[cf. p. 580]}$$

Since this gives rise to a torque equal to $\eta r(\delta r\,.\,\delta s)\,\dfrac{\partial u}{\partial x}$ for the element δs of the annulus defined by r and $r + \delta r$, the torque on the annulus is given by

$$\delta\Gamma = \eta r\,\delta r\left[\oint ds\right]\frac{\partial u}{\partial x}.$$

Now $u = r\omega$, where ω is the angular velocity at any point in the section considered. At a plane determined by $x + \delta x$, and for an element defined by r and $r + \delta r$,

$$u + \delta u = r\left(\omega + \frac{\partial\omega}{\partial x}\,.\,\delta x\right),$$

$$\therefore\ \frac{\partial u}{\partial x} = r\,.\,\frac{\partial\omega}{\partial x}.$$

† Trouton and Andrews, *Proc. Phys. Soc.*, **19**, 47, 1903.

$$\therefore \ \delta\Gamma = 2\pi\eta . r^3 . \frac{\partial\omega}{\partial x} . \delta r. \qquad \left[\because \oint ds = 2\pi r.\right]$$

The torque Γ on any normal cross-section of the cylinder is therefore given by

$$\Gamma = \int_0^a 2\pi\eta r^3 . \frac{\partial\omega}{\partial x} . dr = \tfrac{1}{2}\pi\eta a^4 . \frac{\partial\omega}{\partial x},$$

where a is the outer radius of the cylinder. This couple is equal to the applied couple. If $\bar{\omega} = \dfrac{\partial\omega}{\partial x}$ = the relative angular velocity per unit length of the cylinder,

$$\eta = \frac{2\Gamma}{\pi\bar{\omega} . a^4}.$$

For substances such as pitch the material was cast in the form shown in Fig. 11·31(b) so that its ends could be fixed firmly in specially designed clamps. For glass, available in the form of a tube,

$$\eta = \frac{2\Gamma}{\pi\bar{\omega}(r_1{}^4 - r_2{}^4)}. \qquad [(r_1 > r_2).]$$

In developing this method it must be noted that Trouton and Andrews assumed from considerations of symmetry, that any two planes in the cylinder, normal to its axis, moved over each other about the common axis and that they remained plane.

To apply the couple to the specimen the following method was used. A horizontal shaft, turning freely on anti-friction wheels, had attached to it a pulley; a cord passed round a groove in this pulley and carried a load. The shaft carried a square socket for the purpose of gripping the squared end of the cylinder of the substance which was made to fit exactly. A similar but fixed socket prevented the other end of the cylinder from turning. The rate of rotation was determined by observing the motion of a pointer over a circular scale mounted on the shaft.

The method lends itself to the determination of the coefficient of viscosity of a substance at different temperatures, since the cylindrical specimen can be conveniently surrounded by a jacket and kept at any required temperature.

In discussing this work by Trouton and Andrews, APPLEYARD suggested, in order to avoid end-effects, that two rotating index pointers should be used and that these should be attached, at a known distance apart, to the specimen.

The following results were obtained:—

Substance	Temperature	Viscosity (gm.cm.$^{-1}$sec.$^{-1}$)
Pitch	0°C	$5\cdot1 \times 10^{11}$
	8°C	$9\cdot9 \times 10^{10}$
	15°C	$1\cdot3 \times 10^{10}$
Soda glass	575°C	$1\cdot1 \times 10^{13}$
	660°C	$2\cdot3 \times 10^{11}$
	710°C	$4\cdot5 \times 10^{10}$

Viscosity and its variation with temperature.—(*a*) ***Liquids:*** The variation of viscosity with temperature is so marked in the case of liquids that a statement of the value of the viscosity of a certain liquid is almost meaningless unless the temperature is stated explicitly. The viscosity of water, for example, changes from 0·0101 gm.cm.$^{-1}$sec.$^{-1}$ at 20° C. to 0·0047 gm.cm.$^{-1}$sec.$^{-1}$ at 60° C.; for castor oil an increase in temperature from 10° C. to 20° C. causes a decrease in its viscosity from 24·18 poise. to 9·86 poise. Many empirical formulae have been suggested to represent such a variation of viscosity with temperature, viz.

$$\eta_\theta = \eta_0(1 + \alpha\theta + \beta\theta^2)^{-1}, \quad \text{or} \quad \eta_\theta = \eta_0(1 + b\theta)^{-n},$$

or

$$\log \eta = A + \frac{B}{T},$$

where η_0 is the viscosity at θ° C., T is the absolute temperature, and $\eta_0\alpha$, β, b, A and B are constants.

Each is only an approximation and the second equation is not true for oils. In 1934, ANDRADE,* on a theoretical basis, suggested

$$\eta\rho^{-\frac{1}{3}} = A \exp\left(\frac{c\rho}{T}\right),$$

where ρ is the density, T the absolute temperature, and A and c are constants. This is in good agreement with experiment except in the case of water and some alcohols. SPELLS, working in Andrade's laboratory, found that for gallium, which is liquid at 30° C. and has a boiling point above 1600° C., and whose viscosity was measured over the range 52° C.–1100° C.,

$$\eta\rho^{-\frac{1}{3}} = (2\cdot468 \times 10^{-3}) \exp \frac{79\cdot05\rho}{T},$$

and this agrees with observed values of η to within three per cent.

* Andrade, *Phil. Mag.*, **17**, 497 and 698, 1934.

(*b*) Unlike liquids, gases show an increase in viscosity with temperature. The simple kinetic theory suggests that the variation of η, with the absolute temperature T, is given by

$$\eta = \kappa T^{\frac{1}{2}},$$

where κ is a constant.† In 1893 Sutherland,‡ taking into account the fact that there is a small finite force of attraction between neighbouring molecules, so that molecular collisions are more abundant than in the case of an ideal gas at the same temperature, showed that

$$\eta = \eta_0 . \frac{aT^{\frac{1}{2}}}{1 + \frac{S}{T}},$$

where η_0 is the viscosity at 0° C., while a and S are constants. S is known as ***Sutherland's constant***. For moderate ranges in temperature this formula is applicable to many gases but when observations are made over the full range of temperature which can now be explored in the laboratory the formula fails both at high and low temperatures.

EXAMPLES XI

11·01. Describe and give the theory of *either* the capillary tube *or* the coaxial cylinder method of measuring the viscosity of a liquid.

State the conditions which must be satisfied for the theory to apply.

11·02. In an experiment with Poiseuille's apparatus the following observations were recorded:—volume of water issuing in 10·0 minutes 80·7 cm.^{3}, head of water 39·1 cm., length of tube 60·24 cm., mean internal radius of tube 0·0523 cm. Obtain a value for the viscosity of water at the temperature at which the experiment was carried out.

[0·0139 gm. cm.$^{-1}$.sec.$^{-1}$]

11·03. A capillary tube of length l and of radius a is fixed vertically through the base of an upright vessel. Water in the vessel is maintained at a depth h while it flows out through the tube. Find the rate of flow through the tube in cubic centimetres per sec., assuming that the water leaves the tube at a small velocity, the length of the tube in the water being l'.

Explain how this result can be applied in viscosity measurements.

(G)

11·04. A cylindrical vessel of constant cross-sectional area A is filled with liquid to a depth h above an orifice in its side. A horizontal capillary tube of length l and mean internal radius a is attached to the orifice. Show that the volume of liquid in the vessel and above the side orifice will be halved in time $A.\frac{8\eta l}{\pi a^4 g\rho}.\ln 2$ where η is the viscosity of the liquid, g the intensity of gravity and ρ the density of the liquid. [*N.B.* This time is independent of h—it depends only on the 'halving' of the initial head.]

† cf. Vol. II.

‡ Sutherland, *Phil. Mag.*, **36**, 507, 1893.

11·05. Water ($\eta = 0{\cdot}010$ gm.cm.$^{-1}$sec.$^{-1}$) is escaping from a tank through a horizontal capillary tube 50 cm. long and 0·024 cm. in diameter. What is the rate of flow of water from the tank when the level in it is 50 cm. above that of the tube? How long will it take for the water level to fall a further 10·0 cm., if the cross-sectional area of the tank is everywhere 100 cm.^{2}? [$7{\cdot}98 \times 10^{-4}$ cm.^{3}sec.$^{-1}$; $7{\cdot}40 \times 10^6$ sec.]

11·06. In an experiment to determine the viscosity of hydrogen at 20·6° C., as described on p. 564, the capillary tube was 48·5 cm long and had a mean diameter of 0·0323 cm. The excess pressure was 11·07 cm. of oil, density 0·870 gm.cm.$^{-3}$ and the current 0·2244 A. Assuming the electrochemical equivalent of hydrogen to be $1{\cdot}045 \times 10^{-5}$ gm.-coulomb.$^{-1}$, and that one mole of hydrogen occupies 22·24 litres at s.t.p., obtain a value for the viscosity of hydrogen at 20·6° C. [Atmospheric pressure 75·4 cm. of mercury.] [$9{\cdot}26 \times 10^{-5}$ poise.]

11·07. Describe how you would investigate the variation of the viscosity of water with temperature over the range 0° C. to 100° C., deducing any necessary formula.

Explain the general nature of the variation and how it compares with the variation of the viscosity of air with temperature over the same range.

11·08. Define the terms 'viscosity' and 'steady flow'. Describe how you would determine the viscosity of tap water, drawing special attention to the precautions you would take to ensure an accurate result.

11·09. Assuming the usual formula for the flow of a liquid through a capillary tube, derive the corresponding formula for the case of a gas. Indicate how you would carry out an experimental investigation of η for hydrogen by this method.

11·10. The space between two horizontal discs is filled with a viscous fluid. Find the couple on a circular area at the centre of one of the discs when the other is maintained in a state of steady rotation.

11·11. Two coaxial cylinders, with guard rings, are set up as by Millikan for the determination of the viscosity of air at room temperature. The inner cylinder—without the guard rings—is 20 cm. long and is kept at rest in the usual way. It experiences a couple of 3000 dyne.-cm. when the outer cylinder rotates 10 times per second. If the radius of cross-section of the inner cylinder is 5·00 cm. and the clearance between it and the outer cylinder 0·062 cm., obtain a value for the viscosity of air. [$1{\cdot}85 \times 10^{-4}$ poise.]

11·12. Liquid flows in stream line motion in a horizontal tube of length l and internal radius a. Derive an expression for the volume passing in unit time due to an effective pressure difference p between the ends of the tube. Show that if a cylindrical coaxial rod of radius $a/2$ and length l is inserted into the tube the rate of flow is reduced by about 88 per cent for the same pressure difference between the ends. (S)

11·13. Glacial ice is forced through a horizontal circular cylindrical tunnel of radius and length 10 metres. Assuming that it flows according to the simple law of viscosity, if the coefficient of viscosity of ice is $1{\cdot}2 \times 10^{14}$ c.g.s. units, its density 0·9 gm.cm.$^{-3}$, and its latent heat of fusion 80 cal.gm.$^{-1}$, find the rate of supply of heat at the outlet of the tunnel if the ice melts as fast as it flows through. The pressure difference between the ends of the tunnel is 10^8 dyne.cm.$^{-2}$. Derive the formula used in the calculation. (S) [$2{\cdot}4 \times 10^4$ cal.sec.$^{-1}$]

11·14. Explain how the viscosity of a liquid can be measured by means of the rate of flow through a narrow tube and derive the expression for this rate in terms of suitable quantities.

A narrow tube of radius a and of length l is connected horizontally to a tank of uniform horizontal cross-section A. Water in the tank has initially a depth H and after an interval t it falls to a depth h. The viscosity of water being η, determine the relation between h and t assuming conditions of slow, steady flow. (G)

11·15. Define ***coefficient of viscosity*** and find its dimensions.

Use the method of dimensions to investigate the viscous resistance R to the motion of a sphere of radius r moving in a fluid of viscosity η and density ρ with velocity v, supposing R to be proportional to v^n.

Discuss the significance of the result and any practical importance it may have when (a) $n = 1$, (b) $n = 2$. (G)

11·16. The air in a large vessel, which is closed except for a long capillary tube of uniform cross-section, escapes to the outer atmosphere while the pressure inside the vessel exceeds the atmospheric pressure, p_0. Show that the time required for the pressure in the vessel to fall from p_1 to p_2 is

$$t = \frac{8\eta l V}{\pi a^4 p_0} \ln \frac{(p_2 + p_0)(p_1 - p_0)}{(p_1 + p_0)(p_2 - p_0)},$$

where V is the volume of the vessel, η the coefficient of viscosity of air and a is the radius of the capillary tube. Describe a known method (or suggest one) for finding the viscosity of air by the application of this formula. (S)

11·17. Water, with a coefficient of viscosity η, flows through a horizontal capillary tube of radius a, under a constant pressure head p. If the length of the tube is l, determine the volume delivered from the tube per sec. The correction for the velocity of exit may be neglected.

Explain how the formula is applied in the determination of the coefficient of viscosity of water.

If the pressure, instead of being constant, is maintained by a head of water in a vessel of cross-section A from the base of which the capillary tube projects horizontally, how long does it take for the water to fall from a depth h_1 to a depth h_2? (G)

11·18. Explain the statement, *the viscosity of water is* 0·0101 *poise at* 20° C.

Using the method of dimensional analysis, show how the resistance to motion of a sphere moving through a fluid varies with its velocity. What happens when the velocity becomes large?

Explain how the result can be applied to determine the diameter of a small drop of oil falling, under gravity, through a gas.

11·19. State the quantitative meaning of the term ***viscosity*** and describe how the viscosity of a mobile liquid may be determined by measuring its rate of flow through a capillary tube.

Derive any formula required in the calculation. (G)

11·20. Describe and give the theory of a method for determining the viscosity of a vapour.

11·21. What is meant by the ***dimensions*** of a physical quantity? Use the method of dimensions to show how (a) the volume of a liquid flowing in unit time through a tube of circular cross-section depends on the radius of the tube, the pressure gradient along the tube and the viscosity of the liquid, (b) the weight of a liquid drop depends on the radius of the orifice from which it falls and the density and surface tension of the liquid. Assume in each case that only the variables mentioned need be considered.

How would you investigate the result obtained in (b)? (G)

11·22. An evacuated vessel carries a capillary tube which is sealed into one of its sides and through which it can be filled. If the vessel is exhausted to a very low pressure and one end of the tube is opened to the atmosphere, how long does it take for the vessel to become full of air at half the pressure of the atmosphere? Work out the formula required to obtain this result, introducing the necessary data.

Describe a method suitable for determining the viscosity of air. (S)

11·23. State the assumptions which are made in deducing Stokes' law in regard to the fall of a small sphere through a viscous medium, and give an account of the experiments which have indicated the limitations of this law.

11·24. A steel sphere, diameter 0·30 cm., falls through glycerine and covers a distance of 25 cm. in 10 sec. The densities of steel and glycerine may be taken as 7·8 and 1·26 $gm.cm.^{-3}$ respectively. Obtain a value for the viscosity of glycerine.

11·25. Powdered chalk contains particles of various sizes. If some chalk is shaken up with water and allowed to stand, find a value for the diameter of the largest particles that remain in suspension 50 minutes later, assuming that the depth of the water is 12 cm., the density of chalk is 2·7 $gm.cm.^{-3}$, and the viscosity of water at the temperature prevailing is 0·010 $gm.cm.^{-1}sec.^{-1}$. [$2{\cdot}66 \times 10^{-4}$ cm.]

11·26. State Newton's law of viscosity and discuss any mechanism or theory which has been put forward to account for this law. Derive an expression for the torque necessary to maintain a circular cylinder in rotation about its axis in an infinite fluid. (S)

11·27. Describe an experiment to determine the variation of the viscosity of a liquid with temperature.

A glass sphere falls through a liquid of density 1·21 $gm.cm.^{-3}$ and viscosity 14·0 $gm.cm.^{-1}sec.^{-1}$. What is its terminal velocity if the radius of the sphere is 5·50 mm. and the density of the glass is 2·48 $gm.cm.^{-3}$? [2·98 $cm.sec.^{-1}$]

11·28. In a motor-car trial a certain car develops a fixed maximum power. The resistance to motion is kV^n, where k is a constant, and V is the velocity of the car *relative to the air*. In a trial the observed speeds with and against the wind are v_1 and v_2 respectively. Find the maximum speed in still air.

Actual records are calculated by timing the run over a measured mile in each direction, averaging the times, and calculating the speed from this average time. Show that if $n = 1$, this is an under-estimate of the true speed for still air.

11·29. Assuming that a sphere of radius r travelling through a liquid of density ρ with a velocity u experiences a resistance F given by a formula of the type

$$F = \kappa r^{\alpha}\rho^{\beta}\eta^{\gamma}u,$$

where κ is a constant and η the viscosity of the liquid, determine α, β and γ.

Find the terminal velocity acquired by a steel ball 0·32 cm. in diameter, falling under gravity, through an oil of density 0·92 $gm.cm.^{-3}$ and viscosity 16·4 poise. The density of steel may be taken as 7·82 $gm.cm.^{-3}$ and κ as 6π.

[$\alpha = 1, \beta = 0, \gamma = 1$; $2{\cdot}34 \times 10^{-2}$ $cm.sec.^{-1}$.]

11·30. Two horizontal capillary tubes, A and B, are connected together in series so that a steady stream of liquid flows through them. A is 1 mm. in internal diameter and 1 metre long; B is 0·6 mm. in internal

diameter and 60 cm. long. The pressure in the liquid at the entrance to A is 20 cm. of water above atmospheric. At the exit end it is atmospheric. What is the pressure at the junction of the tubes?

Would the above distribution of pressure be found if air were flowing steadily through tubes equal in length to the above but with diameters reduced ten times, the pressures at the entrance and exit of the system being unchanged? [16·4 cm. of water above atmospheric.]

11·31. The resistance to the motion of a ship through still water is kv^3, where v is the speed and k is a constant. Owing to small changes in the propeller thrust, the speed at time t is $a + b\cos\omega t$, where a, b and ω are constants. If m is the mass of the ship, find the thrust of the propellers at any instant and the power exerted. Show that the average power is greater than would be necessary if the speed were constant and equal to a.

11·32. A particle moves in a medium which offers a resistance proportional to the cube of the velocity; no other forces act on the particle. During a time τ the velocity of the particle diminishes from v_1 to v_2. Show that in this time τ the particle traverses a distance

$$\frac{2v_1v_2\tau}{v_1 + v_2}.$$

11·33. A particle of unit mass is in motion along the x-axis under an applied force $f\cos pt$, a controlling force $-n^2x$ and a frictional force equal to $2k$ times the velocity. Prove that the part of the motion depending on the initial conditions dies away exponentially with the time, and determine the motion remaining after a long interval of time.

Prove that after a long interval of time the ratio of the average potential energy during a complete oscillation (in the field of the controlling force) is to the average kinetic energy during a complete oscillation in the ratio $n^2 : p^2$.

11·34. A projectile moves in a medium resisting with force proportional to the square of its velocity. It is projected vertically upwards with velocity v. Show that it rises to a height $\dfrac{V^2}{2g}\ln\dfrac{V^2 + v^2}{V^2}$ in time $\dfrac{V}{g}\tan^{-1}\dfrac{v}{V}$, where V is the limiting velocity for vertical fall.

Also show that it will be in the air a total time

$$\frac{V}{g}\left[\tan^{-1}\frac{v}{V} + \sinh^{-1}\frac{v}{V}\right],$$

and find the velocity with which it will strike the ground.

11·35. A particle of mass m, initially at rest, is acted on by a constant force fm and by a resisting force $mk\dot{x}$ where $\dot{x}$ is the velocity of the particle at a distance x from the origin. Derive expressions for (a) the terminal velocity acquired, (b) the velocity at a time t after the application of the force, (c) the distance travelled by the particle in this time.

Examine the case when the constant force is replaced by a sinusoidal force $mA\cos\omega t$.

11·36. An oil drop of density 0·95 gm.cm.$^{-3}$ and radius 10^{-4} cm. and carrying a charge of q e.s.u. is driven vertically upwards through air of density 0·0013 gm.cm.$^{-3}$ by an electric field of 2000 volt.cm.$^{-1}$ with a constant velocity of 0·036 cm.sec.$^{-1}$. Calculate the magnitude of q being given that the viscosity of air is 180×10^{-6} gm.cm.$^{-1}$sec.$^{-1}$.

[$24{\cdot}2 \times 10^{-10}$ e.s.u.]

11·37. Describe and explain how you would proceed to determine experimentally a value for the kinematic viscosity of water at 20° C.

Show that it may be expressed in acre.year^{-1}.

The viscosity of water at 4·0° C. is 0·01570 gm.cm.$^{-1}$sec.$^{-1}$. Express its kinematic viscosity in terms of the mile, ton, fortnight system of units. [7·33 × 10^{-7} mile^2. fortnight^{-1}.]

11·38. The viscosity of mercury and its density vary with temperature as shown in the following table. Show that Andrade's equation, cf. p. 604, fits these observations and determine numerical values for the constants A and c.

Temp.(° C.)	−20	0	20	50	100	200	300
$\eta \times 10^2$ (gm.cm.$^{-1}$sec.$^{-1}$)	1·69	1·56	1·86	1·41	1·22	1·01	0·93
$(\rho - 13{\cdot}0) \times 10$ (gm.cm.$^{-3}$)	6·45	5·96	5·46	4·73	3·52	1·13	−1·25

[2·46 × 10^{-3}, 48·4]

CHAPTER XII

ELEMENTARY HYDRAULICS, PLASTICITY AND NON-NEWTONIAN LIQUIDS

Some definitions.—(*a*) ***An ideal liquid is one which has zero compressibility and in which no shearing stress can be maintained.***

(*b*) ***A surface of equal pressure is a surface such that at all points in it the pressure has a constant value.***

The resultant thrust on a small element of volume between two neighbouring surfaces of equal pressure.—Suppose that the liquid is at rest, or else an ideal liquid in motion. Let δS, be a small element of the surface of equal pressure p, and let the normals at every point on the periphery of this element be drawn to terminate on the surface of equal pressure defined by $p + \delta p$. Let δn, Fig. 12·01, be the length of each portion of the normals between the surfaces considered. Let the element of volume thus formed be cut by a plane ABCD normal to the surfaces of equal pressure. Then, if δA is the area ABCD, the forces on the curved portions of the element considered are each equal to $p.\delta A$ in magnitude but are opposed to each other, i.e. the force on this element parallel to the surfaces of equal pressure is zero. Hence the force $\delta p.\delta S$, which arises from the pressure on the flat surfaces of the element, cannot have a tangential component, i.e. the resultant thrust is normal to the surfaces of equal pressure.

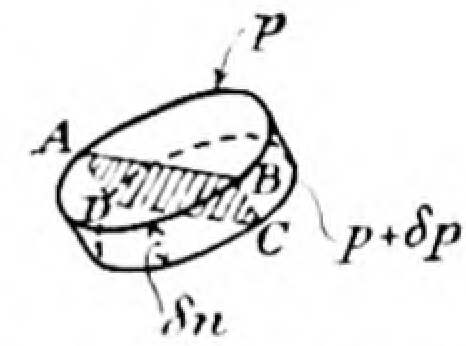


FIG. 12·01.—Resultant thrust on a small element of volume between two neighbouring surfaces of equal pressure.

The shape of the free surface of a rotating liquid.—Let a cylindrical vessel and the ideal liquid within it rotate uniformly with angular velocity ω about a vertical axis. Let the surface of the liquid acquire the shape generated by the revolution of the curve OPA, shown in Fig. 12·02(*a*) when this rotates about the vertical axis ON.

Let δS be a small element of the surface at P, which is essentially a surface of equal pressure, p. Consider a neighbouring surface defined by $p + \delta p$ and at a distance δn from δS, as shown in Fig. 12·02(*b*). The fluid in the element of volume $\delta S.\delta n$ is acted upon by its weight $(\delta S.\delta n)g\rho$, where g is the intensity of gravity and ρ the density of the liquid, and a force $\delta p.\delta S$ at right angles to the

free surface of the element. These forces are shown as PD and PN respectively in Fig. 12·02(c) and their resultant PR must be the centripetal force required to cause the element to move with angular velocity ω about the axis of rotation. If θ is the angle indicated,

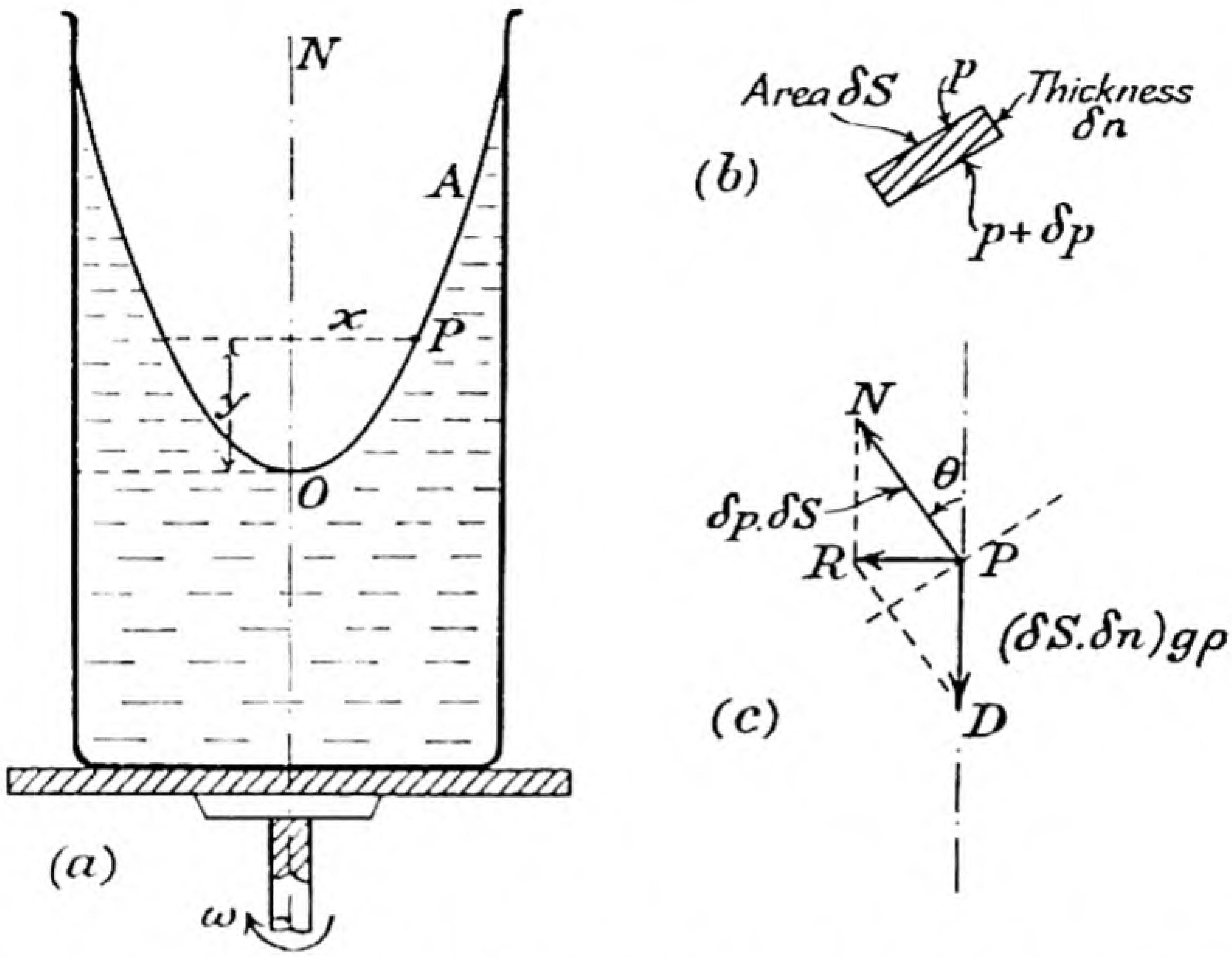


FIG. 12·02.—The shape of the free surface of a rotating (ideal) liquid.

resolving the forces on the element horizontally and vertically, we have

$$(\delta p\,.\,\delta S)\sin\theta = (\delta S\,.\,\delta n)\rho\,.\,\omega^2 x,$$

and

$$(\delta p\,.\,\delta S)\cos\theta = (\delta S\,.\,\delta n)\rho g.$$

$$\therefore\ \frac{\omega^2 x}{g} = \tan\theta$$

$$= \text{slope of tangent at P} = \frac{dy}{dx}.$$

$$\therefore\ y = \frac{\omega^2 x^2}{2g} + \text{C},$$

where C is an integration constant. If we select O as origin we have C = 0, i.e.

$$y = \frac{\omega^2 x^2}{2g},$$

which is the equation to a parabola whose axis is vertical; its focal distance is a, where $4a = \frac{2g}{\omega^2}$, i.e. $a = \frac{g}{2\omega^2}$.

The pressure at any point in a rotating liquid.—Let O, Fig. 12·03, be the lowest point on the free surface of a liquid rotating with angular velocity ω about a vertical axis. Let Ox, Oy be the reference axes and P a point (x, y) in the liquid; suppose P is the centre of a small horizontal area δS. Let the cylinder formed by erecting a vertical line through each point on the boundary of δS intersect the free surface of the liquid in Q. Let PQ $= h$ and p be the pressure excess in the liquid at P, i.e. the atmospheric pressure is taken to be zero. If we consider the external forces acting on the liquid within the cylinder whose axis is PQ, we have

$$p\,.\,\delta\mathrm{S} - (g\rho h)\,.\,\delta\mathrm{S} = 0,$$

i.e.

$$p = g\rho h$$
$$= g\rho(\mathrm{AQ} - y),$$

where A is the point in which QP produced cuts Ox. Since Q is on the free surface of the liquid

$$\mathrm{AQ} = \frac{\omega^2 x^2}{2g}.$$

$$\therefore\ p = g\rho\left(\frac{\omega^2 x^2}{2g} - y\right),$$

or

$$\frac{\omega^2 x^2}{2g} = y + \frac{p}{g\rho} = y + h.$$

Thus if p is constant the point P lies on a surface of equal pressure; it is a paraboloid with an origin at a vertical distance h below the

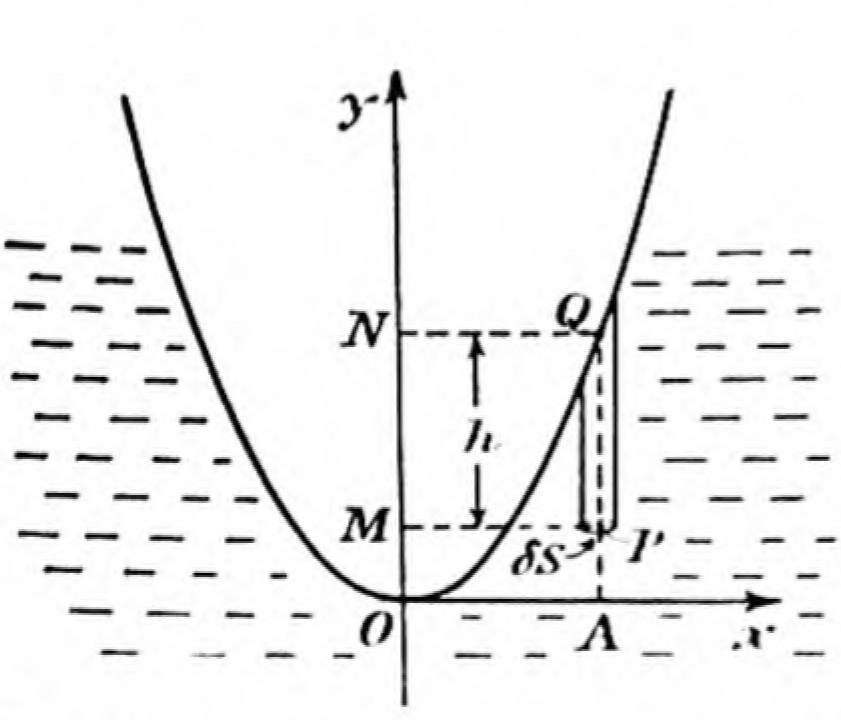


FIG. 12·03.—The pressure at any point in a rotating (ideal) liquid.

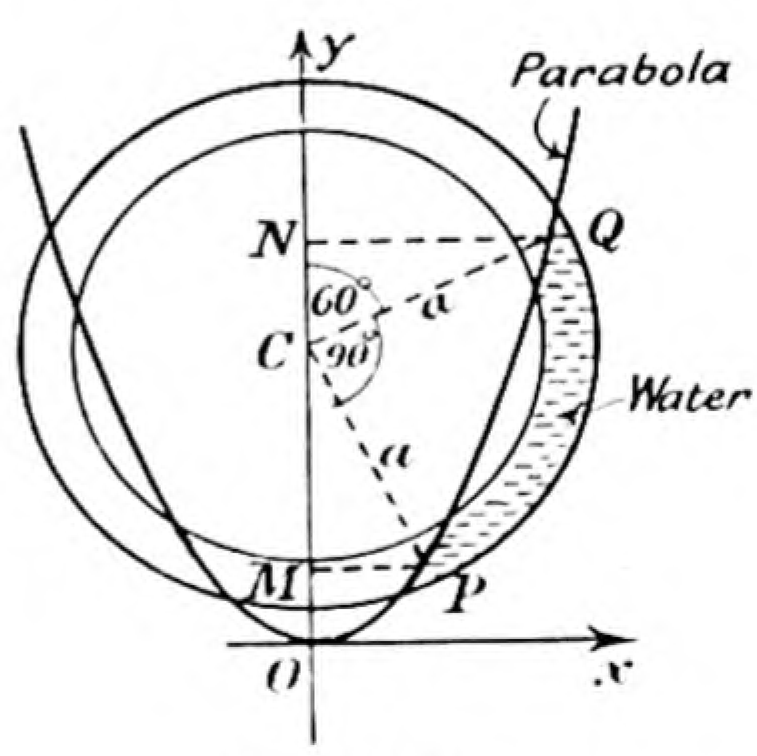


FIG. 12·04.

free surface of the liquid. Thus the surfaces of equal pressure are equal paraboloids.

Example.—A quantity of water occupies a quadrant of a circular tube; a is the radius of this circle and it is large compared with the radius of cross-section of the tube itself. When the tube rotates with

uniform angular velocity ω about a vertical diameter, the highest point in the liquid is at an angular distance of 60° from the highest point in the tube. Neglecting surface tension effects, prove that $a\omega^2 = 2g(1 + \sqrt{3})$, where g is the intensity of gravity.

Let the parabolic free surface of the water meet the tube in points P and Q, cf. Fig. 12·04, and the vertical through C, the centre of the circle, in O. Draw PM and QN perpendicular to OC to cut this line in M and N respectively. Then $\widehat{PCM} = 30°$ and $\widehat{QCN} = 60°$, i.e.

$$PM = \tfrac{1}{2}a; \qquad QN = \tfrac{1}{2}a\sqrt{3}.$$

$$\therefore \frac{3a^2}{4} = QN^2 = \frac{2g}{\omega^2} \cdot ON,$$

and

$$\frac{a^2}{4} = PM^2 = \frac{2g}{\omega^2} \cdot OM.$$

$$\therefore \tfrac{1}{2}a^2 = \frac{2g}{\omega^2} NM = \frac{2g}{\omega^2}\left[\tfrac{1}{2}a\sqrt{3} + \tfrac{1}{2}a\right],$$

i.e.

$$a\omega^2 = 2g(1 + \sqrt{3}).$$

Bernoulli's theorem.—When the particles in a moving fluid which succeed one another at a fixed point have the same density and velocity and the pressure remains constant, the motion of the fluid is said to be steady. Under such ideal conditions which, in practice, are seldom realized absolutely, the motion is said to be streamlined and a ***streamline*** is defined as the actual path of a particle in a moving fluid. In tubes of constant cross-section all the streamlines are parallel to the axis of the tube; in general, the streamlines are curved and are such that at any instant the tangent at any point on it gives the direction of the motion at that point. A tube of flow is such that its boundary is formed by contiguous streamlines.

Now Bernoulli's theorem expresses the manner in which the pressure varies along a streamline and it may be deduced from the principle of the conservation of energy. In Fig. 12·05(a) let AA_1 be the axis of a narrow flow tube, and at A and A_1 let a and a_1 be the areas of the corresponding normal cross-sections. Let A and A_1 be at heights z and z_1 above an arbitrary datum level. Then since the mass of fluid contained between the cross-sections considered is constant, the same mass of liquid crosses every normal section of the tube per unit time. In time δt let the mass of fluid, δm, flowing across a occupy a length δs of the tube; at A_1 the same mass will occupy a length δs_1. Using the energy principle, we have,

$(P.E. + K.E.)_A$ + work done on the mass entering in time δt at A $= (P.E. + K.E.)_{A_1}$ + work done by the mass leaving in time δt at A_1, where the suffixes indicate the two sections of the tube considered.

Thus if p and p_1 are the respective pressures in the fluid, we have

$$\delta m\, gz + \tfrac{1}{2}\delta m\, u^2 + pa\, \delta s = \delta m\, gz_1 + \tfrac{1}{2}\delta m\, u_1{}^2 + p_1 a_1 \delta s_1.$$

But

$$\delta m = (a\, \delta s)\rho = (a_1\, \delta s_1)\rho,$$

where ρ is the density of the fluid, and this is assumed constant.

$$\therefore\ \delta m\left(gz + \tfrac{1}{2}u^2 + \frac{p}{\rho}\right) = \delta m\left(gz_1 + \tfrac{1}{2}u_1{}^2 + \frac{p_1}{\rho}\right),$$

i.e. along a streamline

$$gz + \tfrac{1}{2}u^2 + \frac{p}{\rho} = \text{constant}.$$

This is ***Bernoulli's theorem.***

Alternative proof: Let AB, Fig. 12·05(b), be a very thin flow-tube. At P let the pressure be p, the velocity towards B be u and

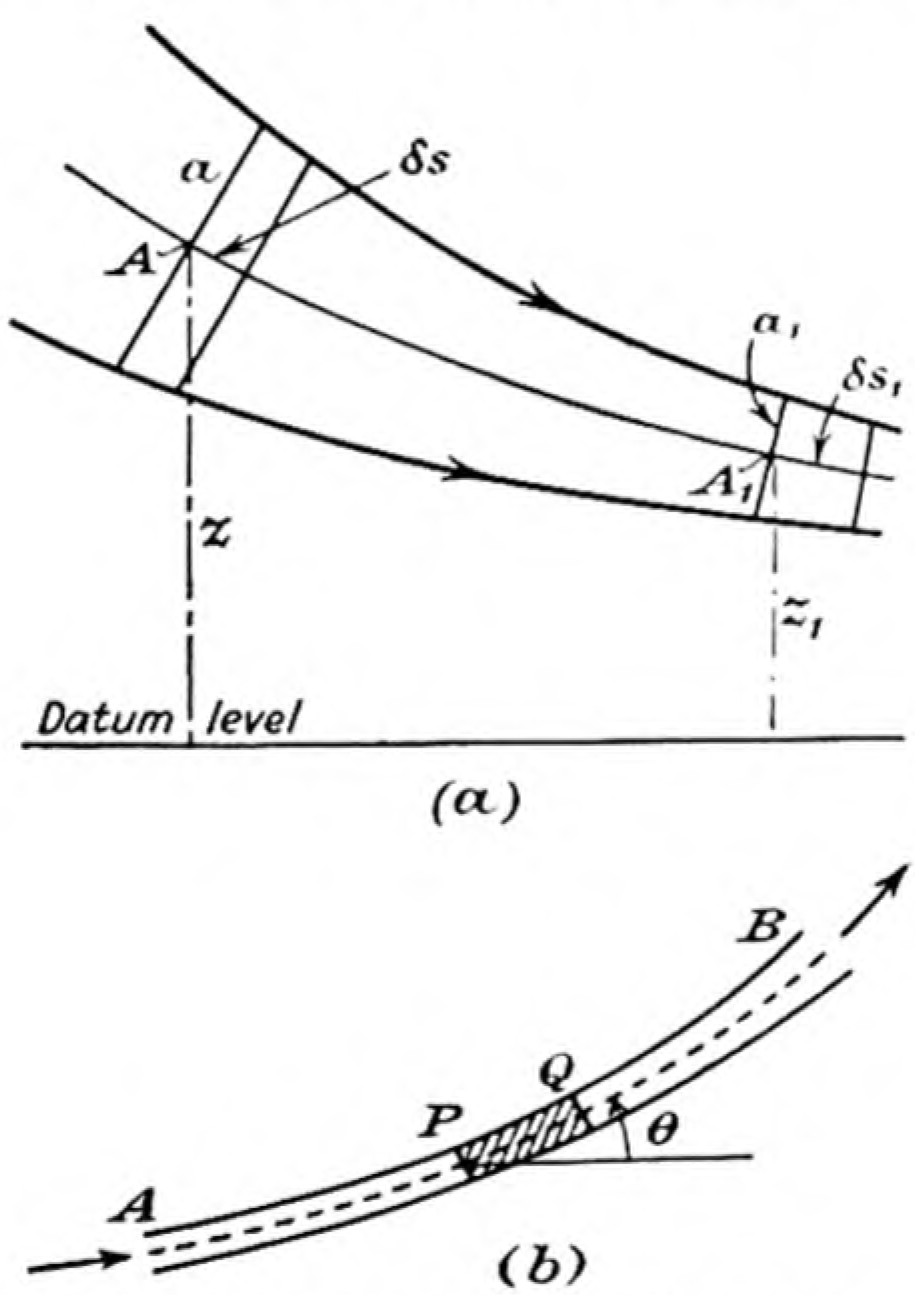


FIG. 12·05.—Bernoulli's theorem.

z the height of P above some fixed level. At Q a point near to P and in the flow-tube, let the above quantities be $p + \delta p$, $u + \delta u$, and $z + \delta z$ respectively. Let the axis of the tube at PQ make an angle θ with the horizontal. If a is the cross-section of the tube at P and ρ the density of the fluid (in general this is a variable), the force urging the fluid in PQ along the tube is

$$pa - (p + \delta p)a - g\rho a(\text{PQ}) \sin \theta,$$

where g is the intensity of gravity. This is equal to the mass of the element times its acceleration. If δt is the time for a particle to pass from P to Q, a distance δs, $u = \lim \frac{\delta s}{\delta t} = \frac{ds}{dt}$, and $\frac{du}{dt}$ is the acceleration.

$$\therefore\ -a\,\delta p - g\rho a\,\delta s.\frac{\delta z}{\delta s} = \rho a\,\delta s.\frac{du}{dt}.$$

$$\therefore\ \rho a\frac{du}{ds}\frac{ds}{dt} + a\frac{dp}{ds} + g\rho a\frac{dz}{ds} = 0.$$

$$\therefore\ u\,\delta u + \frac{\delta p}{\rho} + g\,\delta z = 0.$$

Integrating this, we get

$$\tfrac{1}{2}u^2 + \int\frac{dp}{\rho} + gz = \text{constant},$$

and it is only if ρ is constant that we have

$$\tfrac{1}{2}u^2 + \frac{p}{\rho} + gz = \text{constant}.$$

Experimental illustration of Bernoulli's theorem.—In Fig. 12·06 T is a large tank in which a constant head of water is main-

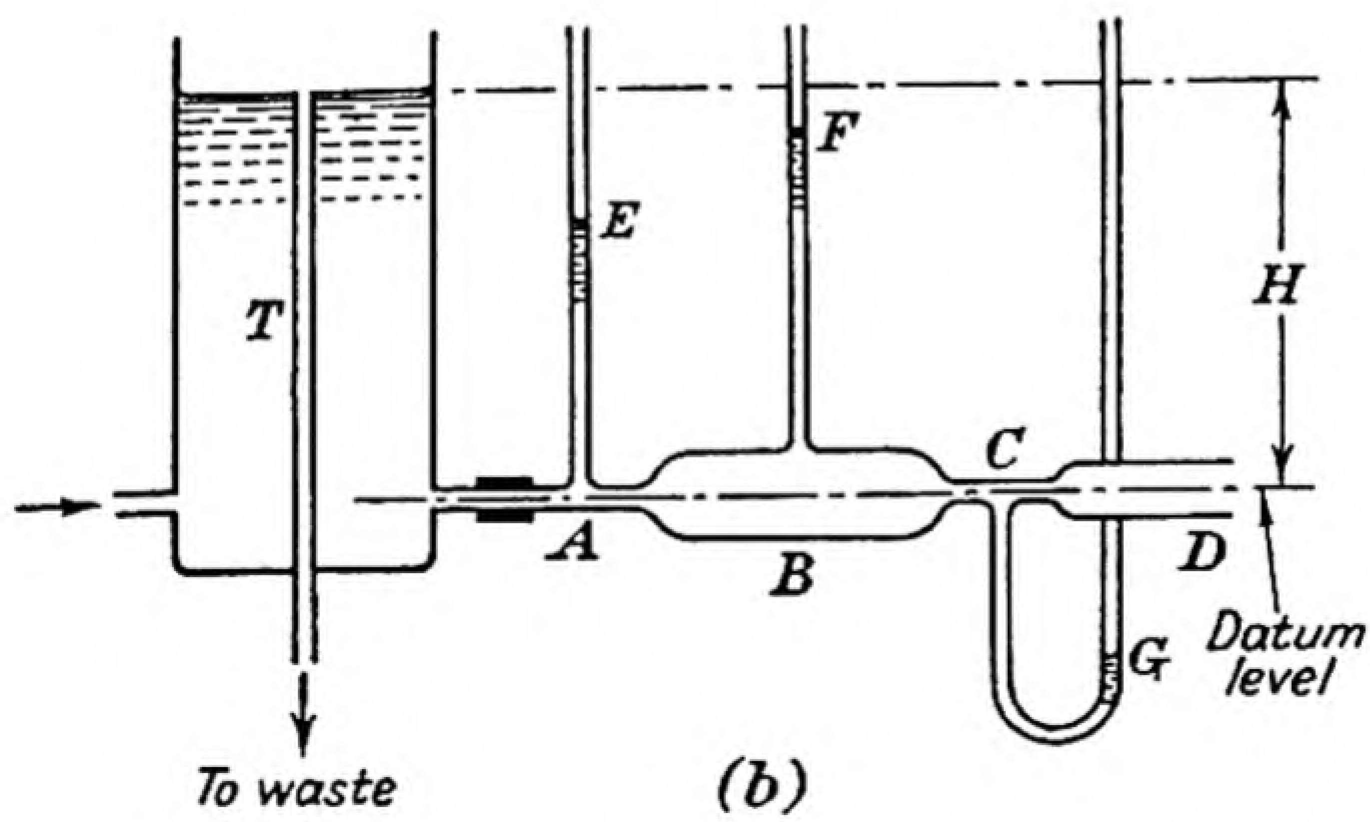


FIG. 12·06.—An experimental illustration of Bernoulli's theorem.

tained in the way indicated. Water from the tank flows through a short horizontal 'pipe' of variable cross-section and fitted with tubes at different sections A, B, C. The tube attached at C, where the main flow tube is very much narrower than elsewhere, is made in the form of a U-tube. When the end D of the pipe is closed the water in all three tubes rises to the same level as the water in T but when D is

opened so that water flows through the pipe, the water surfaces in the tubes at A, B and C will occupy different levels at E, F and G, as shown. If the datum level is the axis of the pipe, then

$$\frac{p_a}{\rho} + \tfrac{1}{2}u_a^2 = \frac{p_b}{\rho} + \tfrac{1}{2}u_b^2 = \frac{p_c}{\rho} + \tfrac{1}{2}u_c^2,$$

and if the cross-sections are such that $u_a > u_b$, then $p_a < p_b$ and at C, where $u_c \gg u_d$, p_c will be less than atmospheric.

To show that when the velocity of flow is large, the pressure is small, Osborne Reynolds devised an experiment in which water under a high pressure flowed through a tube about 2 cm. in diameter at the point of entry but which diminished to about 0·1 cm. diameter near the exit. At this section the velocity of flow was so large that the pressure was reduced to such a low value that the water boiled (*not* at 100° C., of course) and made a hissing noise.

Further illustrations of Bernoulli's theorem.—If a small 'ping-pong' ball is introduced into a stream of air or water issuing from a jet the ball will rise to a fixed position, as shown in Fig. 12·07(*a*), and remain there spinning and turning round without

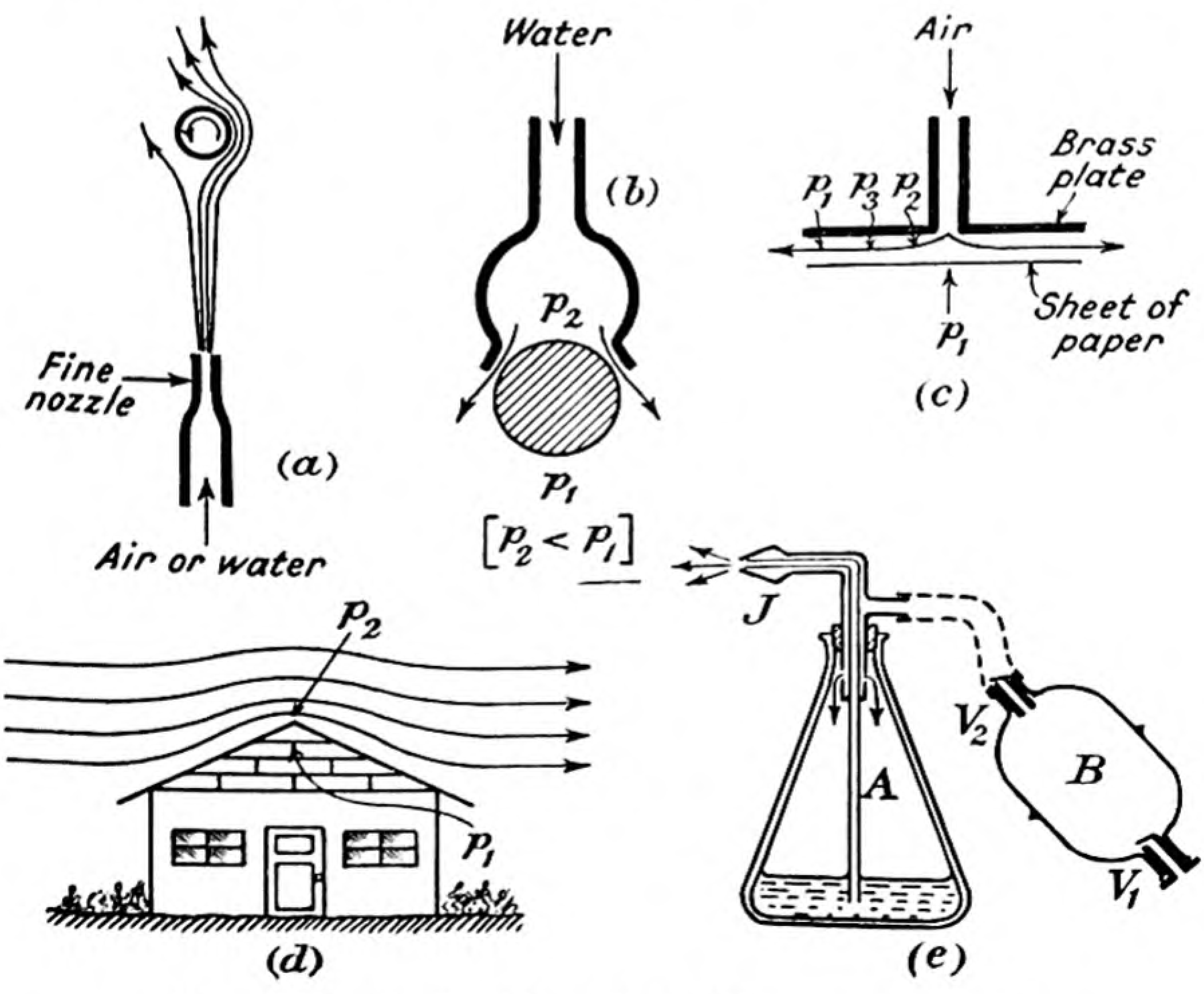


FIG. 12·07.—Further illustrations of Bernoulli's theorem.

falling. The effect is very striking when the jet from which the air stream issues is not vertical and also when it is below a bench-top; in this latter instance the ball floats without visible means of support. If the ball shifts laterally the fluid on its right-hand side causes the ball to spin and since the fluid velocity is high, the pressure within is low; on account of the higher pressure on the left the ball is returned to the stream.

In Fig. 12·07(*b*) there is shown an inverted thistle funnel and a ball which almost fits its mouth. The rapidly moving stream of water passing over that part of the ball close to the mouth of the funnel creates a low pressure within the fluid so that the thrust due to the atmospheric pressure holds the ball in position.

Fig. 12·07(*c*) shows a sheet of paper below a brass plate fitted with a centre tube. If a stream of air passes outwards as indicated the gas moves most rapidly near the centre of the plate; the pressure is therefore lowest near to the centre and since at the edge it is approximately atmospheric the paper does not fall.

In a similar manner if a stream of air issues from a jet placed between two pieces of thin cardboard hanging downwards from suitable supporting threads, the boards will move towards one another due to the lower pressure near the end of the jet.

In a gale the roof of a house is sometimes blown off without damage to the rest of the structure. As the wind blows, cf. Fig. 12·07(*d*), the pressure (p_2) above the roof is reduced and when this reduction is considerable the roof is lifted away as if by suction.

Fig. 12·07(*e*), shows a scent spray or atomizer. When the bulb B is squeezed air issues from the jet J at a high speed so that the pressure here is reduced. The liquid rises in the vertical tube A—the pressure above the liquid being equal to that in B—and is blown out in the form of a fine spray. The valves V_1 and V_2 control the flow of air in and out of B.

Venturi meter.—A practical application of Bernoulli's theorem is also found in a Venturi meter, an instrument for measuring the rate of flow of water, i.e. the volume flowing per unit time, through a pipe. It was invented in 1887 by Clemens Herschel and takes its name from an eighteenth-century Italian who made experiments on the flow of water through conical pipes. In its simplest form it consists of two truncated conical tubes connected together by a short length of cylindrical tubing, known as the 'throat'. The meter, in use, occupies a horizontal position in a pipe-line, the diameter of the large end of each frustum being equal to that of the pipe. Fig. 12·08 is a representation of a Venturi meter in use. As the water flows through the meter its velocity will be a maximum at the throat on account of the reduction in cross-sectional area and consequently the pressure will be reduced. This reduction of

pressure is measured by the difference, H, in the levels of the water as shown in the diagram.

Let a be the cross-sectional area of the throat and A that of the

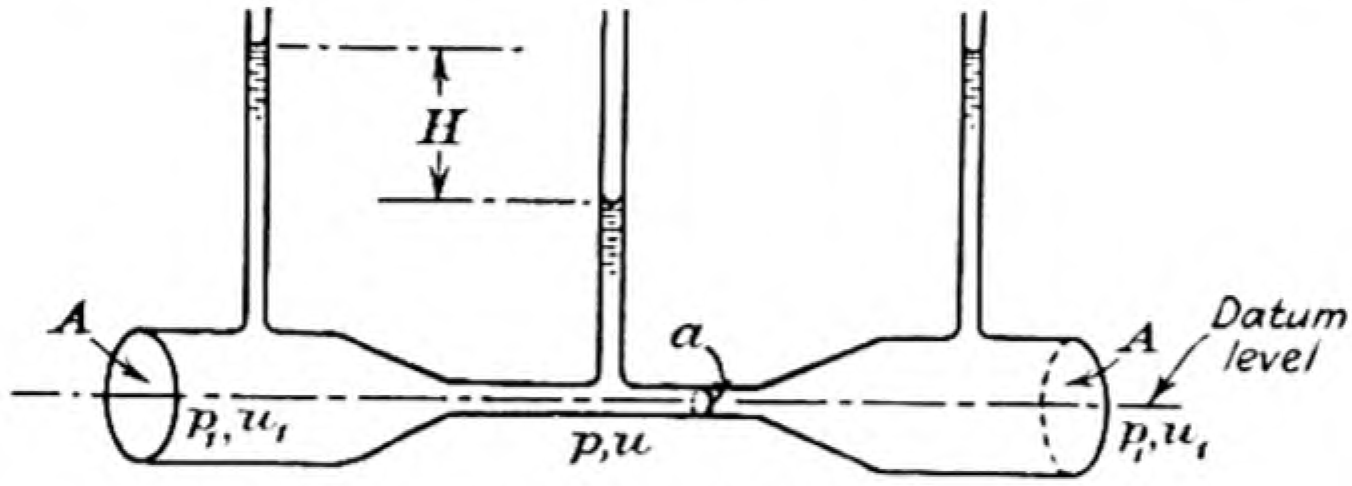


FIG. 12·08.—The principle of a Venturi meter.

main pipe-line. Then if the datum level passes through the axis of the meter, we have, if friction is neglected,

$$0 + \tfrac{1}{2}u_1^2 + \frac{p_1}{\rho} = 0 + \tfrac{1}{2}u^2 + \frac{p}{\rho},$$

where the suffixes refer to the section at A.

Let Q be the volume of water passing per second; then

$$\frac{Q}{A} = u_1 \quad \text{and} \quad \frac{Q}{a} = u.$$

Also $$p_1 - p = g\rho H;$$

hence $$\frac{1}{2}\frac{Q^2}{A^2} = \frac{1}{2}\frac{Q^2}{a^2} - gH,$$

i.e. $$Q = \frac{aA}{(A^2 - a^2)^{\frac{1}{2}}}(2gH)^{\frac{1}{2}}.$$

On account of friction, viscous forces and eddy motions which may be present in the meter, the actual discharge is slightly less than the above theoretical value, so that, in practice

$$Q = \frac{\kappa a A}{(A^2 - a^2)^{\frac{1}{2}}}(2gH)^{\frac{1}{2}},$$

where κ is a constant for any given meter and this must be determined experimentally.

A Pitot tube.—When we consider the flow of a fluid with horizontal stream lines it is convenient to take one of them as the datum level from which the height z in Bernoulli's equation is measured. Then we may write

$$p + \tfrac{1}{2}\rho u^2 = \text{constant},$$

where the symbols have their usual meanings.

Now it must be noted that in this equation p represents the

pressure which actually exists in the moving fluid; it is known as the ***static pressure***. To obtain a value for p it is necessary to measure the force per unit area on a surface whose presence does not in any way modify the motion of the fluid. If a rigid surface is placed in the fluid then it will alter both the direction and speed of flow unless the surface is either (a) parallel to the stream lines or (b) moving through the fluid so that there is no relative motion between the surface and the fluid in its immediate vicinity. By attaching side tubes to a flow tube as in Fig. 12·06, and elsewhere, the first of these two conditions has been tacitly fulfilled.

In 1730 PITOT discovered that if a small tube, now known as a ***Pitot tube***, were inserted in a moving liquid so that the entrance to the tube faced the oncoming stream, then the liquid rose in the small tube as shown in Fig. 12·09(a). To discover the reason for this let Fig. 12·09(b) represent an enlarged view of the immersed end of a Pitot tube. The liquid in the tube is at rest so that at a point B near to the above end of the tube the component of the velocity of flow along a horizontal stream line will be zero. Hence Bernoulli's theorem (omitting the z term) gives

$$p_A + \tfrac{1}{2}\rho u^2 = p_B + 0,$$

so that the static pressure at B measures the so-called dynamic pressure $p_A + \frac{1}{2}\rho u^2$. The difference between these two pressures measures $\frac{1}{2}\rho u^2$, so that if ρ is known a value for u may be found. The point B is known as a ***stagnation point***. Experiment shows that even for gases ρ may be treated as a constant for speeds up to 50 metre.sec.$^{-1}$.

To determine the difference between these two pressures a so-called ***Pitot-static tube*** has been devised. In the first stage of development the tube had a conical end but this was easily damaged so that in later models the end was made hemispherical; to-day the N.P.L. recommends that the nose of the tube should be ellipsoidal. The Pitot tube proper is the copper inner tube of the complete Pitot-static tube shown in Fig. 12·09(c). The outer brass casing forms the static-tube and near to the nose six or eight small holes, such as O_1 and O_2, are drilled in this casing. The tubes T_1 and T_2 are connected to a sensitive manometer so that a value for $\frac{1}{2}\rho u^2$ is easily obtained.

As thus assembled a Pitot-static tube is used to investigate aerodynamic problems, to measure the flow of gases used to cool a nuclear reactor, and very extensively in industry wherever the flow of gases is involved. The actual aperture of the Pitot tube may vary from 0·2 mm. to 2 mm. in diameter for throughout this range its indications are constant. In using the instrument the axis of the nose must be truly horizontal.

When such a tube is applied to the measurements of liquid flow many difficulties arise. In the first place one is never sure that the system is free from air bubbles. Secondly, the tube will disturb the liquid and this effect is large when the tube in which the liquid flows is relatively small in diameter. Thus, as a consideration of Fig. 12·09(d) shows, the liquid will flow more quickly where the effective cross-section of the tube is reduced by the presence of the

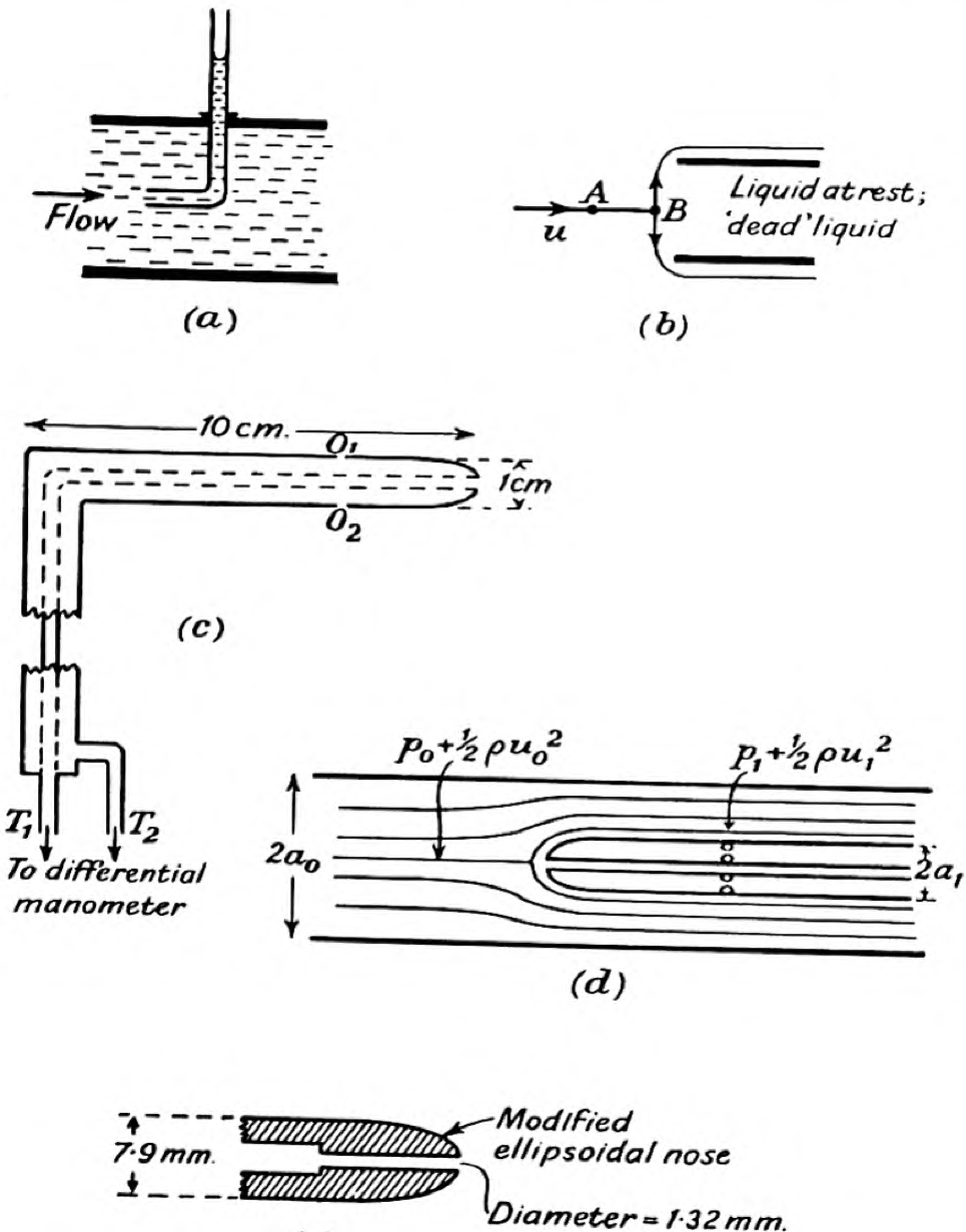


FIG. 12·09.—The principles and development of a Pitot-static tube.

Pitot-static tube; as a first approximation the ratio of the two velocities may be computed from the known values of the diameters of the tubes. Then, with the notation indicated, the measured pressure difference, P, will be given by

$$p_0 + \tfrac{1}{2}\rho u_0^2 - (p_1 + g\rho z_0) = \mathrm{P},$$

where $z_0 \simeq (a_0 - a_1)$ while

$$p_0 + \tfrac{1}{2}\rho u_0^2 = p_1 + g\rho z_0 + \tfrac{1}{2}\rho u_1^2,$$

since Bernouilli's theorem may be applied to the stream line which begins along the axis of the tube and remains in a horizontal plane. Hence

$$\mathrm{P} = \tfrac{1}{2}\rho u_1^2 = \tfrac{1}{2}\rho u_0^2\left(\frac{a_0^2}{a_0^2 - a_1^2}\right),$$

where $2a_0$ and $2a_1$ are the relevant diameters of the tubes.

Another difficulty is that on account of small energy losses in the liquid one ought to write, in general,

$$\text{Measured pressure difference} = \tfrac{1}{2}\rho u^2(1 - f),$$

where $f \to 0$, although it may have both positive and negative values; $(1 - f)$ is known as the ***calibration factor*** and for gases $f \simeq 0{\cdot}003$. When these tubes were first developed the calibration factor was determined by mounting the composite tube on a whirling shaft some 30 ft. in diameter; the speed of the nose was calculated and by a comparison with that deduced from the Pitot-static tube itself, a value for $(1 - f)$ deduced.

In a recent N.P.L. report by Salter (1955), a Pitot-static tube with a nose head not truly ellipsoidal is advocated, cf. Fig. 12·09(e); even when the axis of the tube, to which such a nose is fitted, is 13° from the horizontal it is found that f is practically negligible and the results obtained have an error less than one per cent. In conclusion it may be noted that with air moving at about 50 metre.sec.$^{-1}$ a differential pressure equivalent to 2 cm. of water is created; for very small speeds the pressure difference should be measured with a Chattock manometer.

Resistances to the motion of a fluid in a pipe.—When a fluid is made to flow along a pipe-line, energy is dissipated in overcoming certain resistances which are set up and oppose the motion. Friction, sudden contractions or enlargements in the diameter of the pipe, flow round bends and the presence of valves are some of the causes to which this dissipation of energy must be attributed. In the sequel it will be found more convenient to speak of 'loss of head' rather than loss of energy.

Suppose that a liquid (water) is flowing from a constant pressure

tank through a long narrow pipe, AB, Fig. 12·10, the diameter of the pipe being uniform and its end B open to the atmosphere, pressure P. If it is justifiable to assume that the motion is stream-lined and that all resistance is absent then Bernoulli's equation, applied to the flow at S and at B, gives

$$gz_S + \tfrac{1}{2}u_S^2 + \frac{P}{\rho} = gz_B + \tfrac{1}{2}u_B^2 + \frac{P}{\rho},$$

and since $u_S \rightarrow 0$,

$$gz_S = gz_B + \tfrac{1}{2}u_B^2,$$

or
$$u_B = \sqrt{2g(z_S - z_B)} = \sqrt{2gH},$$

where
$$H = z_S - z_B.$$

Thus, on this basis, the whole head of liquid, H, is utilized in giving energy to the liquid at B. Experiment, however, shows that the mean velocity, u, of the water is less than u_B, so that the energy lost per unit mass of liquid is

$$\tfrac{1}{2}(u_B^2 - u^2) = gH - \tfrac{1}{2}u^2$$

$$= g\bar{h}, \qquad \left[\text{say, where } \bar{h} = H - \frac{u^2}{2g}\right].$$

The quantity '$\bar{h}$' is known as the ***total loss of head*** along the pipe, and includes all causes contributing to this dissipation of energy.

Loss of head by friction.—Suppose now that two vertical tubes C and D are fitted to the pipe-line AB, Fig. 12·10, the lower ends of

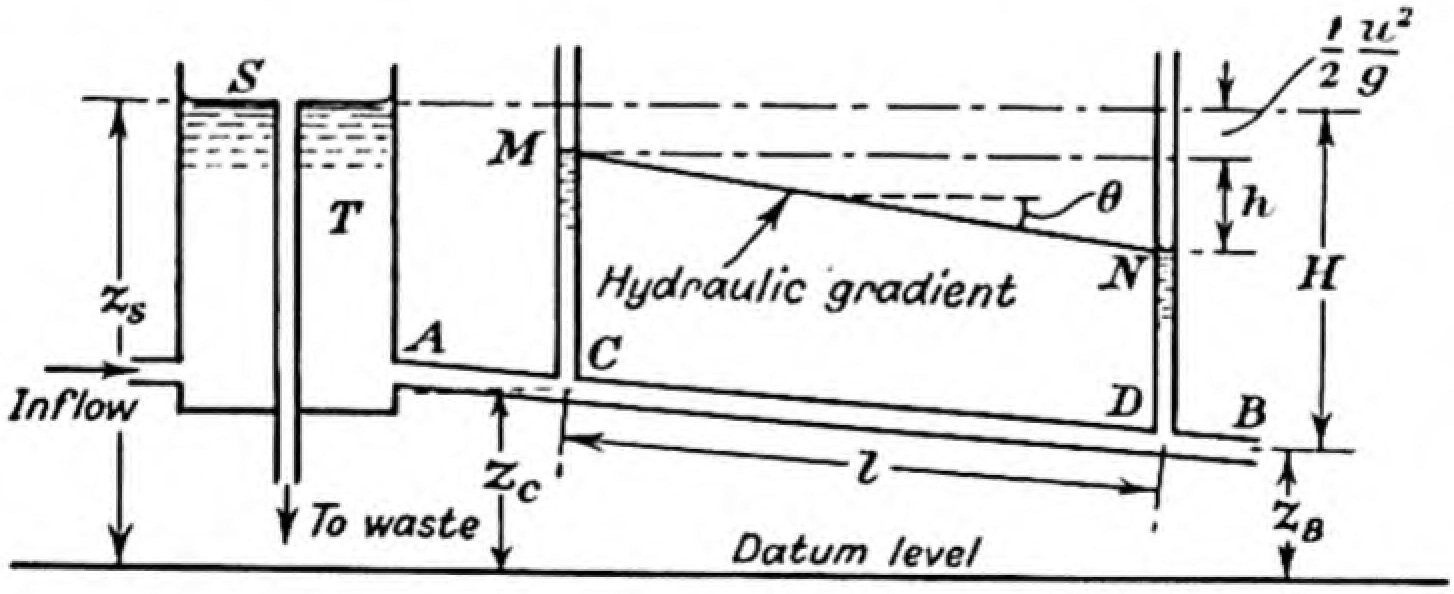


FIG. 12·10.—'Loss of head' due to liquid flow in a pipe.

the tubes being flush with the inside wall of the pipe. If B is closed the liquid rises in each tube until its surface is at the level of the liquid surface S.

Also, if a valve at B is adjusted so that the mean velocity at any

section in the pipe is u, then if there is no resistance to the motion the liquid in both tubes will rise to the same level but the level will be less than that in the tank by a height equal to $\frac{1}{2}\frac{u^2}{g}$. To prove this, we have, by Bernoulli's theorem,

$$gz_c + \tfrac{1}{2}u^2 + \frac{P + g\rho MC}{\rho} = \text{const.} = gz_s + 0 + \frac{P}{\rho}.$$

$$\therefore\ (z_s - \overline{z_c + MC}) = \tfrac{1}{2}\frac{u^2}{g}.$$

$\therefore$ The liquid level above C (or any other point in AB) is less than that in the tank by a height equal to $\frac{1}{2}\frac{u^2}{g}$. Experiment shows that this is not so, the liquid surface above D being at a lower level than that above C. Thus there is a loss of head along the pipe. When the tube is uniform the loss of head, h, is directly proportional to the distance, l, between the sections at which the pressures are measured.

Hydraulic gradient and virtual slope.—The straight line, MN, joining the tops of the columns of liquid in the tubes C and D is known as the ***hydraulic gradient*** and the tangent of the angle, θ, which MN makes with the datum level is called the ***slope, α, of the hydraulic gradient*** or ***the virtual slope***. In practice the slope is small and may be measured by the sin θ, so that

$$\alpha = \sin\theta = \frac{h}{l}.$$

Reynolds' experiments on the flow of water through pipes.—Osborne Reynolds, in 1883, carried out researches, which have become classical in their importance, on the loss of head by friction when water flows through a pipe. The esssentials of his apparatus are shown in Fig. 12·11. A horizontal pipe, AB, 16 ft. long, is connected to the water mains and the flow of water controlled by a suitable regulator inserted in the part of the system between the main and the pipe.

At two points C and D, 5 ft. apart and near to the end B of the experimental tube, two small holes are pierced and short tubes soldered to the pipe at these points. These tubes are connected by indiarubber tubing to a siphon gauge G, made of glass and containing carbon bisulphide [or mercury for the larger head losses].

For small differences between the levels of the liquid interfaces a cathetometer is used but otherwise scales in cm., etc. may be used to determine the height Z. If ρ_0 is the density of the gauge liquid,

ρ that of water, then, if p is the pressure in the water at the level M, the pressure in the other tube at the same level is $p - g\rho h$,

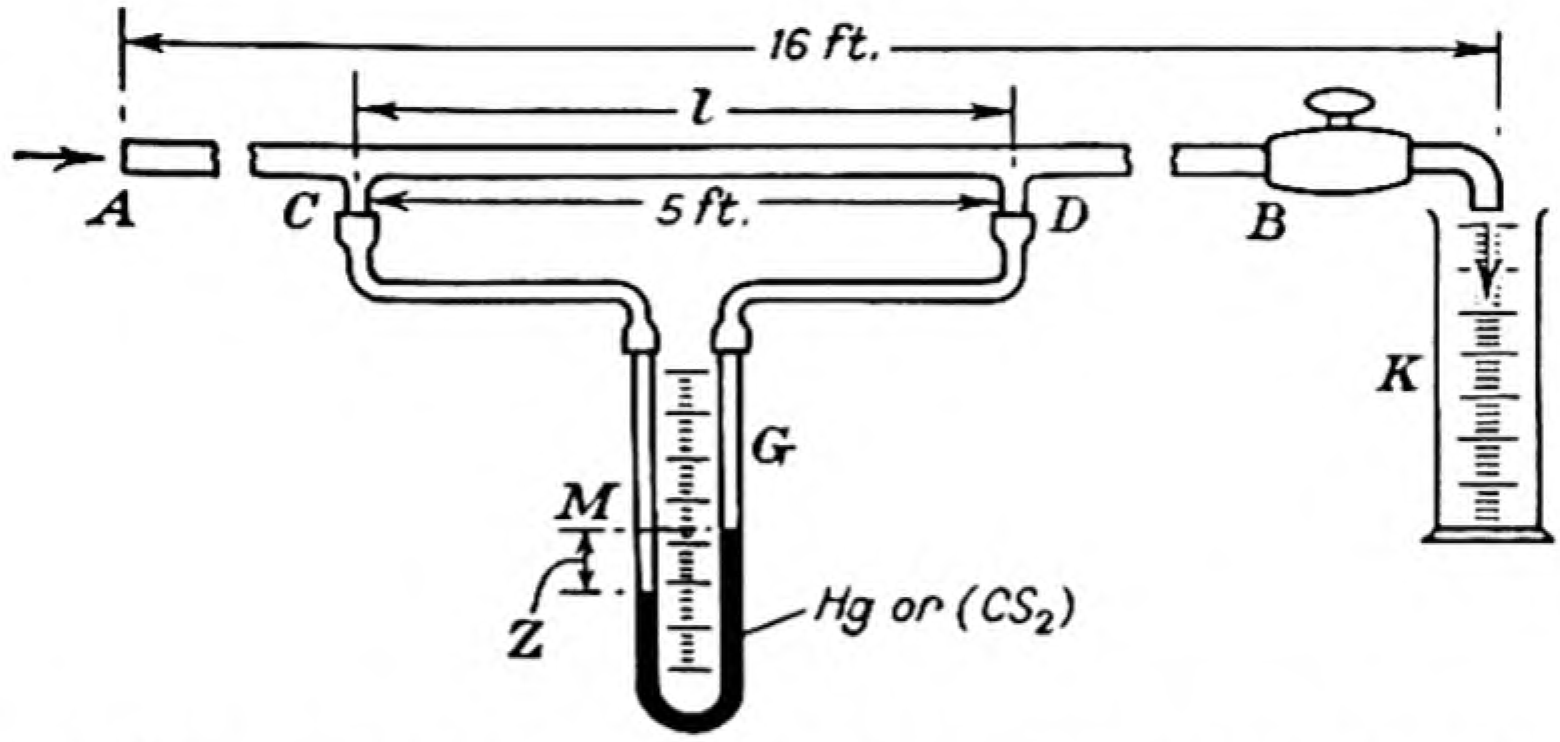


FIG. 12·11.—Reynolds' experiment on the flow of water in a pipe.

where h is the total loss of head along CD, and hence

$$p + g\rho Z = (p - g\rho h) + g\rho_0 Z,$$

or

$$h = Z\left(\frac{\rho_0}{\rho} - 1\right).$$

The mean velocity, u, of the water over any normal section of the pipe is given by

$$u = \frac{\text{Volume of water discharged per second}}{\text{Cross-sectional area of pipe}}.$$

When corresponding values of α, viz. hl^{-1}, and u are plotted, a curve similar to that shown in Fig. 12·12(a) is obtained. The portion OG is a straight line for which

$$\alpha = \kappa_1 u,$$

where κ_1 is its slope; the portion GJ is somewhat irregular at first but the portion towards J follows the law

$$\alpha = \kappa_2 u^n,$$

where κ_2 and n are constants. To determine the constant n, the above equation may be written

$$\log \alpha = \log \kappa_2 + n \log u.$$

By plotting $x = \log u, y = \log \alpha$, the curve shown in Fig. 12·12(b) is obtained. The straight line AB corresponds to the state of flow for which $n = 1$, while for the portion CD, $n = 1{\cdot}75$.

From the graph it follows that the flow of water through a uniform pipe consists essentially of two types:—

(*a*) a steady or ***streamline flow*** up to the point B,

(*b*) an unsteady or eddy flow when the velocity exceeds that at B—in this region we have ***turbulent flow.***

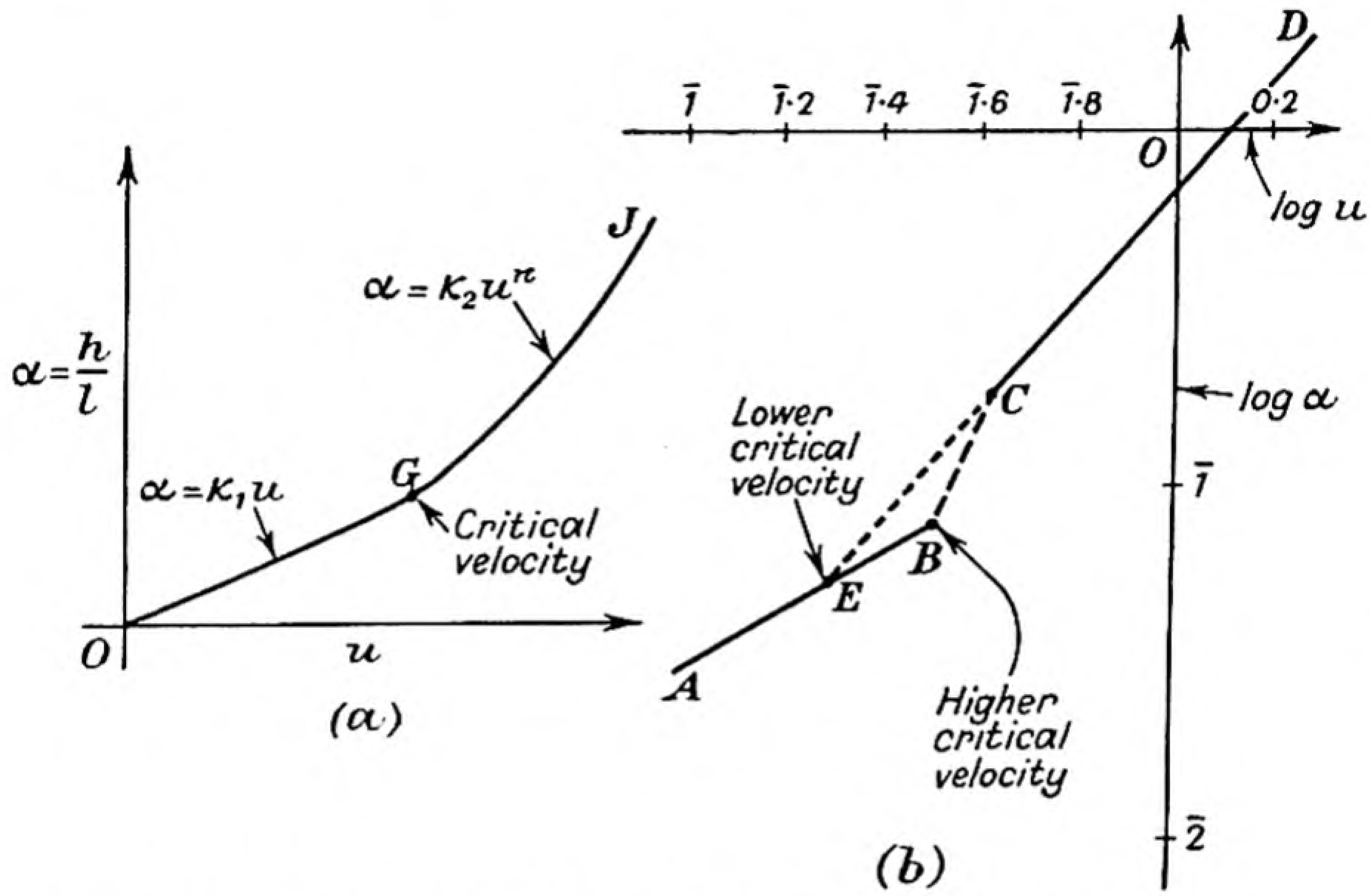


FIG. 12·12.—Graphical determination of lower and higher critical velocities.

The velocity of flow corresponding to the point B, and above which the flow ceases to be streamline, is known as the ***higher critical velocity.***

In the diagram the line BC represents a period of transition when the flow is changing from one type to another. Reynolds argued that the portion EBC of the complete curve exists on account of the inertia which the water possesses and considered that E, the intersection of DC produced with AB, was the ***lower*** or ***true critical velocity.*** The higher critical velocity is that velocity at which water flowing from rest in streamline motion breaks up into eddies, while the lower critical velocity is, for water that is initially disturbed, the velocity above which streamline motion is impossible.

Reynolds' investigations extended to pipes several inches in diameter and some experiments were carried out at temperatures above that of the laboratory. He discovered that the lower critical velocity is inversely proportional to the diameter of the pipe and that it decreased with rise in temperature.

The colour-band method for determining critical velocities.—In addition to the method outlined above for determining the critical velocity of water flowing through a pipe, two other available methods must be mentioned.

(*a*) The earliest of these is due to REYNOLDS, who injected a thin stream of coloured ink along the axis of the tube; the apparatus used is shown diagrammatically in Fig. 12·13. A 'constant head'

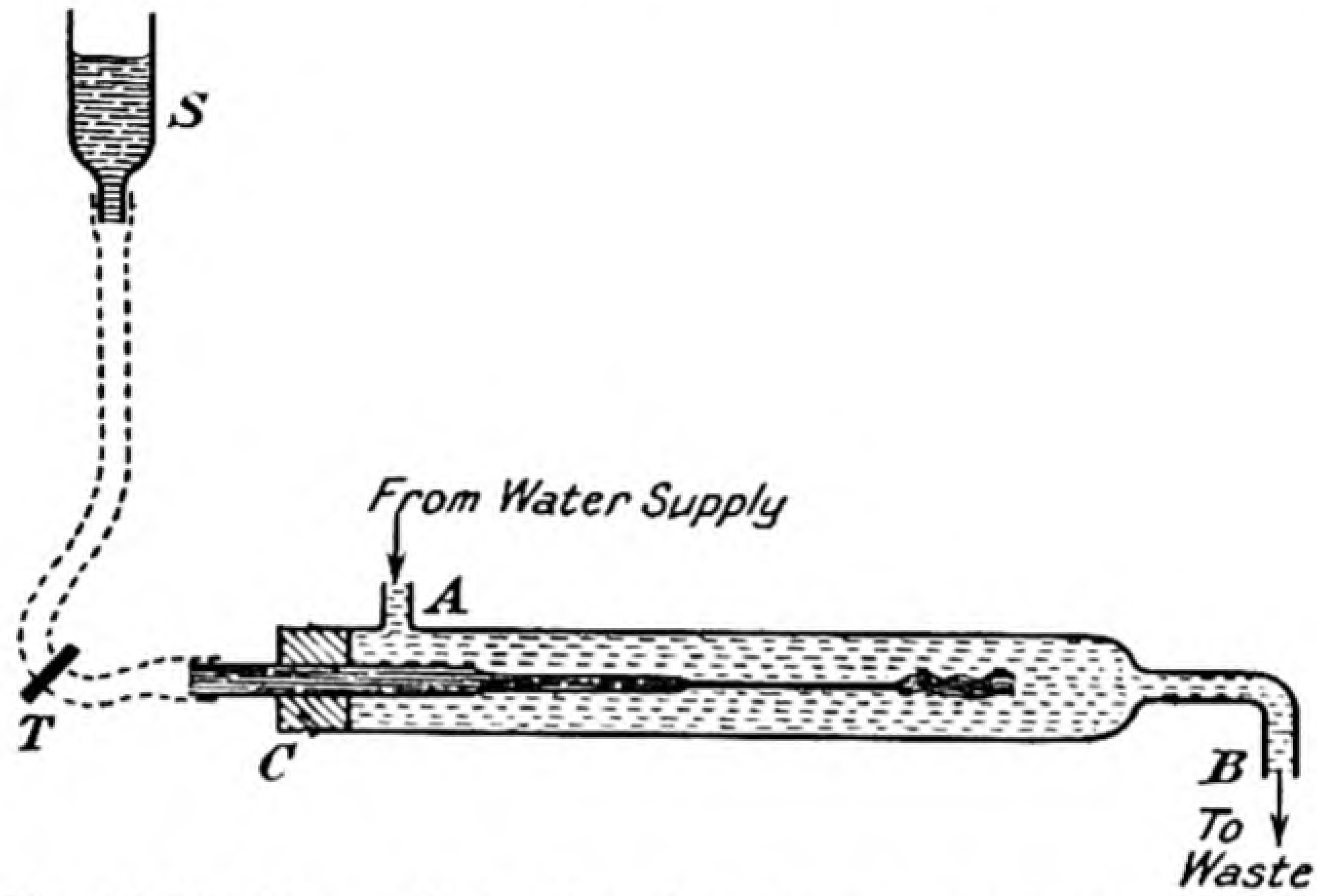


Fig. 12·13—Reynolds' colour-band method for determining critical velocity.

reservoir is connected to a tube, AB, about 2 cm. in diameter for the greater part of its length, the liquid (water) entering at A and escaping at B. At the end, C, of the tube AB is a rubber bung carrying an inlet tube drawn out to a capillary about 10 cm. long and 0·5 mm. in diameter. This is joined to a small reservoir, S, filled with ink. The flow of ink is controlled by a spring clip, T. The position of the reservoir is adjusted by means of the flexible tubing connecting it to A until, with a slow rate of flow of water, a long column of ink is seen escaping from the jet along the central axis of the stream. Under these conditions the liquid is moving with streamline motion, i.e. all filaments are moving parallel to the boundary of the tube. When the state of turbulent flow is reached the coloured band is broken up by eddies and mixes with the water.

(*b*) The second method is due to BARNES and COKER, who in 1904 investigated by means of differential mercury thermometers the variations in the excess temperature of the fluid above its surroundings with the rate of flow. While the motion is streamlined, the frictional resistance is proportional to the velocity u, but when the state of turbulence is reached the resistance is proportional to u^n. It therefore follows that if n exceeds unity the slope of the

graph showing how the excess temperature depends upon the velocity of flow will be much steeper above the critical velocity. The critical velocity is represented by the kink in the curve, cf. Fig. 12·12(a).

Reynolds' number.—From the experiments described in earlier paragraphs, Reynolds found that the lower critical velocity, $[u_c]_1$, is given by

$$[u_c]_1 = \frac{2000\eta}{\rho d},$$

where η is the viscosity of the liquid, ρ its density, and d is the internal diameter of the pipe. Calling $\dfrac{\eta}{\rho} = \nu$, the kinematic viscosity,

$$\frac{[u_c]_1 d}{\nu} = 2000.$$

The quantity $\dfrac{[u_c]_1 d}{\nu}$ is known as ***Reynolds' number***; it is a pure numeric, i.e. it is independent of the system of units used.

For the higher critical velocity $[u_c]_2$, Reynolds' number is 2500.

Motion of a body in a viscous fluid.—If a body of given geometrical shape and size, specified by a length l of some portion of it, is moving with velocity v in a fluid of density ρ and kinematic viscosity ν, then the velocity u, at some point in the fluid and geometrically fixed with reference to the body, may be expressed by the functional equation

$$u = \phi_1(v, \rho, \nu, l),$$

while the slope of the stream line at that point is given by an expression of the form

$$\theta = \phi_2(v, \rho, \nu, l).$$

Each term in ϕ_1 will be of the form v^α, ρ^β, ν^γ, l^δ and must have the dimensions of a velocity. Hence

$$[\mathrm{LT}^{-1}] = [\mathrm{LT}^{-1}]^\alpha[\mathrm{ML}^{-3}]^\beta[\mathrm{L}^2\mathrm{T}^{-1}]^\gamma[\mathrm{L}]^\delta.$$

Comparing coefficients, we find

$$\left.\begin{aligned} 1 &= \alpha - 3\beta + 2\gamma + \delta \\ 0 &= \beta \\ -1 &= -\alpha - \gamma \end{aligned}\right\}.$$

Since $\beta = 0$, $$\alpha + 2\gamma + \delta = 1,$$

and $$\alpha + \gamma = 1.$$

$$\therefore\ \alpha = 1 - \gamma, \quad \text{and} \quad \delta = -\gamma.$$

$\therefore$ Each term has the form

$$v^{\alpha}\nu^{\gamma}l^{\delta} = v^{1-\gamma}\nu^{\gamma}l^{-\gamma} = v\left(\frac{\nu}{vl}\right)^{\gamma}.$$

$$\therefore\ u = v\mathrm{F}_1\left(\frac{\nu}{vl}\right) = vf_1\left(\frac{vl}{\nu}\right). \qquad \text{(i)}$$

Similarly,
$$\theta = f_2\left(\frac{vl}{\nu}\right). \qquad \text{(ii)}$$

[If we write the function for u so that each term in it has the form $v^{\alpha}\, l^{\beta}\, \eta^{\gamma}\, \rho^{\delta}$, we find

$$[\mathrm{LT}^{-1}] = [\mathrm{LT}^{-1}]^{\alpha}[\mathrm{L}]^{\beta}[\mathrm{ML}^{-1}\mathrm{T}^{-1}]^{\gamma}[\mathrm{ML}^{-3}]^{\delta}.$$

Whence $\delta + \gamma = 0$, $1 = \alpha + \beta - \gamma - 3\delta = \alpha + \beta + 2\gamma$, and $1 = \alpha + \gamma$. Thus $\beta = -\gamma$, and $\alpha = 1 - \gamma = 1 + \beta$.

$\therefore$ Each term has the form

$$v^{1+\beta}l^{\beta}\eta^{-\beta}\rho^{\beta} = v\left(\frac{vl}{\nu}\right)^{\beta}.$$

$$\therefore\ u = vf_1\left(\frac{vl}{\nu}\right),$$

as before.]

The second equation indicates that the shape of the stream lines and the direction of motion at any point depend only on the value of the non-dimensional group $\left(\frac{vl}{\nu}\right)$ and not on the separate constituents of the group. The first equation shows that the velocity depends only on v and $\left(\frac{vl}{\nu}\right)$. It is therefore established that, provided a system of bodies are all geometrically similar their sizes alone being different, the systems of stream lines are likewise similar provided $\left(\frac{vl}{\nu}\right)$ is kept constant.

The results of the brilliant series of experiments by Reynolds, described on pp. 624 to 628, led him to the conclusion that a non-dimensional group of the form $\frac{vl}{\nu}$ determines the process that takes place during the motion of a fluid through a cylindrical tube. Thus if d is the diameter of such a pipe and u the velocity of the liquid passing through it, $\frac{ud}{\nu}$ is a critical variable and when it attains

a certain value the flow changes from a steady so-called stream-line flow to a turbulent one in which eddies are formed. If R is Reynolds' number (2,000)†,

$$u = \frac{2000\eta}{\rho d} = \frac{2000\nu}{d}.$$

$$\therefore \frac{ud}{\left(\frac{\eta}{\rho}\right)} = \frac{ud}{\nu} = 2000.$$

Frequency of eddy formation.—Let f be the frequency of eddies formed in a fluid when it flows past a solid object or emerges from a tube with velocity u. Then, using the other symbols as defined in the previous paragraph, we may write

$$f = \psi(u, l, \nu, \rho).$$

Each term in this function will be of the form

$$u^{\alpha} l^{\beta} \nu^{\gamma} \rho^{\delta},$$

and have dimensions T^{-1}.

$$\therefore [T^{-1}] = [LT^{-1}]^{\alpha}[L]^{\beta}[L^2T^{-1}]^{\gamma}[ML^{-3}]^{\delta}.$$

$$\therefore \delta = 0, \quad 0 = \alpha + \beta + 2\gamma, \quad \text{and} \quad -1 = -\alpha - \gamma.$$

$$\therefore \alpha = 1 - \gamma, \quad \beta + \gamma + 1 = 0.$$

$$\therefore \alpha = 2 + \beta.$$

Hence each term in ψ has the form

$$u^{2+\beta} l^{\beta} \nu^{-1-\beta} = \frac{u}{l} \cdot \left(\frac{ul}{\nu}\right)^{1+\beta}.$$

$$\therefore f = \frac{u}{l}\, \psi\left(\frac{ul}{\nu}\right).$$

PART II—THE FLOW OF SOLIDS

Plasticity or the flow of solids.—In practice it is found that there exists in nature a number of substances—solids in ordinary parlance and of which pitch and Chatterton's compound are familiar examples—which do not begin to flow until the applied stress exceeds a certain value. On a stress-strain diagram this value is represented by the yield-point [cf. Fig. 7·09, p. 273]. Such substances always behave as elastic bodies when the stress is below

† Sometimes Reynolds' number is expressed in terms of the radius of cross-section of the tube; then its value is 1,000.

the value corresponding to the yield-point, but for larger stresses the amount of deformation depends upon the time for which the stress is operative. The substance is then said to be in a ***plastic state***. Such a state must be carefully distinguished from that of a liquid when its flow is controlled only by viscous forces, for in this latter instance there is always a finite flow however small the applied stress. BINGHAM, an early worker on the plastic state, visualized an ideal plastic solid as one for which the rate of flow is always zero for stresses below a critical value but beyond it the rate of flow is directly proportional to the excess stress. Such a substance is known as a 'Bingham solid' but it is very doubtful if such is ever realized in practice for with all actual materials the yield point is not uniquely defined; it is found that between the region of perfect elasticity and the region of plastic flow there is a region of imperfect elasticity.

Plasticity.—Many workers on this subject have endeavoured to give a quantitative definition of plasticity ($\pi\lambda\alpha\sigma\tau\iota\kappa\acute{o}\zeta$) but none is entirely satisfactory. Here it must suffice to say that plasticity is that property of a substance which enables it to be deformed continuously and permanently without rupture during the application of a stress exceeding the yield-value of the material.

The yield-value of a material.—The definition now to be given is a little crude, but it will serve our present purpose. Suppose that a body is subjected for some time to a shear stress which is then reduced to zero. When the stress has thus been removed the body may not recover its original size and shape—it is then said that the body has acquired a ***permanent set*** and that the applied stress exceeded its ***yield-value***. This is not a precise statement for no mention is made of the manner of loading or the magnitude of the permanent deformation.

Early work on the plastic deformation of materials.—Strains produced within the range of perfect elasticity do not give rise to any visible change in the structure of the material. It is known, however, that there are small changes in such physical properties as the resistivity or thermoelectric power [with respect to lead] of the material but the difficulties of investigating such changes even experimentally are manifold. On the other hand the permanent alterations in size and shape which are a feature of the production of permanent set are always accompanied by visible changes both in the microscopic and macroscopic structure of the material.

The fact that metals could be caused to flow through small apertures by subjecting them to high stresses and so behave like liquids of high viscosity was first established, in 1864, by TRESCA.

In these experiments the lines of flow were made apparent by using thin sheets of metal, superposed to form a solid block. Tubes and rods of lead and copper, alloys of lead and tin, have been produced by extruding the material through annular openings. Alloys of copper and zinc, at temperatures much below their melting points, yield to the same treatment.

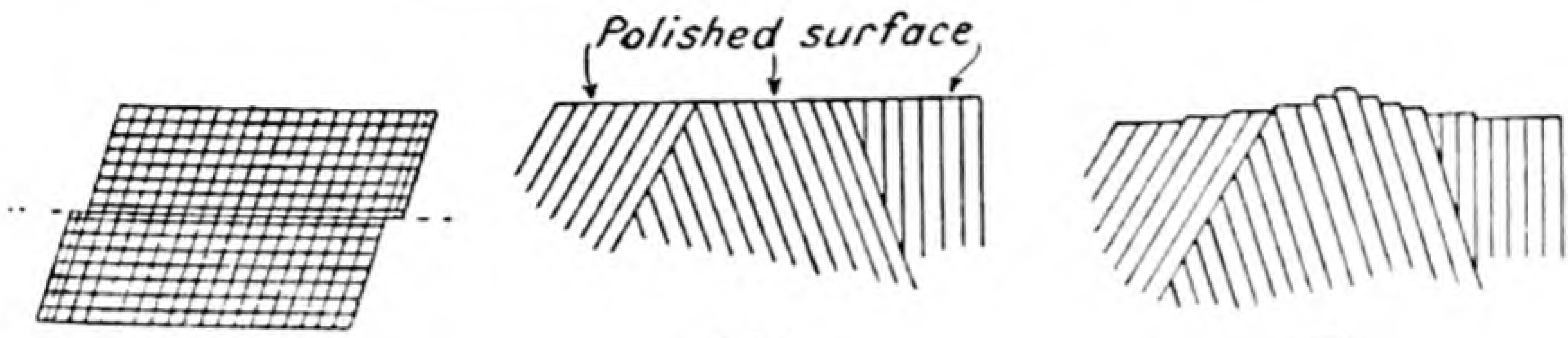
The question which arises at once is 'What is the internal mechanism of plastic deformation?' Let us consider a poly-crystalline metal such as lead and a substance like pitch. Each substance can be made to flow but the superficial resemblance between the phenomena is not due to an identity of the processes. Pitch does behave, as regards its mode of flowing, like a highly viscous liquid, but with the aid of a microscope it is soon apparent that the phenomenon in metals is essentially different. The problem was first studied in connexion with the flow of large masses of ice, e.g. glaciers. In 1845 FORBES asserted that the flow of a glacier was due to the plactic flow of ice-crystals under the action of large stresses. In opposition to this theory TYNDALL, 1857, attributed the flow to the well-known phenomenon of regelation. Such an explanation must be confined to substances such as ice, the melting point of which is lowered when pressure is applied. The well-known 'creeping' of lead sheets on a sloping roof—Tyndall reported that the movement of the lead covering on the choir of Bristol Cathedral amounted to nine inches per year—is due to its flow under gravity, and even ice is plastic at temperatures below which water exists in liquid form, so that plasticity must be independent of regelation.

Writing on this subject Professor DESCH says 'Many crystals are capable of being deformed without losing their crystalline character, from the oleates, which are so soft that two crystals brought into contact at a point unite to form a single one, or potassium manganous chloride [$MnCl_3 . 2KCl.H_2O$], a crystal of which may be pressed into lenticular shape by the fingers without being cracked or broken, to the apparently rigid crystals of calcite or rock salt. Even these may be deformed to an extraordinary extent if certain precautions are taken.'

If a piece of metal with a highly polished surface is subjected to stretching forces it is found that a number of fine parallel lines make their appearance on each grain; the direction of these lines varies from one grain to another. At the boundaries of these grains an abrupt change in the direction of these fine lines reveals the crystalline structure of the metal. 'It is not easy, at first sight, to recognize the true character of these lines; they are not cracks or ridges. Under oblique illumination, lines having any given direction are found to be visible only when the incident light

falls on them from a particular direction, so that they alternately appear and disappear when the stage of the microscope is rotated. The lines are therefore shown to be steps, and the name of "*slip-bands*" has been given to them, as best expressing their character.'

The formation of slip-bands takes place in the following way. The regular arrangement in planes of the atoms or ions in a crystal permits slipping of these ions relative to each other to take place more readily along certain planes and these are known as ***glide-planes*** or planes of easy gliding. Fig. 12·14(*a*) is a diagrammatic representation of the structure of a crystal and it is at once apparent that a translatory motion in a horizontal plane may occur without



(*a*) (*b*) (*c*)

FIG. 12·14.—The formation of slip-bands in a metal.

any disturbance of the crystalline structure. Such motion may occur along a number of such planes parallel to one another. Slip-bands have their origin when successive slips occur in parallel glide-planes. Fig. 12·14(*b*) represents a polished surface before straining and on it portions of some crystal grains are indicated. After straining the grains are deformed as in Fig. 12·14(*c*). The inequalities in level are caused by the slipping which has taken place along the glide-planes and it is the steps produced on the surface in this way that are the slip-bands previously described. The phenomenon exactly resembles the formation of repeated 'faults' in geological strata.

The plastic yielding of a metal under stress beyond the yield-value is therefore attributable to the slipping which occurs on innumerable glide-planes and the only disturbance in the crystalline structure is that certain atomic layers suffer a translatory displacement relative to their neighbours.

More about the plastic deformation of metals—early work with lead.—In 1904 TROUTON and RANKINE observed that for a lead wire loaded well beyond the elastic limit, the extension after some time becomes directly proportional to the time, i.e. the flow is viscous in character. In 1910 ANDRADE carried out the first careful investigations on the phenomenon and in them it was realized that as the wire stretches, the cross-section diminishes and hence, for a constant load, the stress increases. To overcome this

difficulty the following automatic method of establishing a constant stress in the wire was devised; the essential details of the apparatus used are shown in Fig. 12·15(*a*).

The 'weight' attached to the lower end of the wire was allowed to sink into water as the wire stretched, the geometric form of the weight being chosen so that the upthrust at any moment was

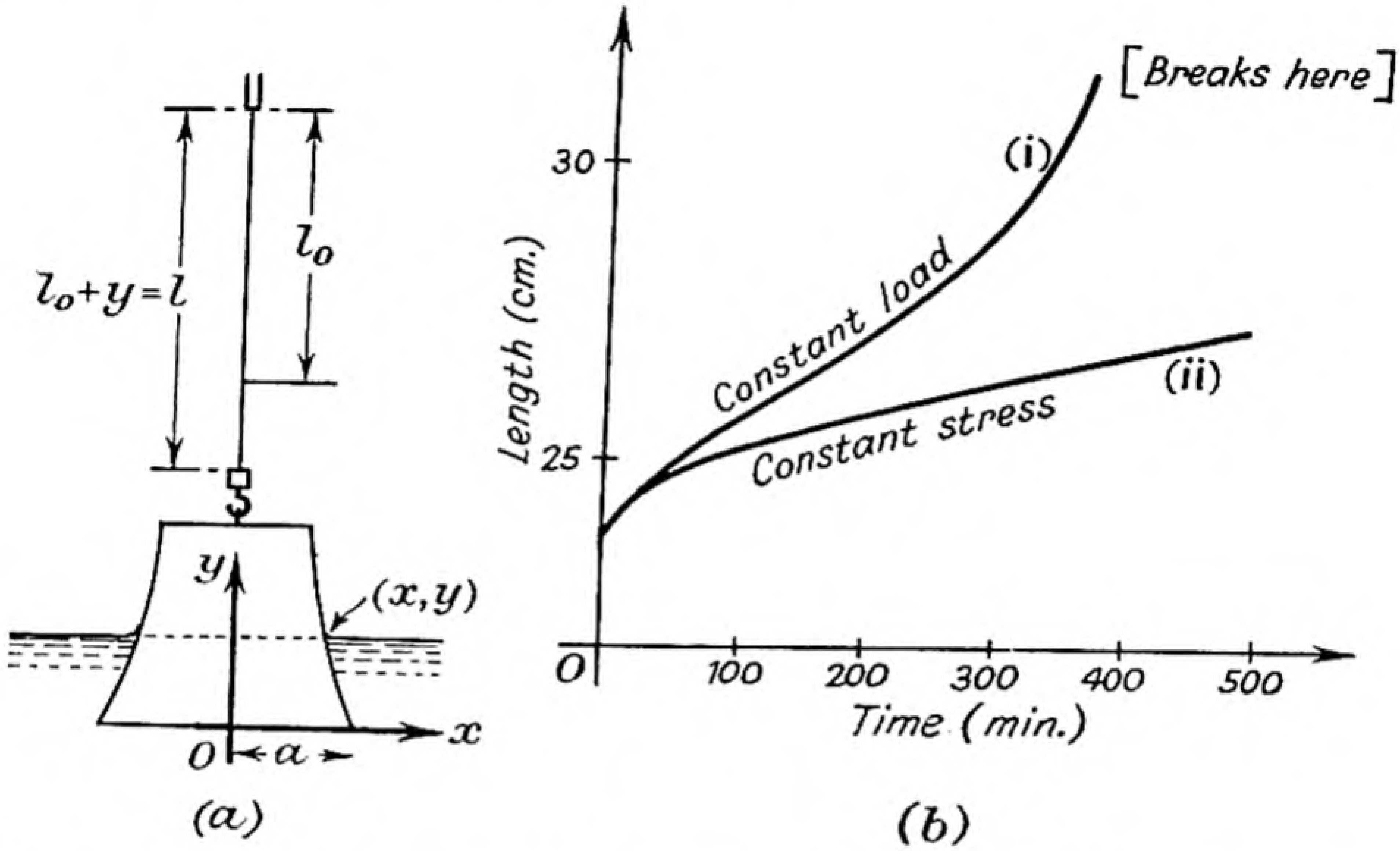


Fig. 12·15.—Andrade's method for producing a constant stress in a wire.

inversely proportional to the length of the wire at that instant; the effective stretching force thus varied directly as the cross-section of the wire. If M is the mass of the load and V the volume of the solid, i.e. of the load, immersed at any instant,

$$(\mathrm{M} - \rho\mathrm{V}) \propto \frac{1}{l_0 + y},$$

where ρ is the density of water, l_0 the initial length of the wire and y is the depth of immersion of the load and hence the elongation of the wire. Thus

$$(\mathrm{M} - \rho\mathrm{V})(l_0 + y) = \text{constant} = \mathrm{M}l_0.$$

$$\therefore\ \mathrm{V} = \frac{\mathrm{M}}{\rho}\left(\frac{y}{l_0 + y}\right).$$

With the notation indicated in Fig. 12·15(*a*) we have

$$\mathrm{V} = \int_0^y \pi x^2\, dy = \frac{\mathrm{M}}{\rho}\left(\frac{y}{l_0 + y}\right).$$

Differentiating with respect to y, we obtain

$$\pi x^2 = \left[\frac{M}{\rho} \cdot \frac{(l_0 + y) - y}{(l_0 + y)^2}\right],$$

or

$$x = \frac{1}{l_0 + y}\sqrt{\frac{Ml_0}{\pi\rho}}.$$

Thus the weight must be given the form of a portion of a hyperboloid of revolution about the axis Oy.

The results obtained with lead are shown by the curves reproduced in Fig. 12·15(b). The experiments were carried out at room temperature and under the conditions of constant stress the initial instantaneous stretch is followed by a flow which finally becomes constant for large stresses. At constant load the rate of flow rapidly increases.

ANDRADE found that the flow under constant stress is represented by the formula

$$l = l_0[1 + \beta t^{\frac{1}{3}}] \exp(\kappa t),$$

where β and κ are constants.

In 1932 ANDRADE and CHALMERS carried out similar experiments on polycrystalline cadmium and the excellent agreement of the above formula with the observations is revealed by Fig. 12·16.

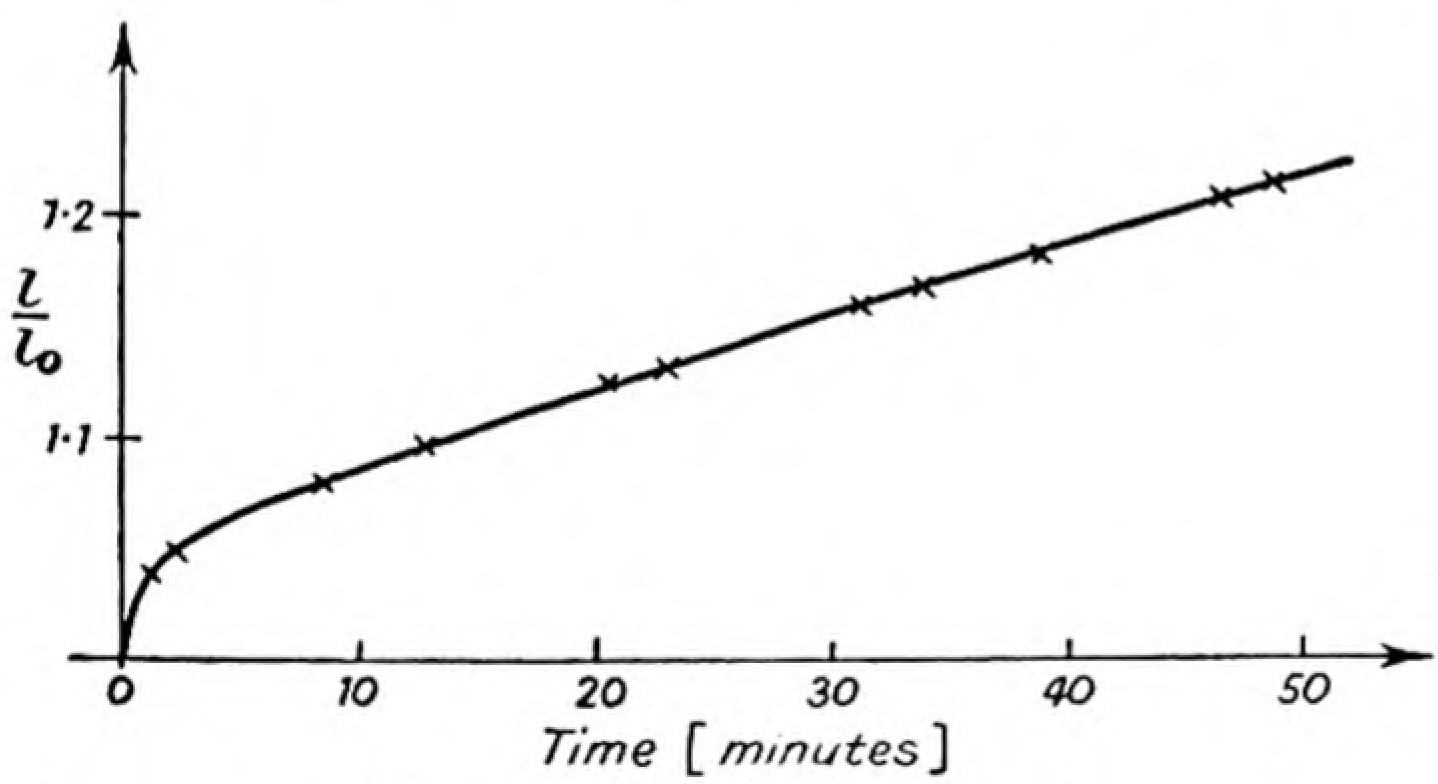
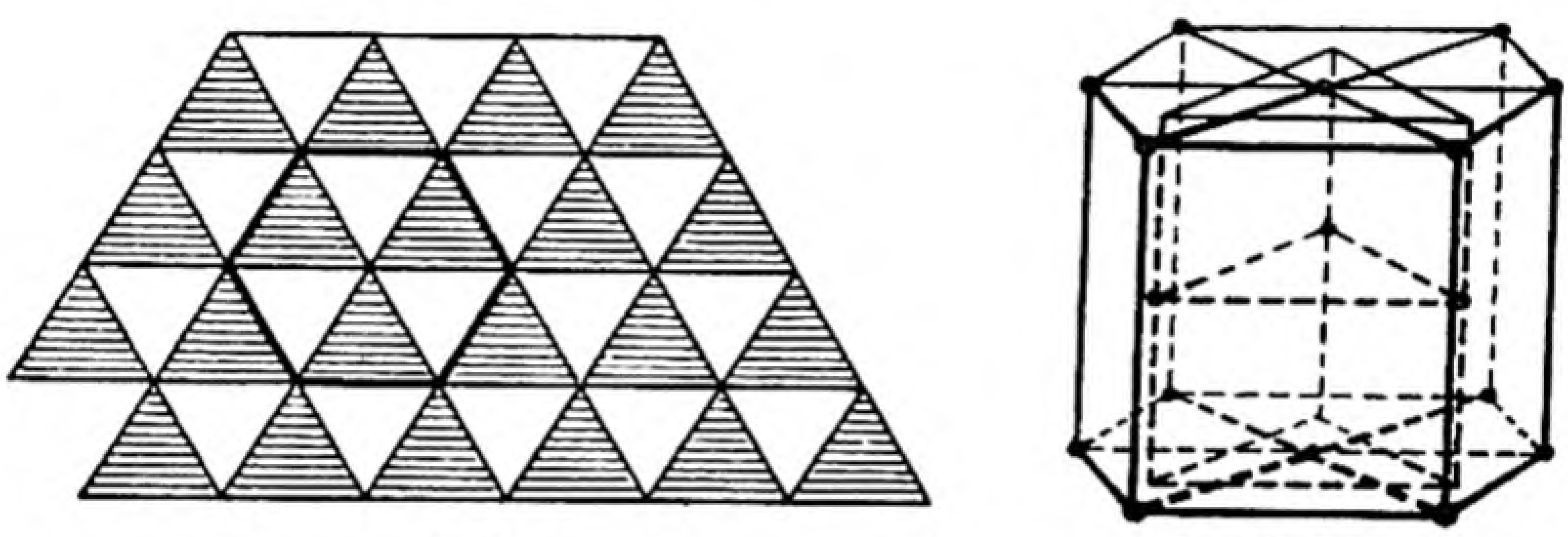


FIG. 12·16.—Fit of the formula $l = l_0[1 + \beta t^{\frac{1}{3}}] \exp(\kappa t)$ to the flow of polycrystalline cadmium at constant stress.

Note on crystals.—Until about 1912 a crystal was regarded as an object with a well-defined geometrical shape such as that of a cube, tetrahedron, dodecahedron, hexagonal prism, etc. Curiously enough the external shape of a crystal is not one of its fundamental characteristics but merely a manifestation of the fact that its ionic, atomic, or molecular constituents are arranged in a regular or systematic pattern in space.

For our present purpose only crystals whose external form is that of a hexagonal prism will be considered. By means of X-rays it has been shown that this name is not an ideal one because in such crystals the ions or atoms are situated on a lattice in which the unit prisms have equilateral triangles for their bases. Fig. 12·17(*a*) shows how the triangular bases (shaded) of the unit prisms fit together so that, under suitable conditions, a crystal with a hexagonal form may be obtained. [Only the prisms whose bases are shaded are considered to contribute to the structure of a crystal.]

In a simple triangular lattice the atoms are arranged at the



(*a*) (*b*)

FIG. 12·17.—Unit triangular close-packed prism.

corners of the triangular prisms and each atom is shared by six prisms. This means that each prism whose base is shaded in Fig. 12·17(*a*) has one atom associated with it.

If an additional atom is placed at the body-centre of each prism, the lattice on which the atoms occur is known as a 'hexagonal close-packed' lattice; such a lattice is shown in Fig. 12·17(*b*). In order to gain a clearer picture of such a lattice we can imagine that each atom (or ion) is represented by a sphere, all spheres being equal. If these spheres are arranged in a layer and packed as closely as possible together the centres of the spheres will lie at the corners of equilateral triangles and each 'interior' sphere will be surrounded by six immediate neighbours whose centres lie at the corners of a hexagon. In Fig. 12·18(*a*) these spheres are marked A.

Now let an identical sphere, B, be placed on this layer so that it nestles in one of the dimples or umbilical recesses between three A spheres. A second layer, identical with the first, may be built up by continuing to lay B spheres in position. When this is done it will be found that every alternate umbilical depression in the first layer is covered by a sphere in the second layer—cf. Fig. 11·40(*b*).

Other spheres, C, may now be added to form a third layer and this may be arranged in one of two ways. Either the remaining depressions in the first layer may be covered by the C spheres, as in

Fig. 12·18(*c*), or each such sphere may be placed over a sphere in the layer A, as shown in Fig. 12·18(*d*).

In the first arrangement we have a face-centred cubic lattice, as seen by looking along the diagonal of the cube. Each sphere is in contact with a ring of six neighbours in its own layer and three in

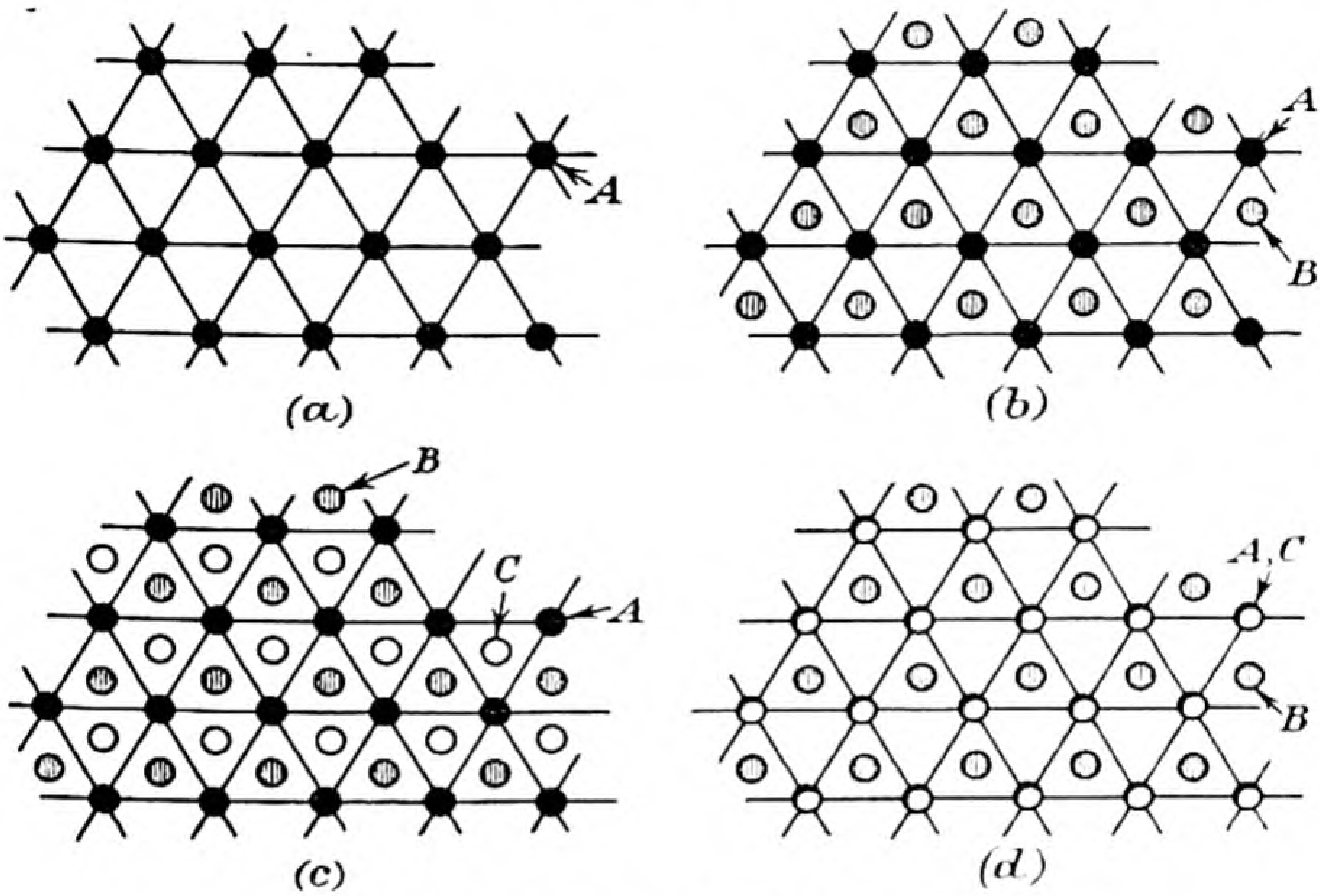


FIG. 12·18.—The formation of close-packed lattices.

each layer immediately above and below its own; thus the total number of spheres in contact with any other is twelve.

In the second arrangement we have the hexagonal close-packed structure and again each sphere is in contact with twelve others. The planes of closest packing are the basal planes.

Single crystals.—Crystals of metals have their origin in nuclei which are present in the cooling mass of molten metal. In 1913 ANDRADE succeeded in preparing large single metallic crystals. These were made by heating metallic wires to a temperature just above the melting point and allowing them to cool very slowly. In another method, due to CZOCHRALSKI, the lower end of a suitable grafting rod is immersed in the molten metal at a temperature a few degrees in excess of its melting point and a crystal thread is slowly drawn upwards out of the melt; the speed of withdrawal must be found by trial. In 1921, CARPENTER and ELAM produced single crystals of aluminium by a method founded upon the principle of recrystallization after stretching. Initial strains within the material are removed by an annealing process and then a two per cent plastic

stretch is produced. This is followed by a process involving alternate heating and straining, the growth, promoted by the heat treatment, taking place from centres created by the strain.

At a later date ANDRADE and ROSCOE enclosed a wire somewhat loosely in a horizontal glass tube surrounded by a small electric furnace travelling slowly along the tube. The wire is just melted locally over a short length which moves with the furance and in this way strain-free crystals of such metals as cadmium can be produced.

The mechanical deformation of a single crystal.—When a single crystal in the form of a wire is subjected to a very small stretching force it is found that the crystal elongates and the appearance of a line round the external surface of the crystal shows that a displacement between the 'upper' and 'lower' parts of the wire has taken place but no further elongation occurs. When the stress is raised another line appears, i.e. another displacement occurs and then the stretching stops. So the process may be continued, each successive extension giving rise to another line which is the intersection of a slip-plane with the surface of the wire; in the process, the surface of the wire, originally smooth, becomes 'scaly'. After the formation of many such lines the wire, originally cylindrical in section, becomes ribbon-shaped, so that it appears somewhat like a pack of cards which has been pushed lengthwise. The deformation of the wire is due to whole sections of the crystal moving as entities along certain crystallographic planes, the so-called planes of 'easy glide'. Finally, when the wire breaks, the fracture is parallel to the slip planes and it possesses a mirror-like surface.

A remarkable feature of this phenomen is that the yield-point of the material concerned has a very low value and that plastic deformation begins when the stress is several times lower than that for the same material in the polycrystalline state, i.e. single crystals are extraordinarily weak and plastic. Also, the extension may be several times the length of the original crystal. With a single crystal of copper, a permanent set may be caused by a tensile stress as small as 150 gm.-wt.mm.$^{-2}$ and when the fractional deformation along the length of the specimen amounts to about fifty per cent, the mechanical strength increases about fifty times.

The gliding of two sections of a single crystal past each other during slip differs from the gliding of two smooth well-lubricated metal plates in that (*a*) the plates can glide in any direction but the direction of glide in a crystal is determined by some crystallographic axis and its inclination to the direction of maximum stress; (*b*) the frictional resistance to gliding in crystals increases as the amount of deformation increases but if the area of contact of the plates remains constant so does the resistance to gliding; (*c*) the work spent in causing the plates to glide is converted into heat whereas when a

crystal is deformed some of the energy becomes stored as potential energy in the specimen.

The deformation in single crystals is caused, as we have noted already, by the shearing of the crystal upon certain 'glide planes'. The gliding process is most easily studied in hexagonal close-packed crystals such as zinc. Now zinc is a metal with a close-packed hexagonal arrangement of atoms and glide normally occurs on a basal plane, i.e. the plane with the greatest density of atomic population. As glide proceeds the resistance to extension increases, i.e. the metal becomes strain hardened. The direction of glide in

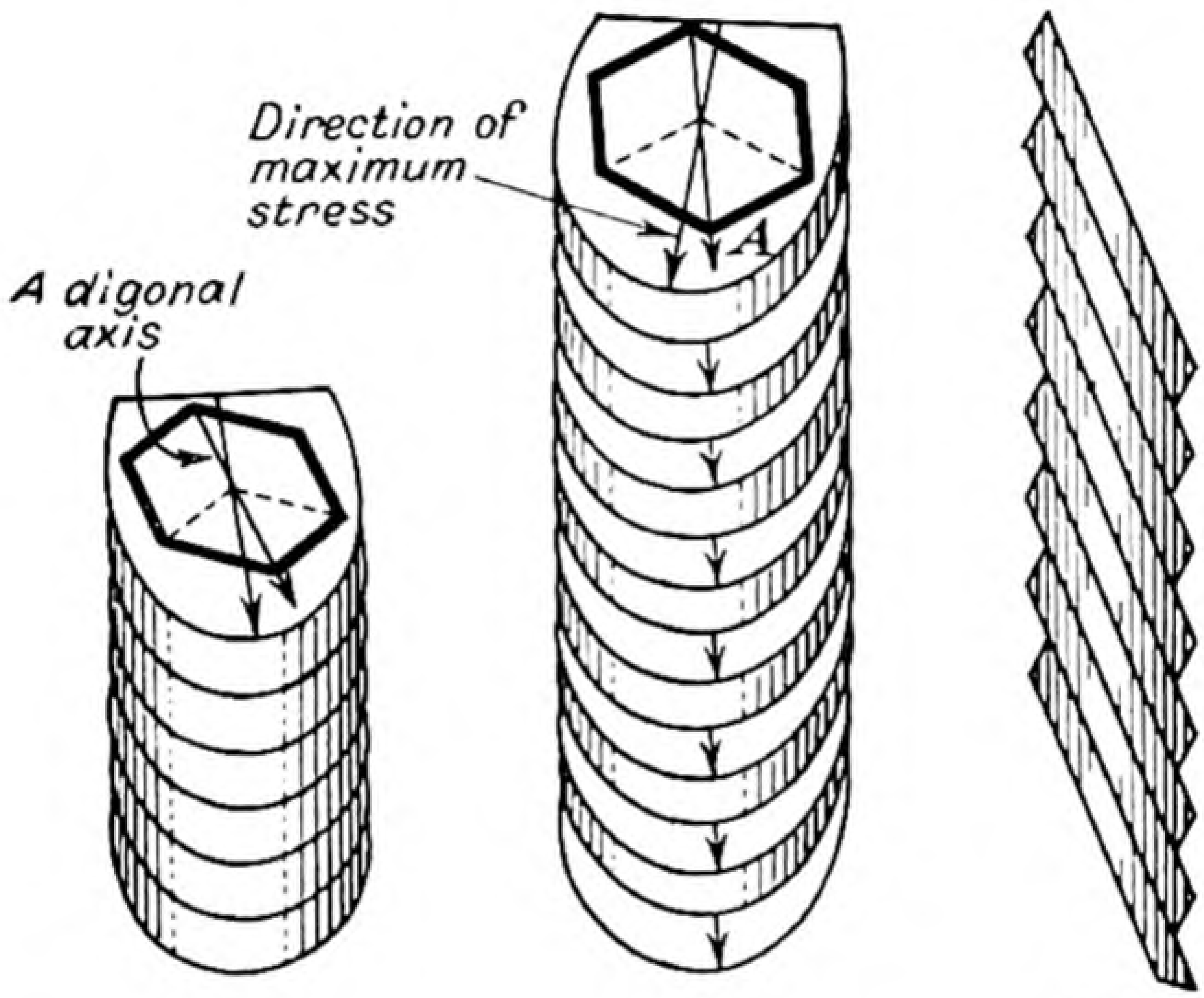


FIG. 12·19.—Polanyi's model of a single crystal in the form of a wire.

a basal plane is usually parallel to that digonal axis, which is a direction of greatest row density in the lattice, and which is nearest to the direction along which the maximum stress exists. As slip proceeds the direction of slip gradually alters and the planes of slip tend to become parallel to the direction of the stretching forces.

In a polycrystalline specimen some of the crystals will be orientated in such a way that slip actually occurs in them but the shifting process is greatly hampered by the surrounding crystals if they are less favourably orientated.

To render apparent the processes at work during the deformation of a single crystal of zinc POLYANI (1924) constructed a wooden model. A diagram of this model is shown in Fig. 12·19. In Fig. 12·19(*a*) there is shown a pile of wooden discs of equal thickness, the flat surfaces of the discs being the basal planes of the

crystal. It will be noted that these flat surfaces are not normal to the axis of the wire. The hexagon on the uppermost surface indicates the crystallographic orientation of the zinc atoms or rather of the lattice upon which they are situated in orderly manner. One of the three digonal axes is also shown. Fig. 12·19(*b*) shows the model after the crystal it represents has been deformed to a certain extent. The direction of glide is indicated by the arrow A; the second arrow shows the direction of maximum stress in a basal plane. As slip proceeds the direction of slip tends to become parallel to that of the stretching forces. The elongation of the single crystal is caused by the plane of slip tending to become parallel to the direction of stretch—cf. Fig. 12·19(*c*). If the basal plane is, by chance, initially normal to the direction of the applied stretching forces, no slip occurs and as these are increased fracture eventually takes place without slip.

Now the basal plane of a hexagonal lattice is unique for only one plane of slip is possible, but in other lattices, e.g. the face-centred cubic lattice, the four octahedral planes may each serve as a slip plane. In such instances slip may occur first on one set and then on another. The stress necessary to produce the first slip is the true measure of the elastic limit and in contrast to this the pseudo-elastic limit determined by experiments on polycrystalline specimens is of little theoretical importance.

As the temperature of the single crystal specimen is raised the hardening produced by the deforming process is less noticeable but the tendency to flow steadily becomes more prominent.

PART III NON-NEWTONIAN LIQUIDS

Introduction.—Newton's hypothesis concerning the steady flow of a viscous fluid is epitomized by the equation

$$\frac{F}{A} = \eta \frac{du}{dz} = \eta \frac{d\phi}{dt}. \qquad \text{[cf. pp. 536 and 579]}$$

In all simple liquids, i.e. those in which there is only one phase and which do not have a structure that is threadlike, the stress, $\frac{F}{A}$, is directly proportional to the rate of shear, $\frac{d\phi}{dt}$, provided that the flow is laminar. Liquids for which η is a constant at a given temperature (and pressure) are known as Newtonian liquids. In 1921 Griffiths showed that with rates of shear varying from 0·002 rad.sec.$^{-1}$ to 10,000 rad.sec.$^{-1}$, there is no measurable change in the viscosity of air-free water.

In this connexion it is interesting to recall that originally Poiseuille's law was deduced from observations on the flow of

water and aqueous solutions through tubes of small bore and although one of the main purposes of the investigation was to discover laws that govern the flow of blood through the veins and capillaries of the human body, yet it is now known that for blood Poiseuille's law does not hold. The flow of many impure liquids, semi-liquids and semi-solids, which are the liquids important to everyday life, is found to be anomalous in that their behaviour is similar to that of blood. Such liquids are termed ***non-Newtonian***, i.e. the rate of shear is not proportional to the shearing stress and neither Poiseuille's nor Stokes' law is obeyed.

It is found that liquids which behave anomalously in this way are heterogeneous; for example, particles of a different nature may float about in the body of the liquid (***sols***), or the substance may be a ***gel***, i.e. it may possess a structure which binds the mass together to form a semi-solid.

The 'apparent viscosity' of a non-Newtonian liquid.—For a non-Newtonian liquid the quantity

$$\frac{\text{Stress}}{\text{Velocity gradient}}$$

will have a fixed value for a given velocity gradient and this quantity is known as the ***apparent viscosity*** of the liquid. A common feature of anomalous liquids is that the apparent viscosity decreases as the rate of shear increases.

Now blood is not a homogeneous liquid for it contains corpuscles but not in sufficient concentration to show much change of viscosity with the rate of shearing. Sometimes the presence of ambient particles may be detected visually with the unaided eye but often

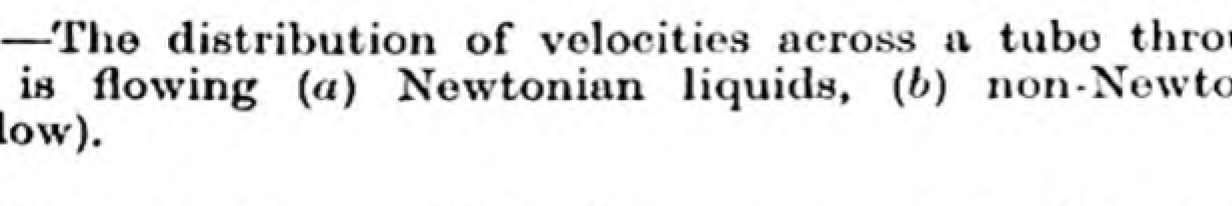


FIG. 12·20.—The distribution of velocities across a tube through which a liquid is flowing (*a*) Newtonian liquids, (*b*) non-Newtonian liquids (plug-flow).

the particles are too small to be seen even under a microscope, yet the heterogeneous nature of the liquid may always be revealed by its ability to scatter light waves.

Fig. 12·20(*a*) shows how the velocity of flow of a Newtonian liquid varies across a tube in which it is flowing without turbulence. For a markedly non-Newtonian flow the distribution of velocities

is shown in Fig. 12·20(*b*). The velocity is nearly constant across the major portion of the tube and falls in a steep gradient to the walls; this type of flow is accordingly known as ***plug flow***.

Another important difference between the two types of liquid under discussion is this; in accordance with Newton's hypothesis, a homogeneous liquid is characterized by the property that it begins to move under the smallest possible rate of shear that may be created within the liquid. When, however, the concentration of the material in suspension in the liquid is high the substance will withstand small shearing stresses without beginning to move. When the stress is increased so that the substance begins to flow, the stress at which this occurs is known as the ***yield value*** for the substance concerned. Somewhat stiff pastes behave in this way. Likewise table jellies may be set in a mould and yet when the mould is removed the jelly retains its shape in spite of the gravitational forces that are endeavouring to make it flow.

In connexion with this Richardson writes,† 'The setting of a gelatine solution is an interesting case in which a non-Newtonian liquid—the ***sol***—acquires internal structure by the reduction of its temperature and becomes a ***gel***, possessing both yield point and elasticity. The latter factor is evidenced by the way in which jelly partly displaced from equilibrium, "shivers" to and fro and eventually returns to equilibrium. The gel can be transformed into the sol again by heating, or, sometimes, by shaking. A system which possesses gel characteristics when left alone, but becomes—temporarily—a sol when disturbed is said to be ***thixotropic***. A good paint should possess this property, so that it can be brushed or sprayed over a surface smoothly and retain a uniform texture and gloss when left. Further, although it has a high viscosity when settling down slowly over the surface it has to cover (and this promotes a good "finish"), yet when forced out of the spray nozzle at high speed its viscosity is low and energy losses in the nozzle are so reduced to a minimum.'

An experimental study of non-Newtonian liquids.—In the study of anomalous viscosity the fundamental parameter is the rate of shear. Accordingly it is necessary to design an apparatus in which different known rates of shear can be produced. At present capillary tubes cannot be used for the subject is not sufficiently far developed. It is necessary to use a cylindrical viscometer. When such a viscometer contains a true liquid its viscosity is given by the formula

$$\eta = \frac{b^2 - a^2}{4\pi a^2 b^2 l} \cdot \frac{\Gamma}{\omega_0}, \qquad \text{[cf. p. 581]}$$

† 'Endeavour', **4**, No. 14, April 1945.

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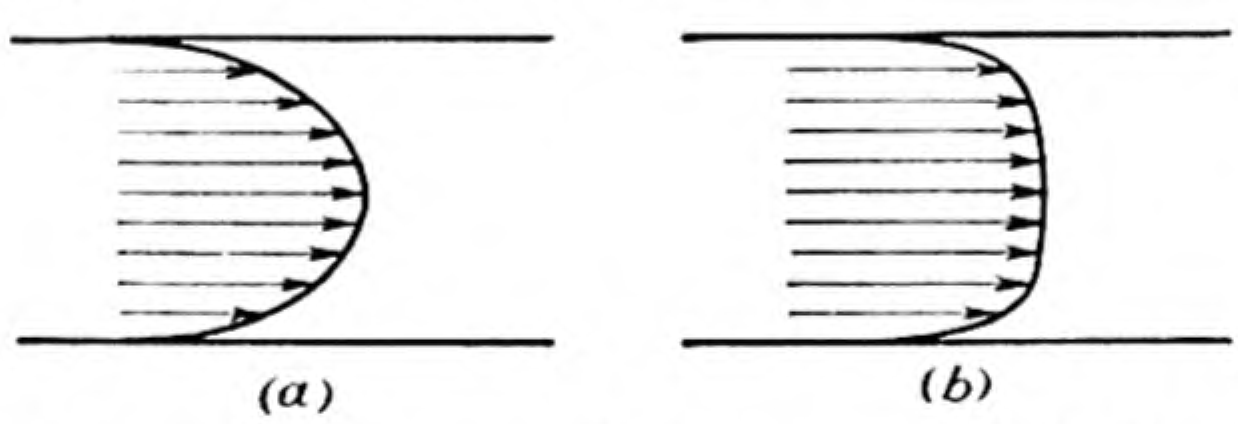

FIG. 12·20.—The distribution of velocities across a tube through which a liquid is flowing (*a*) Newtonian liquids, (*b*) non-Newtonian liquids (plug-flow).

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Fig. 12·20(*a*) shows how the velocity of flow of a Newtonian liquid varies across a tube in which it is flowing without turbulence. For a markedly non-Newtonian flow the distribution of velocities

When two different Newtonian liquids are used, one in the upper and the other in the lower viscometer, the ratio of the speeds of rotation may be adjusted so that the suspended yoke is not deflected.

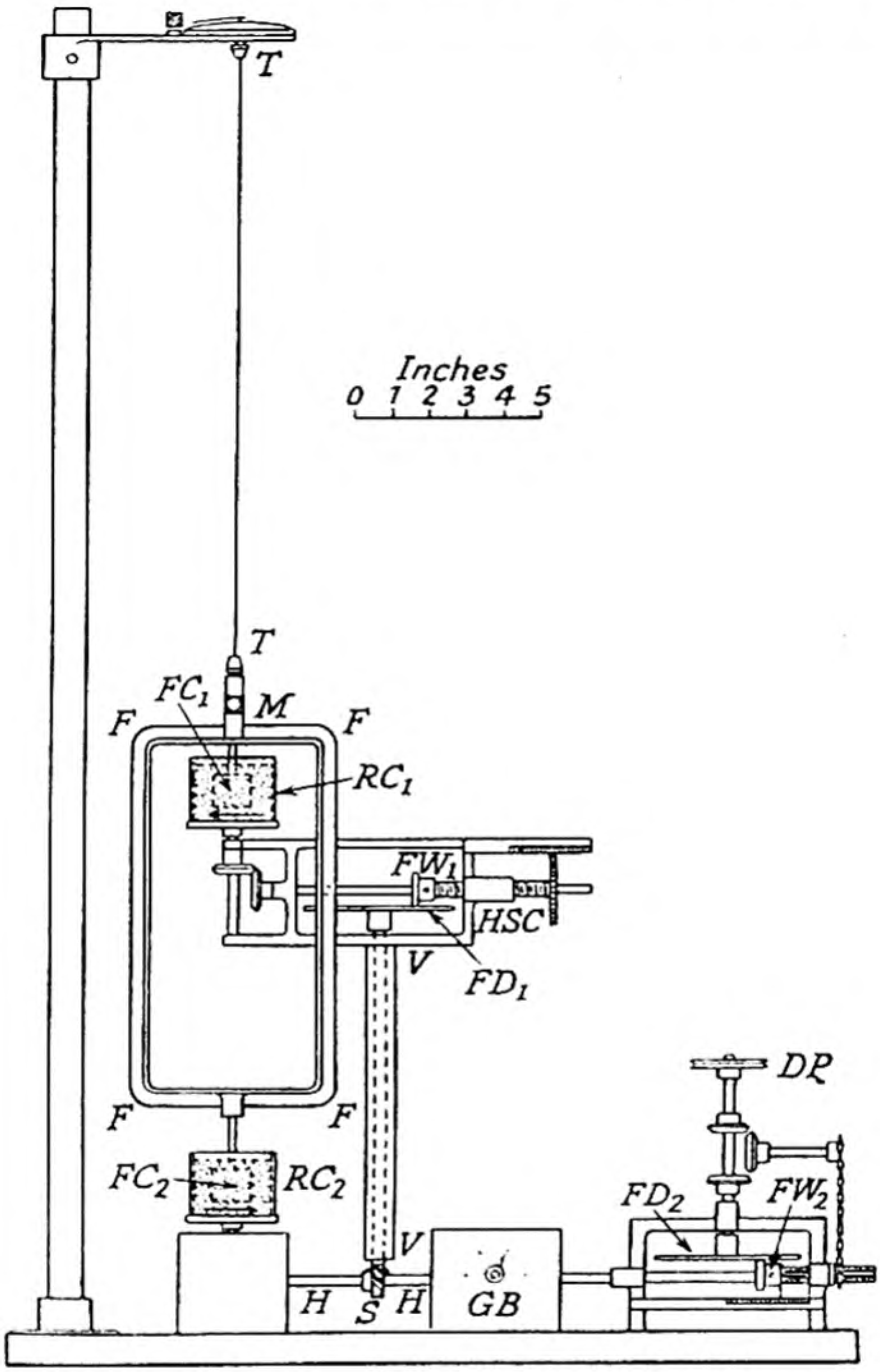

FIG. 12·21.—A viscometer for the study of non-Newtonian liquids.

If the speeds are increased but the ratio kept constant, the deflexion will remain zero, for the rate of shear will increase in the same ratio. This is not so if one liquid is non-Newtonian.

Also, for true liquids, if the speeds of rotation of the outer cylinders are identical the deflexion will vary linearly with the speed as shown by the straight line OA, Fig. 12·22. This line can be represented by the equation

$$\theta = \kappa(\eta_T - \eta_S)\sigma,$$

and if η is constant $\dfrac{\Gamma}{\omega_0}$ is also constant. If the liquid used is non-Newtonian then $\dfrac{\Gamma}{\omega_0}$ will not be constant.

In this type of viscometer the rate of shear is

$$r\frac{d\omega}{dr} = \frac{\Gamma}{2\pi\eta l}\left(\frac{1}{r^2}\right), \qquad \text{[cf. p. 580]}$$

and hence varies across the gap. Discussing this subject ANDRADE writes, 'If the liquid is non-Newtonian we obtain a value for the viscosity which corresponds to some kind of characteristic rate of shear within the range. The question is, how should we take the value most appropriate . . . We can take an average rate of value of the shear by multiplying each elementary surface by the appropriate rate, integrating and dividing by the area between the cylinders, which gives'

$$\text{Mean rate of shear} = \frac{\displaystyle\int_a^b 2\pi r\,.\,r\frac{d\omega}{dr}\,.\,dr}{\pi(b^2 - a^2)}$$

$$= \frac{4a^2b^2\omega_0}{(b^2 - a^2)^2}\ln\left(\frac{b}{a}\right),$$

'or we can take the linear velocity of the outer cylinder and divide by the gap, obtaining for the rate of shear a value $\dfrac{b\omega_0}{b-a}$. The former is the thing to do.'

In 1940 PRYCE-JONES designed the apparatus shown in Fig. 12·21 Essentially it is a double cylindrical viscometer, i.e. it consists of two Couette viscometers, one vertically above the other as shown in the diagram. The upper unit contains a true liquid of known viscosity and the lower unit the material under test. The non-rotating inner cylinders FC_1 and FC_2 are rigidly connected together by means of an aluminium yoke FFFF to which there is rigidly attached a concave mirror M. This yoke is suspended from the torsion wire TT.

The pulley DP, connected to a constant speed motor, drives the gearbox GB through a continuously variable gearing. This consists of a spring-operated friction disc FD_2 and a friction wheel FW_2. The position of this wheel may be adjusted with respect to the centre of the disc so that the speed of the latter may be varied continuously. The rest of the apparatus consists of devices for setting the outer cylinders of the two viscometers in rotation in opposite senses. They may be run at the same angular speed or at speeds bearing a definite but adjustable ratio to one another.

12·02. Show that for stream-line motion in a fluid

$$g\delta z + u\delta u + \frac{\delta p}{\rho} = 0,$$

and hence show that if ρ is constant, $gz + \frac{1}{2}u^2 + \frac{p}{\rho} = \text{constant}$.

For a gas (when z may be neglected) in which $pv^{\gamma} = \text{constant}$, where γ is a constant, show that

$$u^2 + \mathrm{A}p^n$$

is constant, where A is a constant and $n = 1 - \gamma^{-1}$.

Hence show that if the gas flows from a vessel where $u = 0$, $p = p_1$, to a place where $u = \mathrm{U}$ and $p = p_2$, then

$$\mathrm{U}^2 = \mathrm{A}(p_1{}^n - p_2{}^n).$$

12·03. Derive an expression relating the pressure and velocity of a fluid along a streamline (Bernoulli's theorem).

Describe an instrument for measuring the rate of flow of a liquid along a tube and, assuming Bernoulli's theorem, obtain an expression for the rate of flow.

Water flows through a horizontal glass tube whose radius at A is 2·0 cm. and at B is 0·5 cm. Neglecting energy losses due to viscosity and turbulence, calculate a value for the rate of flow of water through the tube when the pressure difference between A and B is 1·8 cm. of water. [46·7 cm.^{3} sec.$^{-1}$]

12·04. Particles of water in a basin, flowing very slowly towards a hole at the centre, move in paths which are approximately circular, so that if u is the speed of a particle, $u = \frac{a}{x}$, where a is a constant and x the distance from the central axis. Show that at the surface of the water

$$z = \mathrm{A} - \frac{a^2}{2gx^2}.$$

Assume A = 1 ft., $a = 0·75$ ft.^{2}sec.$^{-1}$, and plot the curve giving the shape of the curved surface. What does (A $-$ z) represent?

Chapter XIII

BROWNIAN MOVEMENT, OSMOSIS AND DIFFUSION IN AQUEOUS SOLUTIONS

Molecular and colloidal solutions.—When, for example, sugar is added to an excess of water the sugar dissolves; it has disappeared from sight and passed into solution and with the relatively small volumes of water that can be used experimentally it is found that the sugar is uniformly distributed throughout the solution. When solution is complete even the most powerful microscope available fails to detect any particle of sugar. From this fact it is concluded that the sugar has been broken up into its ultimate indivisible entities, i.e. sugar molecules.

On the other hand, when aqueous solutions of sodium chloride and of silver nitrate are mixed together, a white precipitate of silver chloride begins to form at once and the solid particles grow to a limiting size. The particles are readily visible and, under the action of gravity, they eventually settle down on the base of the containing vessel. The particles of silver chloride are so large that they do not pass through the pores of ordinary filter paper and hence the precipitate may be removed from the mixture by filtration. When, however, an aqueous solution containing about 0·01 per cent of gold chloride is made slightly alkaline by the addition of magnesia [magnesium oxide is very slightly soluble in water] and a few drops of formaldehyde are added, the mixture acquires a ruby-red colour. This coloration is due to metallic gold present in the form of minute particles, which do not settle under the influence of gravity, and the gold cannot be removed from the aqueous mixture by filtering. In such instances as this, the precipitate, produced by the interaction of two clear solutions, consists of molecular aggregates which no microscope is sufficiently powerful to render visible to us, and yet when a strong beam of light is caused to traverse the liquid and the liquid is examined in a direction normal to that of the incident beam a turbidity, due to scattered light, is easily seen. This turbidity signifies that the liquid medium has suspended in it aggregates which are small but not of molecular dimensions. Such a suspension is known as a ***colloidal solution.***

To discover the reason why the particles, which comprise a colloidal solution and which almost invariably have a density greater than that of water, do not settle down it is necessary to

recall Stokes' law, which was discussed in the last chapter [cf. p. 592]. There it was shown that every small sphere descending under gravity through a viscous medium acquires a terminal velocity, u, given by

$$u = \frac{2}{9}\frac{(\rho - \sigma)gr^2}{\eta},$$

where the symbols have their usual meanings. For gold particles, 10^{-6} cm. in radius, suspended in water at 20° C., when $\eta = 0{\cdot}01$ poise, the terminal velocity is approximately 4×10^{-7} cm.sec.$^{-1}$, i.e. $2{\cdot}5 \times 10^6$ sec., or 30 days, is the time required for the particles to fall 1 cm. in water.

The Brownian movement.—About 1828 Brown, an English botanist, observed through a microscope some grains of pollen suspended in water. He discovered that these grains were taking part in a much-agitated dance which continued without any decrease in its vigour. The particles were seen to move hither and thither, apparently under no control, in the field of view of the microscope. Further experiments revealed that any kind of particle, provided that it was sufficiently small, exhibits this same remarkable phenomenon, and Brown concluded that some inanimate cause was the agent responsible for it. After many hypotheses concerning the ultimate cause of this motion had been made, Delsaulx in 1877 and Gouy in 1888 advanced the view that the irregular motion of the suspended particles is due to the thermal agitation of the molecules of the surrounding liquid.

As a result of many researches between 1828 and 1905 it was established that:—

(*a*) the suspended particles undergo very irregular motions and these take place completely at random;

(*b*) the greater the viscosity of the liquid medium, the more slow is the motion of the particles suspended in it;

(*c*) the smaller the particles the more quickly do they move;

(*d*) the motions exhibited are continuous and eternal; the Brownian movement of suspended particles has been observed in liquids contained in the enclosed cavities of some varieties of quartz and these cavities and the liquids in them will have been there for thousands of years.

Thus it was established that this so-called Brownian movement is a characteristic of individual particles, each one moving independently of its neighbours. Since the motion is eternal, the only source of energy necessary to maintain the motion must be in the ever-present thermal motion of the liquid molecules themselves.

With these ideas for a background, a mental picture of the processes at work can be formed. When the suspended particle is large

in comparison with the liquid molecules [the ratio being of the order 10^6], the molecular bombardment to which the particle is subjected is practically uniform, and for any one of these particles its motion will be determined solely by gravity and the viscosity of the liquid. With smaller particles, the number of bombarding particles per unit time is less and the laws of probability indicate that the bombardment will be less uniform. This inequality in the distribution of the attacking molecules will give rise to a force causing the suspended particle to move; its motion will be determined in any given instance by the viscosity of the medium in which it moves.

Brownian movements and Laplace's law of atmospheres.—From many researches made in connexion with the physical chemistry of solutions it has been concluded that when substances are in solution the molecules of the dissolved substance behave in many respects as if the same molecules were moving freely as molecules of an ideal gas in free space. It will now be assumed that the particles in a suspension and undergoing Brownian movements are comparable to an atmosphere of molecules in otherwise empty space. To proceed, however, it is necessary to establish theoretically ***Laplace's law of atmospheres.***

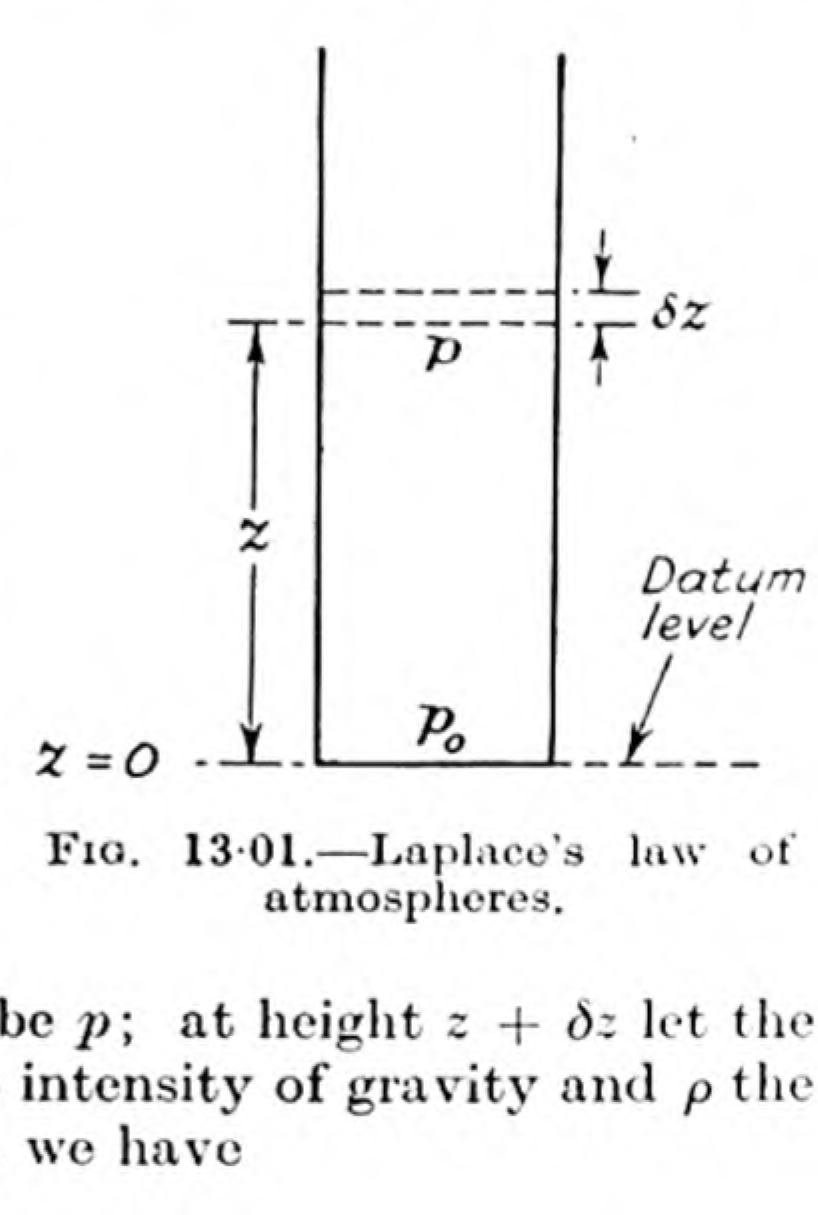

Fig. 13·01.—Laplace's law of atmospheres.

To do this, consider an ideal gas enclosed in a tall vertical cylinder as shown in Fig. 13·01. Let the pressure at a height z in the cylinder be p; at height $z + \delta z$ let the pressure be $p + \delta p$. Then if g is the intensity of gravity and ρ the density of the gas at this pressure p, we have

$$\delta p = -g\rho\,\delta z = -g\,\frac{M}{V}\,\delta z,$$

where M is the mass of a gram-molecule, or mole, and V is the volume occupied by one mole of the gas at pressure p. We have, also,

$$pV = RT,$$

where T is the absolute temperature of the gas and R is the universal gas-constant.

$$\therefore\ \frac{\delta p}{p} = -\,\frac{Mg}{RT}\,\delta z,$$

and on integrating this becomes

$$p = p_0 \exp\left(-\frac{Mg}{RT}z\right),$$

where p_0 is the pressure of the gas at the datum level, i.e. at $z = 0$. In carrying out this integration it has been assumed that the temperature is everywhere constant.

Further, since n and n_0, the number of particles per unit volume at height z and at the datum level, respectively, are directly proportional to p and to p_0, we have

$$n = n_0 \exp\left(-\frac{Mg}{RT}z\right) = n_0 \exp\left(-\frac{mNg}{RT}z\right),$$

where N is Avogadro's constant, i.e. the number of molecules in a gram-molecule, and m is the mass of one molecule.

This is Laplace's well-known law of atmospheres and it expresses precisely how, in a gaseous atmosphere, the number of particles per unit volume varies with height.

From the above argument it follows that since the mass of an actual gas molecule is extremely small, considerable heights must be reached before the density varies by a measurable amount. On the other hand, for molecules like sugar, a fractional change in the density of their distribution equal to that for a given change in the density of nitrogen, is reached in a height equal to exp (0·1) times the corresponding height in nitrogen. For colloidal particles with a mass 10^5 times that of a sugar molecule, a fractional change in the density of distribution will occur in a few millimetres whereas a change in height of several kilometres would be necessary to produce a corresponding change in a truly gaseous atmosphere.

Before use of Laplace's equation, expressing the distribution of particles in a gas with height, can be made in connexion with colloidal particles, it must be remembered that each particle is buoyed up so that g must be replaced by $\left(1 - \frac{d}{D}\right)g$, where d is the density of the liquid and D that of the particles. Then

$$\frac{n}{n_0} = \exp\left[-\frac{N}{RT}m\left(1 - \frac{d}{D}\right)gz\right] = \exp\left[-\frac{N}{RT}U(D - d)gz\right],$$

where U is the volume of one particle. The above equation may be written

$$\ln\frac{n_0}{n} = \frac{N}{RT}U(D - d)gz.$$

The expression $n = n_0 \exp\left(-\frac{Mg}{RT}z\right)$ shows that the elevation

required to produce a given rarefaction varies with the nature of the gas. Moreover, if Mz remains constant, the fraction $\frac{n}{n_0}$ is also constant. This signifies that if the molecular weight of one gas is sixteen times that of another, the elevation required to produce the same rarefaction will be sixteen times less for the first gas than it is for the second. Since it is necessary to rise to a height of 5 km.

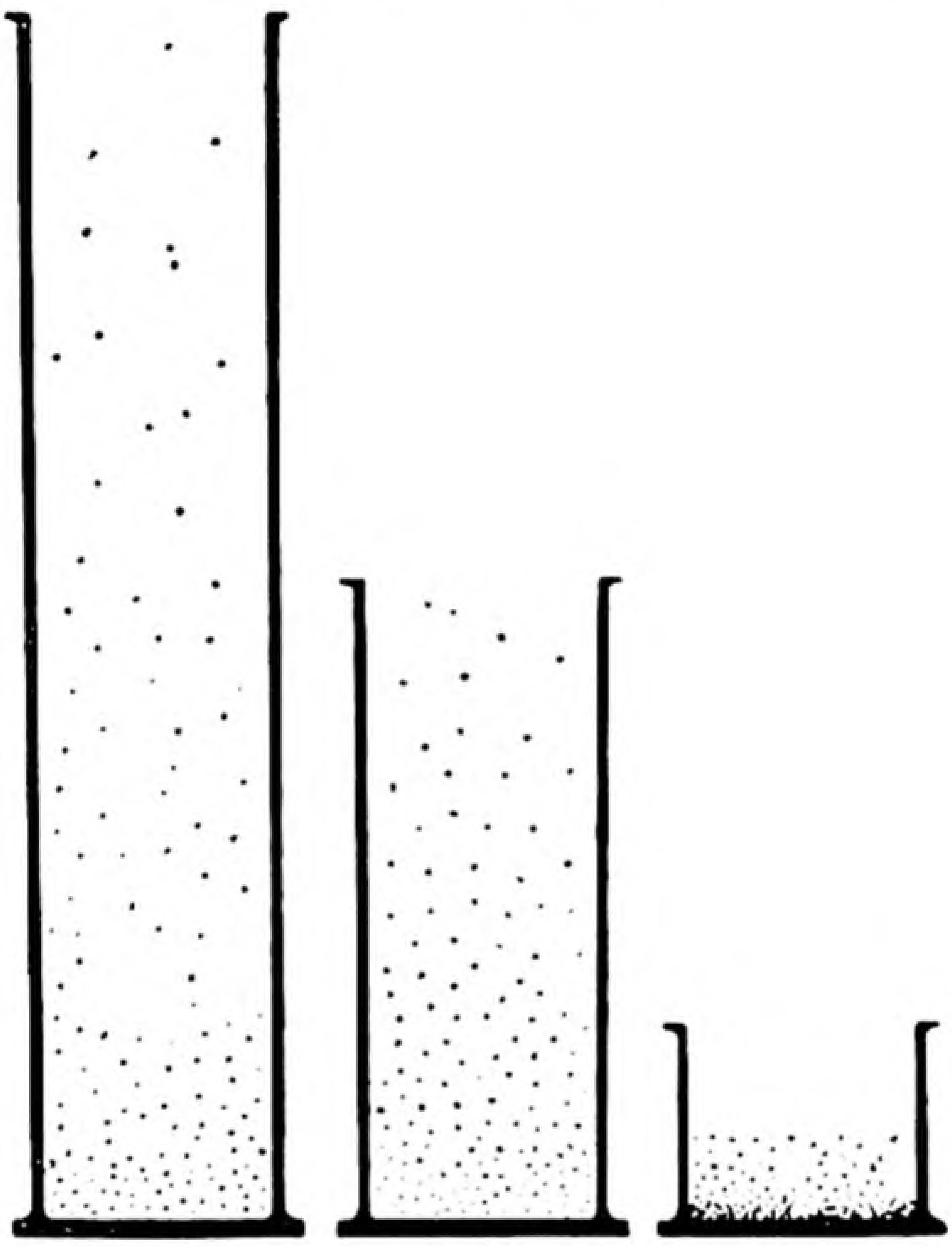

FIG. 13·02.—The distribution of equal numbers of molecules of hydrogen, helium, and oxygen in very tall gas jars.

in oxygen at 0° C. before its density is halved, a height of 80 km. must be reached in an atmosphere of hydrogen at 0° C. before the same fractional reduction in density occurs.

Fig. 13·02 represents three gigantic vertical jars [the tallest being 300 km. high], containing equal numbers of molecules of hydrogen, helium and oxygen, respectively. Assuming the temperature to be constant, the molecules will distribute themselves as shown; the more massive the molecules, the more do they collect together at the base of each jar.

When we look closely into the above argument it is found that the ideal gas laws have been applied to suspended particles in a

liquid. To justify this it is necessary to make use of Maxwell's theorem of the equipartition of energy. According to Maxwell the energy of a system in dynamic equilibrium is equally distributed among all the possible degrees of freedom. Thus for bromine vapour, ${}_{35}Br^{80}$, mixed with hydrogen, ${}_{1}H^{1}$, where the relative masses of the particles are as eighty to one, the average energy of a bromine molecule is equal to that of a hydrogen molecule. The theorem is equally true for still heavier gaseous molecules and there seems no reason to question its validity for macromolecules and hence for minute colloidal particles in suspension. This implies that such particles may be regarded as a very dilute gas for which the ideal gas laws are valid.

Perrin's experimental test of the law of atmospheres.—To test Laplace's law experimentally PERRIN found that colloidal solutions of such substances as ferric hydroxide and arsenic sulphide were unsuitable and therefore used emulsions of gamboge or of gum mastic.

Gamboge, prepared from a dried vegetable latex, was treated with alcohol which dissolved the yellow matter making four-fifths of the crude material. When excess water was added to this solution a yellow emulsion composed of tiny spheres formed. This emulsion was subjected to a vigorous centrifuging action and from the purified emulsion thus yielded Perrin obtained spherules of gamboge which were identical in size and suitable for the experiments contemplated.

The density of the particles was then determined by three different

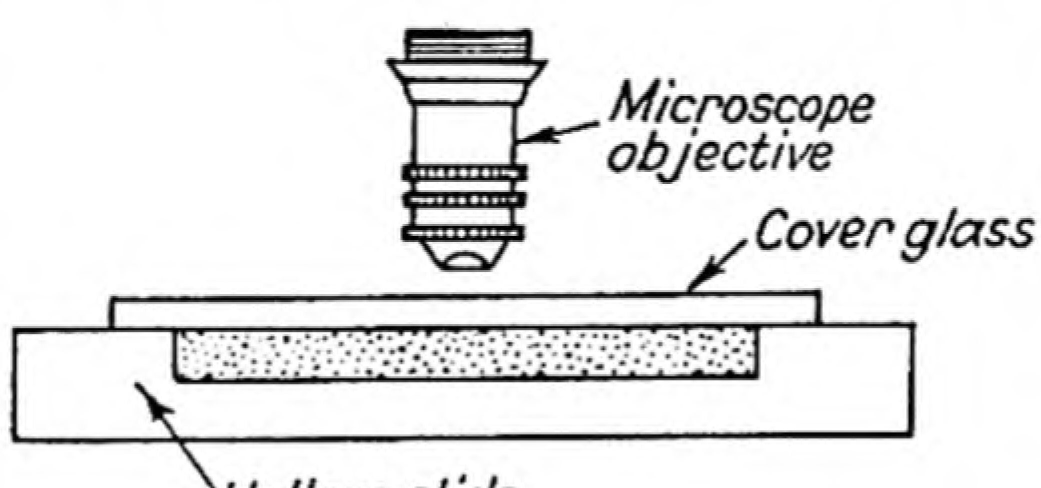

FIG. 13·03.—Perrin's first method for determining Avogadro's constant.

methods which gave concordant results, viz. D = 1·194 gm.cm.$^{-3}$. To determine the volume of a spherule the length of a long column of them was measured and the result checked by making use of Stokes' law.

An emulsion containing the selected particles was then placed in a small cell, as shown in Fig. 13·03, kept at constant temperature by means of a water bath. The cell and its adjuncts were then placed on the stage of a microscope and the field of view restricted

by means of a small diaphragm. The field was of such a size that not more than five or six grains were visible at once. At regular intervals the field was observed and the number of particles present noted. A mean of a very large number of observations was obtained. The microscope was then raised and a count made at the new level in the emulsion. From these experiments the validity of the law of atmospheres was established and Avogadro's constant N was found to have a value between 6·5 and $7{\cdot}2 \times 10^{23}$ mole.$^{-1}$.

The displacement in a given time of particles undergoing Brownian movements: Einstein's equation and its verification.—It has been shown in the foregoing paragraphs that a particle of not too large a size and in suspension in a liquid experiences unbalanced forces due to molecular bombardment. As a consequence of the finite value of the resultant of these unbalanced forces the particle moves in the liquid and then, as it wanders hither and thither through the liquid, its motion is opposed by viscous forces. By means of a microscope, whose field of view is crossed by two systems of parallel and equidistant lines cutting orthogonally, the position of any one particle at a given instant can be specified. The position of one selected particle is noted at the ends of some chosen sequence of time intervals; 30 seconds is a convenient time interval. Thus the average distance, $\overline{\Delta[x]}$, cf. Fig. 13·04, which a

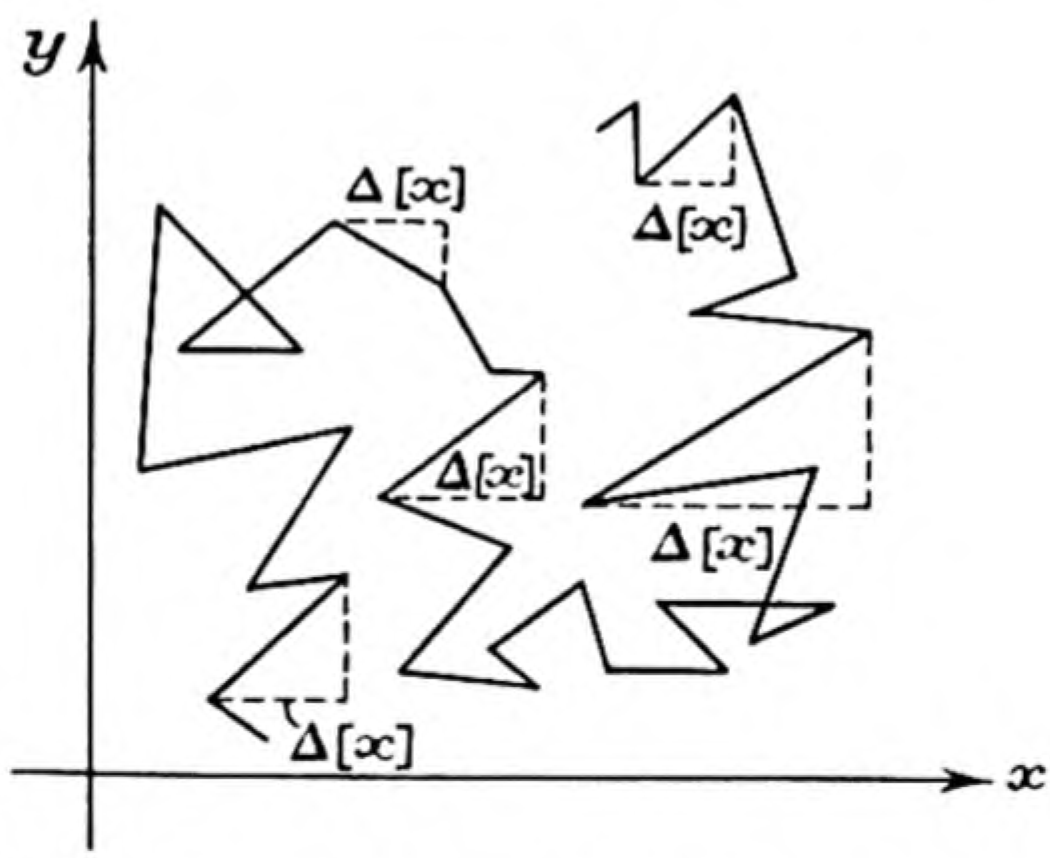

FIG. 13·04.—$\overline{\Delta[x]}$ from a Brownian pattern.

particle moves through in this interval may be determined. To develop a theory of this motion it is necessary to assume that the average distance moved through by a particle in a stated time depends on the forces acting upon a particle; these are caused by its motion in a viscous medium and by the molecules of the liquid which are bombarding it from all directions, but not quite uniformly.

Then, if the results of experiment are in accord with the theory which embodies these assumptions, the underlying assumptions will be justified. Moreover, it will be found that a value for Avogadro's constant can be derived from the observations and if this value agrees with other determinations of this constant additional confirmation will be forthcoming.

The theory was first given in 1905 by EINSTEIN and a little later it was extended by SMOLUCHOWSKI. In 1908 LANGEVIN simplified the argument which is as follows. If x, y, z denote the cartesian coordinates of a particle at a given instant, the origin always being the position of the particle at time $t = 0$, its motion in the x-direction during the time t is given by the equation

$$m\ddot{x} = \mathrm{X} - b\dot{x},$$

where X is the instantaneous x-component of the resultant force which the particle experiences from molecular bombardment and the term involving b represents the retarding force due to the motion of the particle in a viscous medium. The force denoted by X will be highly irregular in value and as often negative as it is positive, while if the particle is a spherule of radius a, then according to Stokes, $b = 6\pi a\eta$, where η is the viscosity of the liquid.

Multiplying the above equation throughout by $x\,dt$ and integrating, we get,

$$m\int_0^t x\ddot{x}\,dt = \int_0^t \mathrm{X}x\,dt - b\int_0^t x\dot{x}\,dt.$$

Now $x\dot{x}\,dt = d(\frac{1}{2}x^2)$, and $\int_0^t x\ddot{x}\,dt$ can be integrated by parts; in fact it is $\Big[x\dot{x}\Big]_0^t - \int_0^t \dot{x}^2\,dt$. We therefore have

$$m\Delta[x\dot{x}] - m\int_0^t \dot{x}^2\,dt = \int_0^t \mathrm{X}x\,dt - \tfrac{1}{2}b\Delta[x^2],$$

where $\Delta[\alpha]$ denotes the change in a quantity α in the interval of time from $t = 0$ to $t = t$.

Now
$$\int_0^t \dot{x}^2\,dt = \overline{\dot{x}^2}\,t,$$

where $\overline{\dot{x}^2}$ is the average value of $\dot{x}^2$ in the time interval. On the other hand since $\dot{x}$ fluctuates rapidly, it can be shown that $\Delta[x\dot{x}] \rightarrow 0$. Also, since X is as often negative as it is positive, the $\int_0^t \mathrm{X}x\,dt \rightarrow 0$ provided the value of the integral is averaged for a large number of particles. Hence if we restrict the argument to a large number of particles, we find

$$m\overline{\dot{x}^2}t = \tfrac{1}{2}b\overline{\Delta[x^2]},$$

where $\overline{\Delta[x^2]}$ denotes the average of $\Delta[x^2]$ for all the particles considered.

Also, approximately, $pV = RT = \frac{1}{3}Nm\,\overline{c^2}$, where the symbols have their usual meanings. Hence the mean kinetic energy of a particle is $\frac{3}{2}\frac{RT}{N}$. Now the kinetic energy due to the motion along the x-direction only is one-third of the total kinetic energy of a particle or $\frac{1}{2}\frac{RT}{N}$. The above equation may therefore be written

$$\frac{RT}{N}t = \tfrac{1}{2}(6\pi a\eta)\overline{\Delta[x^2]},$$

or

$$\overline{\Delta[x^2]} = \frac{RTt}{3\pi a\eta N}.$$

Smoluchowski has emphasized that this theoretical formula cannot be expected to agree rigorously with experimental results because it has been assumed that the particles are spherical in shape and that they are without mutual attraction.

In actual experimental work a value for $\overline{\Delta[x]}$ is first found; the corresponding value of $\overline{\Delta[x^2]}$ is then calculated by using the relation

$$\overline{\Delta[x]} = \sqrt{\frac{2}{\pi}\overline{\Delta[x^2]}},$$

which is a consequence of the elementary kinetic theory.

The Brownian movement in gases: Millikan's accurate verification of Einstein's equation.—It was Smoluchowski who first drew attention to the fact that since Einstein's equation is valid wherever Stokes' law is applicable, then we would expect a Brownian movement to be present when particles are suspended in gases as well as in liquids. The first direct measurements on the Brownian movement of particles in gases were made by Ehrenhaft in 1907. He found that the motion, as theory indicates, is much more lively.

By 1911 MILLIKAN and his co-workers had attained an order of exactitude in experiments on Brownian movements in gases which enabled a very reliable value for Avogadro's constant to be obtained. The reasons for this remarkable increase in the accuracy of measurement are as follows. In all Perrin's work the accuracy of the final result was very much restricted by the difficulty of determining with precision the radius of the particle and some doubt as to the validity of Stokes' law must be contemplated. Perrin himself believed that Stokes' law was only valid for liquids. Finally, it must be remembered that the calculation was made with the aid of an equation which had been derived from analogy between

colloidal suspensions and gaseous molecules. Millikan overcame these objections by working with droplets of oil in air or hydrogen. Each droplet was a sphere and the radius of any one selected drop was determined by a technique which this investigator had developed in his classical determination of the charge of an electron.

In the present research the drop was charged and an experimental study made of:

(*a*) the lateral Brownian displacements of the drop when suspended in the gas—a suitably chosen electric field balanced the weight of the drop.

(*b*) its rate of fall under gravity alone, and

(*c*) its rate of fall or ascent when gravity was opposed by a force arising from the presence of the charged drop in an electric field.

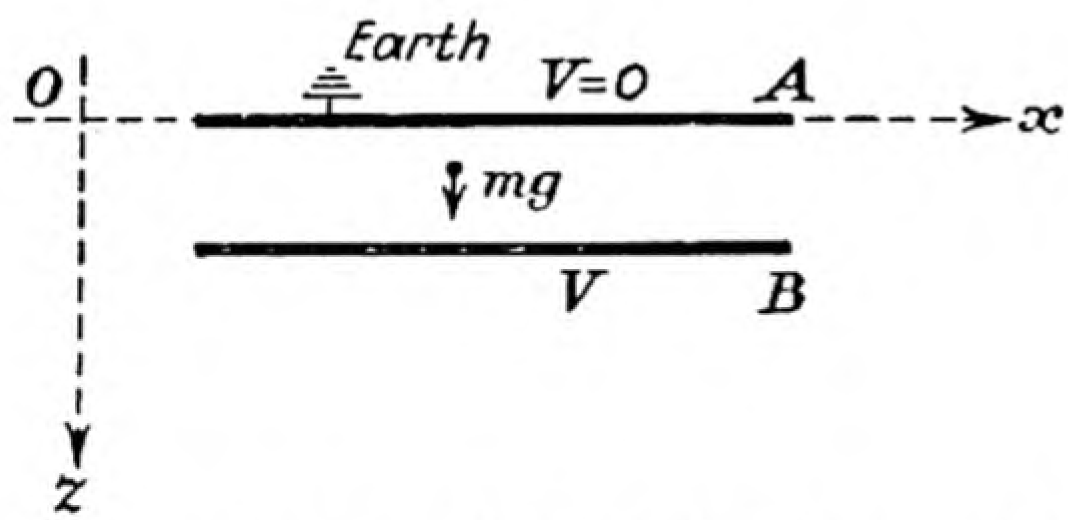

FIG. 13·05.—The motion of a charged drop between the plates of a horizontal condenser.

The theory of the experiment is as follows. Let A and B, Fig. 13·05, be the upper and lower plates respectively of a condenser and suppose that between these plates there is a droplet of mass m. The plates of the condenser are made horizontal so that the electric field shall be parallel to the earth's gravitational field. Let Ox and Oz be the rectangular axes to which the motion of the drop is referred. All quantities will be considered algebraically. Let V be the potential of the lower plate. Then the electric field is $\vec{\mathrm{E}} = \hat{\imath}0 + \hat{\jmath}0 + \hat{k}\mathrm{E}$, where $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ are three unit vectors. Let q be the charge on the drop. Then when the field is on, the motion of the drop is determined by the equation

$$q\mathrm{E} + mg - b(\dot{z})_{\mathrm{F}} = 0,$$

since the drop moves without acceleration. In this equation m is the mass of the drop, b the quantity already defined, g the intensity of gravity and $(\dot{z})_{\mathrm{F}}$ the velocity of the drop when the electric field is applied.

When there is no electric field applied, the motion is given by

$$mg - b(\dot{z})_0 = 0.$$

Millikan selected a drop for which $q = e$, the electronic charge. Under these conditions we have

$$b = \frac{e\mathrm{E}}{(\dot{z})_{\mathrm{F}} - (\dot{z})_0}.$$

Now Einstein's equation is

$$\overline{\Delta[x^2]} = \frac{2\mathrm{RT}}{\mathrm{N}} \cdot \frac{t}{b},$$

and since the value of b has been found experimentally, we have

$$\overline{\Delta[x^2]} = \frac{2\mathrm{RT}}{\mathrm{E}}\left[\frac{(\dot{z})_{\mathrm{F}} - (\dot{z})_0}{\mathrm{N}e}\right]t.$$

Millikan and Fletcher first held a drop suspended between the plates of the condenser and measured its Brownian motion along a line normal to the line of sight and to gravity. This was done by timing the transits of the drop across cross-hairs at a known distance apart in the eye-piece of the observing low-power microscope. By using gases of low viscosity and working at lower pressures the values of $\overline{\Delta[x^2]}$, which were dealt with, were fifty times as large as those used by Perrin for a drop of the same size and nature. In this way much more accurate measurements could be made. Another advantage that these workers had was that the temperature of the apparatus could be controlled more easily. As a result of these investigations it was found that

$$\mathrm{N}e = -2{\cdot}88 \times 10^{14} \text{ e.s.u. mole.}^{-1}.$$

OSMOSIS AND OSMOTIC PRESSURE

The nature of osmosis.—When red blood corpuscles are placed in water they expand rapidly and ultimately burst, but if they are placed in a strong salt solution they shrivel up. This phenomenon is characteristic of the membranes surrounding many animal and vegetable cells, for these allow water to pass through freely but retard or entirely prevent the passage of solids. ***Osmosis*** is the name given to this spontaneous passage of a liquid through a membrane. Its effects were first observed by the ABBÉ NOLLET in 1748, who discovered that when an animal bladder separated alcohol from water there is a passage of water through the skin of the bladder into the alcohol but that no alcohol flows in the reverse direction. This spontaneous and differential flow of water is found to take place when a membrane of the above type separates an aqueous solution from water. The following simple experiment illustrates this phenomenon. A piece of wet parchment paper is stretched over the end of a large thistle funnel and when nearly dry it is coated with glue along the boundary. The inverted funnel is partly

filled with a solution of sodium chloride, cane-sugar, or some other substance, and immersed in water as shown in Fig. 13·06. After standing for some time the level of the solution will have risen considerably; water must have passed through the parchment into the solution. This statement is not complete, for water will have passed from the solution into the water in the beaker at the same time as water passed from the beaker into the solution. This osmotic flow arises from the bombardment of the molecules upon the membrane; on the one side there are only molecules of water arriving at the membrane, while on the other there are molecules of water and solute as well. The water rises in the tube until the excess hydrostatic pressure due to the column of liquid thus produced causes the water to flow outwards at a rate equal to that at which osmosis causes it to flow into the solution.

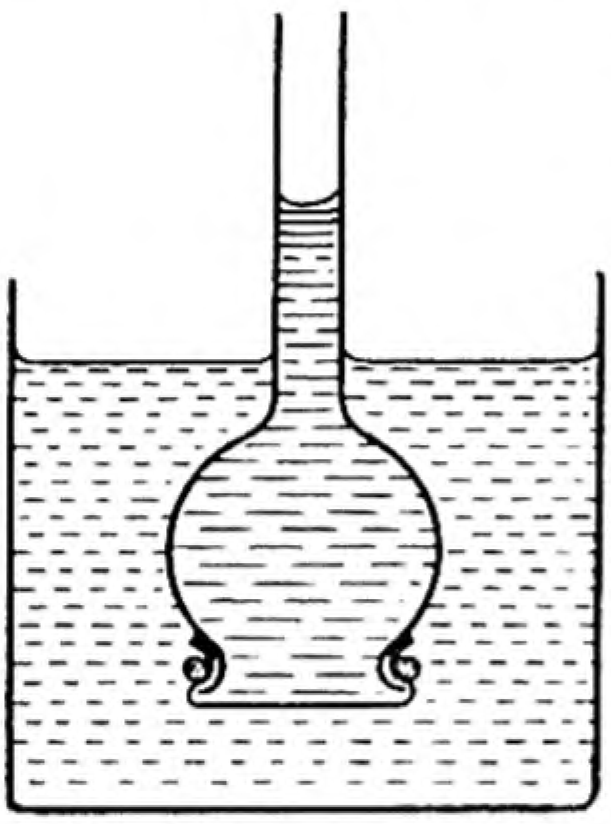

FIG. 13·06.—Osmosis.

Quantitative observations are not possible with such simple apparatus for the membrane used is not truly ***semi-permeable***; this is a term first used in 1886 by VAN'T HOFF to describe a membrane which allows the free passage of water but entirely prevents that of a dissolved substance through it. Such membranes as those used in the simple experiment just described do not fulfil these requirements for they do permit some of the dissolved substance to pass through them.

An osmotic flow of the solvent is also observed when a membrane separates two solutions of the same nature but differing in concentration. The flow of solvent is such that the concentrations of the solutions tend to become equal, i.e. there is an excess of solvent passing from the weaker to the stronger solution.

Artificial semipermeable membranes.—In 1864 TRAUBE discovered that copper ferrocyanide, $Cu_2Fe_2(CN)_6$, is an excellent semipermeable membrane and it is still considered as the best membrane for experiments on osmosis.

Experiment.—Place a weak solution of potassium ferrocyanide in the bottom of a beaker and when it has ceased to move introduce a strong solution of copper sulphate so that it lies below the ferrocyanide solution. A thin gelatinous precipitate of copper ferrocyanide is formed; it separates the two solutions. The membrane does not increase in thickness since the dissolved substances cannot pass through it, but after the lapse of about two hours it will be seen that the membrane has a distinct bulge upwards. This proves that more water passes downwards than flows upwards, and hence that the copper solution has the greater osmotic pressure.

Osmotic pressure.—The membrane of copper ferrocyanide prepared in the above experiment is too fragile to support more than a small pressure difference, but its strength is very considerably increased if it is produced within the walls of a porous pot. The technique of preparing the copper ferrocyanide membrane in the walls of a porous pot so that it acquired additional mechanical strength was first worked out by PFEFFER in 1877, and he then succeeded in making the first quantitative measurements in connexion with osmosis. To prepare this membrane the porous pot is boiled in distilled water for several hours to remove air bubbles. A 0·25 per cent solution of copper sulphate is then placed inside the pot and a 0·21 per cent solution of potassium ferrocyanide outside. Each solution should reach very nearly to the top of the pot. Diffusion occurs and the two dissolved substances meet inside the walls of the pot where a membrane of copper ferrocyanide is formed. This process should be allowed to continue without interruption for two days. The pot thus prepared is boiled in several changes of distilled water and is then ready for use. If allowed to become dry it should be boiled for several hours to expel all air again.

If such a pot, provided with a rubber bung carefully waxed in position and fitted with a long capillary tube, is filled with a saturated solution of sugar cane and then immersed in water, the change in level of the liquid in the capillary is very rapid. After several days a tube 1 mm. in diameter must be several metres long if the liquid is not to exude from it. This spontaneous differential flow of liquid through the membrane can be completely stopped by the application of a suitable pressure; the flow is reversed if the pressure is increased beyond this value.

Definition.—That pressure which must be applied to a solution to prevent the spontaneous differential flow of liquid through a semi-permeable membrane separating the solution and solvent is termed the osmotic pressure of the solution.

Pfeffer's experimental method for determining the osmotic pressure of weak solutions.—To determine the osmotic pressure of a weak aqueous solution the apparatus shown schematically in Fig. 13·07 may be used. It is similar to an apparatus originally used by PFEFFER. A mercury manometer, M, with one limb closed and containing air, or better, nitrogen, is connected to the porous pot, A, containing the solution. This solution is introduced through the tube B, the air in the connecting tubes being displaced by some of the solution; after this operation the tube B is hermetically sealed. Water enters the solution and the pressure inside the pot increases; it is transmitted to the gas in M. Ultimately the pressure within the apparatus ceases to change, and this constant pressure is the osmotic pressure of the solution. It is

calculated from the change in volume of the gas (nitrogen) in the closed limb of the manometer. The serious objection to this method lies in the fact that the water entering the solution changes the concentration of the latter so that the readings do not correspond to the osmotic pressure of the original solution; neither do they to the final solution, for its concentration is not uniform and it is the concentration of the solution in the immediate vicinity of the membrane which determines the osmotic pressure which is measured. It is better to measure the external pressure which must be applied to a solution to prevent the passage of the solvent through the semi-permeable membrane separating it from the solvent. Such a method must always be used in accurate work, and especially if the solutions are concentrated. Details will be given later.

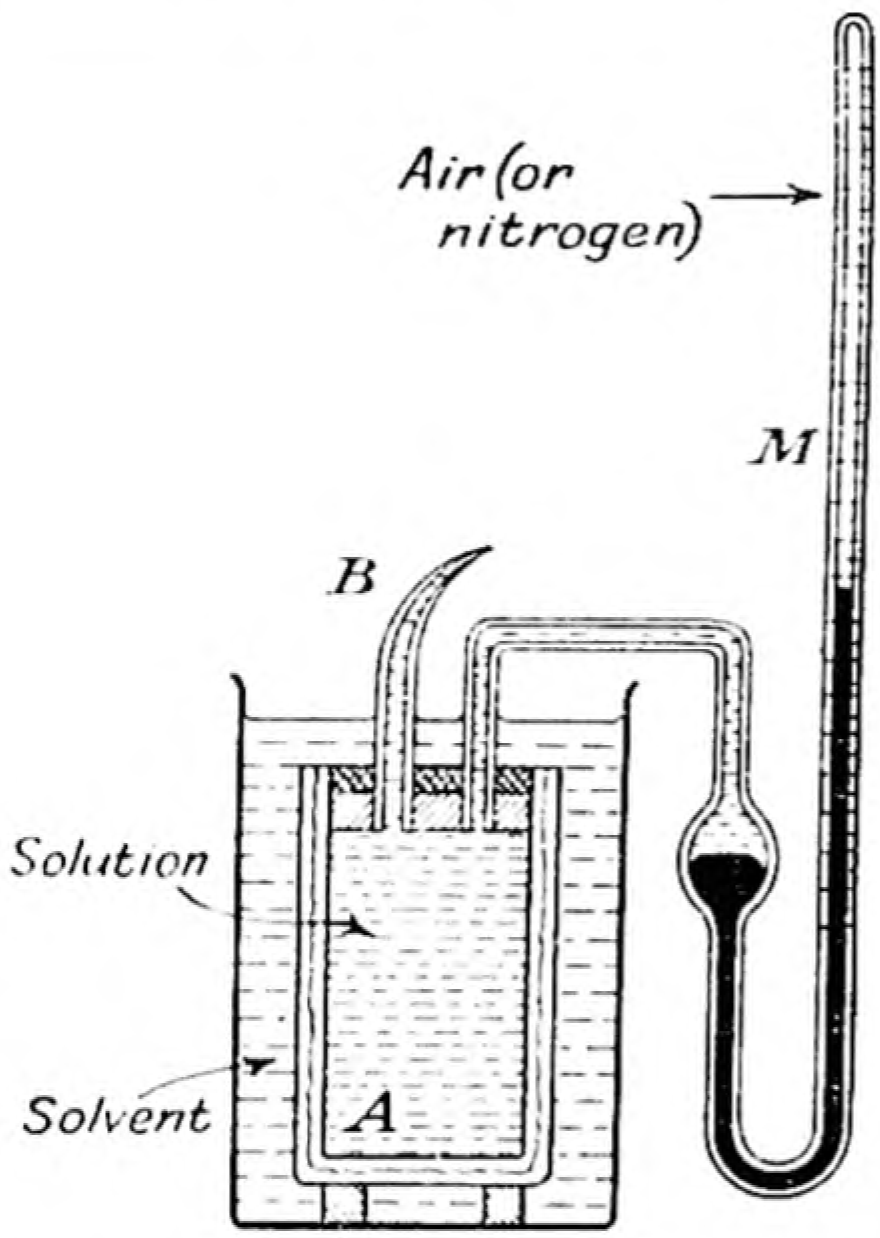

Fig. 13·07.—The measurement of osmotic pressure [dilute solutions].

The fundamental laws of osmosis.—Among the aqueous solutions examined by Pfeffer are those of sucrose, dextrose and cane sugar. Some of his results for cane-sugar in aqueous solution at about 15° C. are recorded in Table A.

Table A

Concentration (c) [gm. per 100 cm.^{3} of solution]	Osmotic pressure, p [cm. of mercury]	$\left\|\frac{p}{c}\right\|$
1·003	52·1	52·0
2·014	102	50·5
2·767	152	55·0
4·060	209	51·5
6·138	307	50·0

The last column in the above table shows that $\frac{p}{c}$ is approximately constant, i.e. ***the osmotic pressure of a dilute solution at constant temperature is directly proportional to the concentration of the dissolved substance*** [Law I]. Nowadays the existence of this law would be revealed by a graphical method.

Table B shows how the osmotic pressure of an aqueous solution of sucrose, of constant strength—one per cent—varies with its absolute temperature, T.

TABLE B

Absolute temperature [T°K.]	Osmotic pressure [cm. of mercury]	$\frac{p}{T}$
279·8	50·5	0·180
286·7	52·5	0·183
295·0	54·8	0·186
305·0	54·4	0·178
309·0	56·7	0·184

From this table the fact emerges that $\frac{p}{T}$ is practically constant and, here again, a better test could be made graphically. Thus ***the osmotic pressure of a dilute solution is directly proportional to its absolute temperature*** [Law II].

The two laws stated above were first enunciated in 1886 by van't Hoff after he had made a careful study of the experimental results obtained by Pfeffer. If C is the concentration of the dissolved substance in mole.cm.$^{-3}$, then $C = V^{-1}$, where V is the volume (cm.^{3}) of the solution containing one mole of solute. Thus the first law may be written,

$$pV = \text{constant},$$

while the second law may be expressed by the equation

$$p = \text{constant} \times T.$$

Combining these two equations we have,

$$pV = R_0T,$$

where R_0 is a constant. By comparing the osmotic pressure of a sucrose solution with the pressure of hydrogen at the same temperature and concentration, R_0 was shown by van't Hoff to be identical with R, the universal gas constant. Later on this investigator established this fact by thermodynamical reasoning [cf. Vol. II]. The excellent agreement between theory and experiment which later work has abundantly confirmed, indicates that a law, analogous to Avogadro's law for gases, is applicable to dilute solutions. It states that ***when equal numbers of molecules of different solutes are dissolved in equal volumes of solution (dilute), at a constant temperature, the osmotic pressure of the solution is independent of the nature of the dissolved substance.***

If Ω is the volume of a solution containing χ moles of dissolved substance, $\chi V = \Omega$, so that

$$p\left(\frac{\Omega}{\chi}\right) = RT.$$

If $C = \frac{\chi}{\Omega}$, the concentration in mole.cm.$^{-3}$, then

$$\frac{p}{C} = RT.$$

If c is the concentration in gm.cm.$^{-3}$, $c = MC$, so that

$$\frac{p}{c} = \frac{RT}{M}.$$

Thus, if the osmotic pressure, in absolute units, of a solution at temperature T and concentration c is known, it is possible to determine the molecular weight of the dissolved substance.

Osmotic pressure and the determination of molecular weight.—It is customary in experimental work on osmosis to measure the pressure in atmospheres and to consider the volume in cm.^{3} occupied in solution by one mole of the dissolved substance. The characteristic equation for an ideal gas, when the pressure is measured in atmospheres and 1 gram-molecule occupying a volume V is considered, then becomes

$$PV = \overline{R}T,$$

and $\overline{R}$ is a universal constant for all gases. It must be noted, however, that $\overline{R}$ is different from the universal gas constant R which appears in the ideal gas equation $pV = RT$, where the pressure p is expressed in absolute units. It is known that 1 gram-molecule of a gas at s.t.p. occupies 22,415 cm.^{3}. Hence

$$1 \times 22{,}415 = \overline{R} \times 273,$$

or
$$\overline{R} = 82{\cdot}06 \text{ cm.}^3 \text{ atmos. deg.}^{-1} \text{ K. mole.}^{-1}.$$

This enables us to calculate a value for the osmotic pressure of a non-electrolyte in solution or, knowing the osmotic pressure, to determine a value for the molecular weight of the dissolved substance. Let m gm. of a substance of molecular weight M be dissolved in 100 cm.^{3} of water at $\theta°$ C. Then the number of gram-molecules in this volume is $\frac{m}{M}$, so that 1 gram-molecule would occupy $\left(\frac{M}{m} \times 100\right)$ cm.^{3}. Let P be the osmotic pressure in atmospheres.

Then
$$P \times \left(\frac{M}{m} \times 100\right) = \overline{R} \times (273 + \theta).$$
$$\therefore\ P = \frac{0{\cdot}821m(273 + \theta)}{M} \text{ atmos.}$$

The measurement of the osmotic pressures of concentrated solutions: the work of Berkeley and Hartley.—In 1906 the EARL OF BERKELEY and his assistant HARTLEY published an account of a precision method of measuring the osmotic pressure of a concentrated solution. Instead of measuring the pressure developed in a cell by the passage inwards of the solvent, when the contents of the cell were separated from the solvent by a semipermeable membrane, these investigators measured the pressure which had to be applied to the solution to prevent the differential flow between the solvent and the solution. In principle, the pressure on the solution was gradually increased until it was just sufficient to reduce the resultant flow of solvent to zero; this pressure was taken to be the osmotic pressure of the solution.

The apparatus is represented in principle in Fig. 13·08. A is a

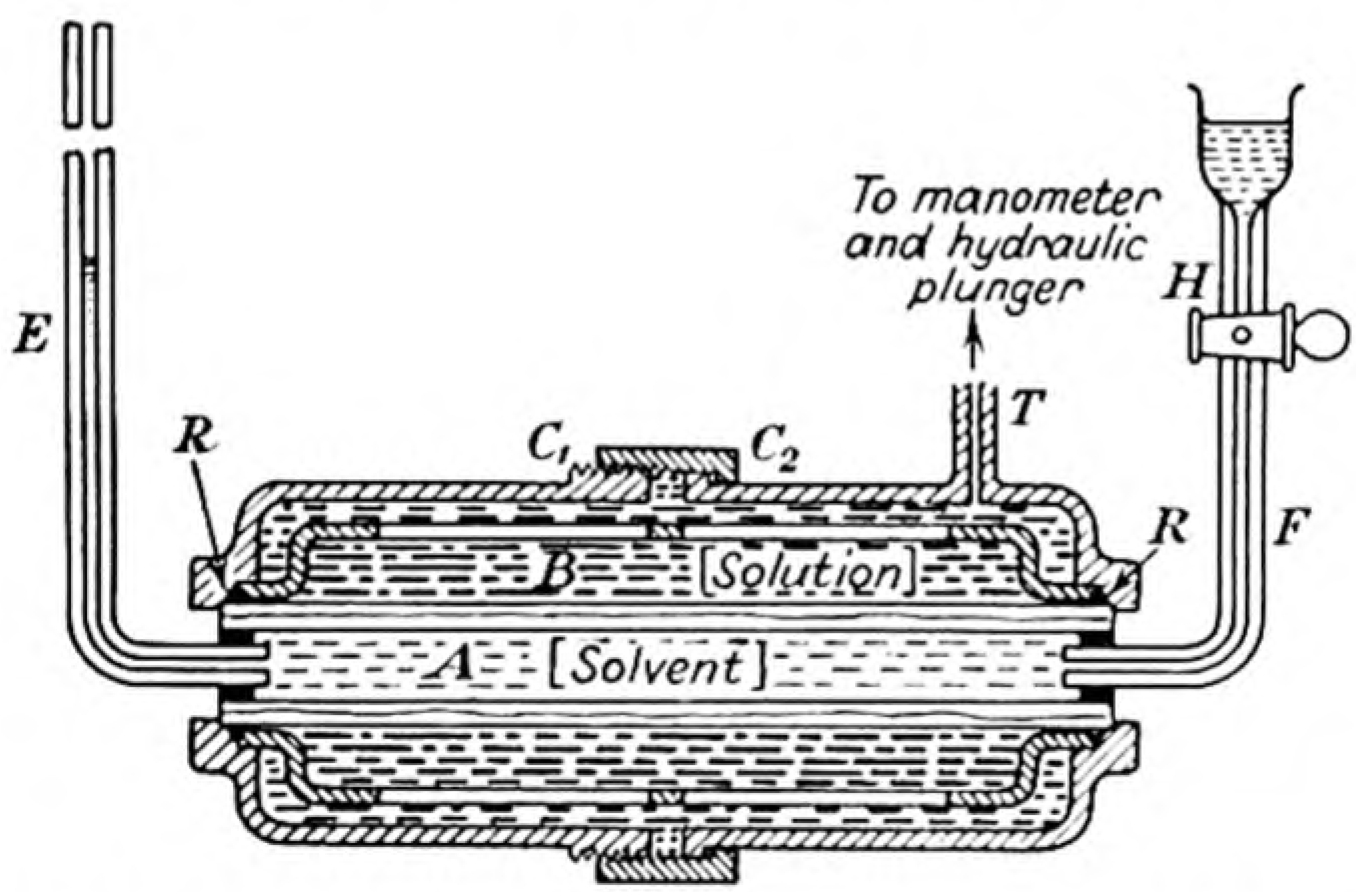

FIG. 13·08.—Principle of method used by Berkeley and Hartley to measure osmotic pressures of concentrated solutions.

porous pot; actually it is a porcelain tube 15 cm. long, 2 cm. external diameter and with walls 0·4 cm. thick. The vertical ends are glazed. The semipermeable membrane is as near as possible to the outer wall of the tube. B is a gun-metal cage, slotted along its cylindrical portion, while C_1 and C_2 are the two portions of an outer gun-metal vessel which can be screwed together. When thus screwed, a thrust is exerted on the rubber-like rings, R, which separate the cage from the outer vessel. The length of the cage is such that, when the

apparatus is finally set up, the rings just overlap the ends of the tube A.

The inner tube A contains water (or the solvent) and the capillary tubes E and F permit the water to be introduced into the apparatus. During an experiment the tube F is closed by means of a stop-cock, H, while E serves as a water-gauge.

A copper ferrocyanide membrane was used. The air was removed from the porous pot by immersing it in an aqueous solution of copper sulphate contained in a vessel which was then exhausted. The tube was then dried superficially, its open ends plugged, and then immersed in a solution of potassium ferrocyanide. The membrane was formed near to the outer surface of the tube A and it was subsequently strengthened by an electrolytic process.

The space between A and the outer vessel was filled with the solution under investigation and a gradually increasing pressure was applied to it; the tube T led to a manometer and a device for increasing the pressure. As long as the applied pressure was less than the osmotic pressure of the solution, water passed from A to the outer vessel and the meniscus in E fell; the motion of the meniscus was reversed when the applied pressure exceeded the osmotic pressure.

Berkeley and Hartley found a great difficulty in deciding the exact point at which the meniscus in the water-gauge remained stationary. The equilibrium pressure was therefore determined by observing the rate of fall and the rate of rise of the meniscus at pressures slightly above and slightly below the equilibrium pressure, respectively. By assuming that the rate of movement of the meniscus is directly proportional to the difference between the applied pressure and the equilibrium pressure, a value for the latter was calculated from the observations. This pressure was corrected for a so-called 'guard-ring leak'; this is the leak through the glazed ends of the porcelain tube.

The following table shows some of the results obtained and also the values for the osmotic pressure calculated on the assumption that the van't Hoff theory is valid. [Cf. also p. 667.]

The Osmotic Pressure of Aqueous Solutions of Glucose at 0°C.

Concentration in mole.lit.$^{-1}$	Osmotic pressure (atmos.)	
	Observed	Calculated
0·504	13·21	12·42
1·005	29·17	24·82
1·611	53·19	39·72
2·262	87·87	55·83
2·465	121·18	68·27

Vegard's osmometer.—In this method (1908) it was decided that the applied pressure should be outside the porous pot so that the stresses to which the latter should be subjected were always compressive. Under these conditions breakages are less likely to occur. Such a principle had already been used by Berkeley. The essential features of the osmometer designed by VEGARD are shown in Fig. 13·09(*a*). A is the porous pot with a copper ferrocyanide

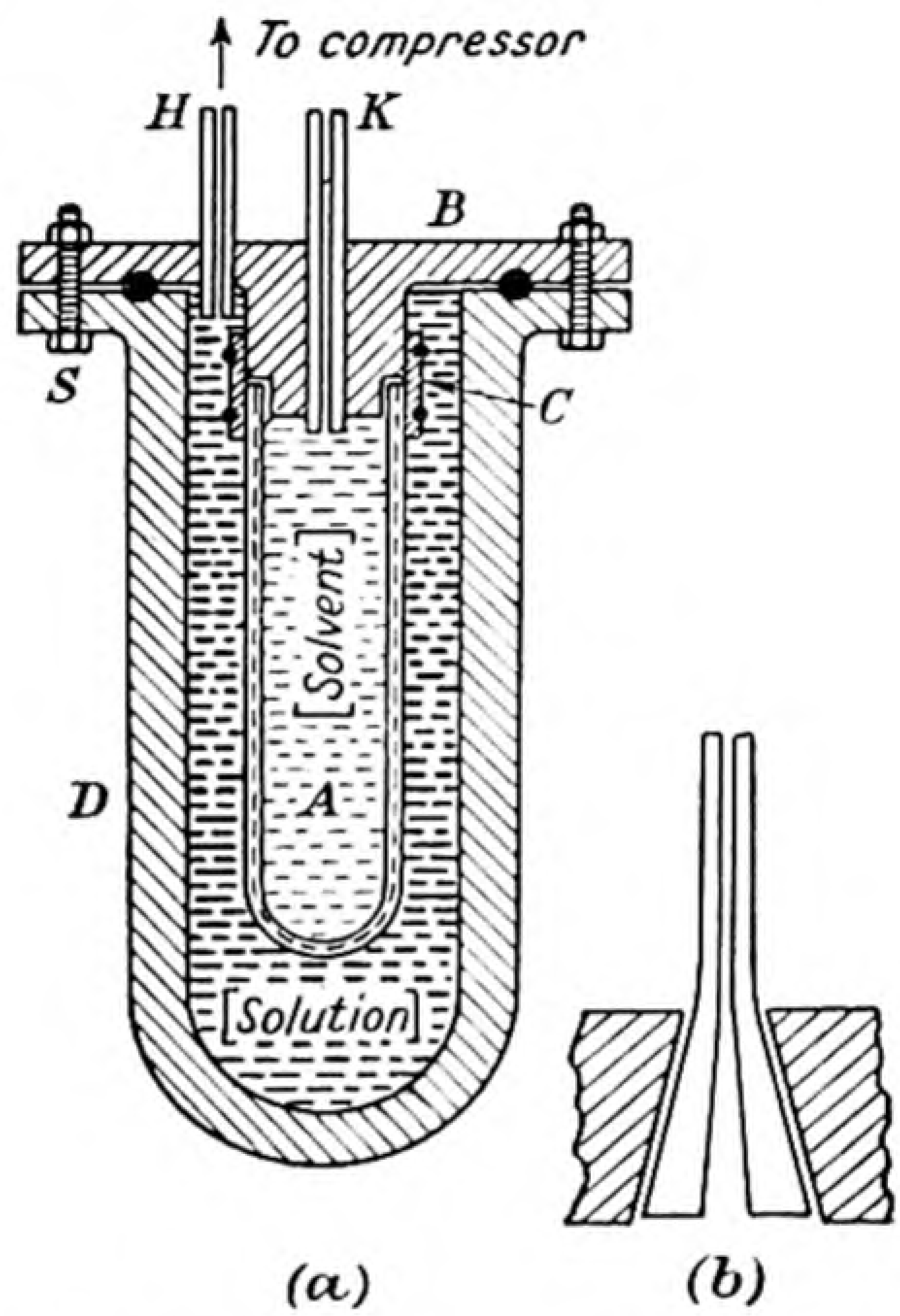

FIG. 13·09.—Vegard's osmotic cell.

membrane near its outer wall. This cell is attached to the lid of the apparatus by a piece of well-fitting rubber tubing, C; the ends of C are secured with cotton threads. B and the outer container are made of cast steel. To represent the arrangement used to prevent leakage between B and D it may be imagined that equal circular grooves are cut in B and the flange of D and that when fitted together a rubber ring, circular in cross-section, fills the grooves cut in B and the flange of D. When the apparatus is assembled and the six screws (S is one of them) inserted, a leak-tight joint is obtained.

H and K are two glass capillary tubes passing, as shown, through the lid B. They are slightly conical at the lower ends and are fitted from below into the lid. The space between H, or K, and the lid is filled with molten sealing wax and when cold a leak-tight

joint is obtained; the arrangement is shown, much exaggerated, in Fig. 13·09(*b*).

The space between A and D is filled with solution and the pressure on the solution is increased by means of a steel cylindrical plunger, operated by means of a screw; the tube H is in direct connexion with the compressor and a gas-filled manometer. The pressure on the solution is increased until the liquid meniscus in K remains stationary, or, more generally, just reverses. The pressure at which reversal of the motion of the meniscus takes place is determined by measuring the velocity of the meniscus on both sides of the equilibrium pressure.

The work of Morse and his co-workers on osmotic pressure.—In 1911 MORSE began his experimental work on the determination of osmotic pressures. The method he adopted was essentially that of Pfeffer but the apparatus was improved in detail and made capable of withstanding higher pressures. By 1916 FRAZER and MYRICK extended this work to still higher pressures.

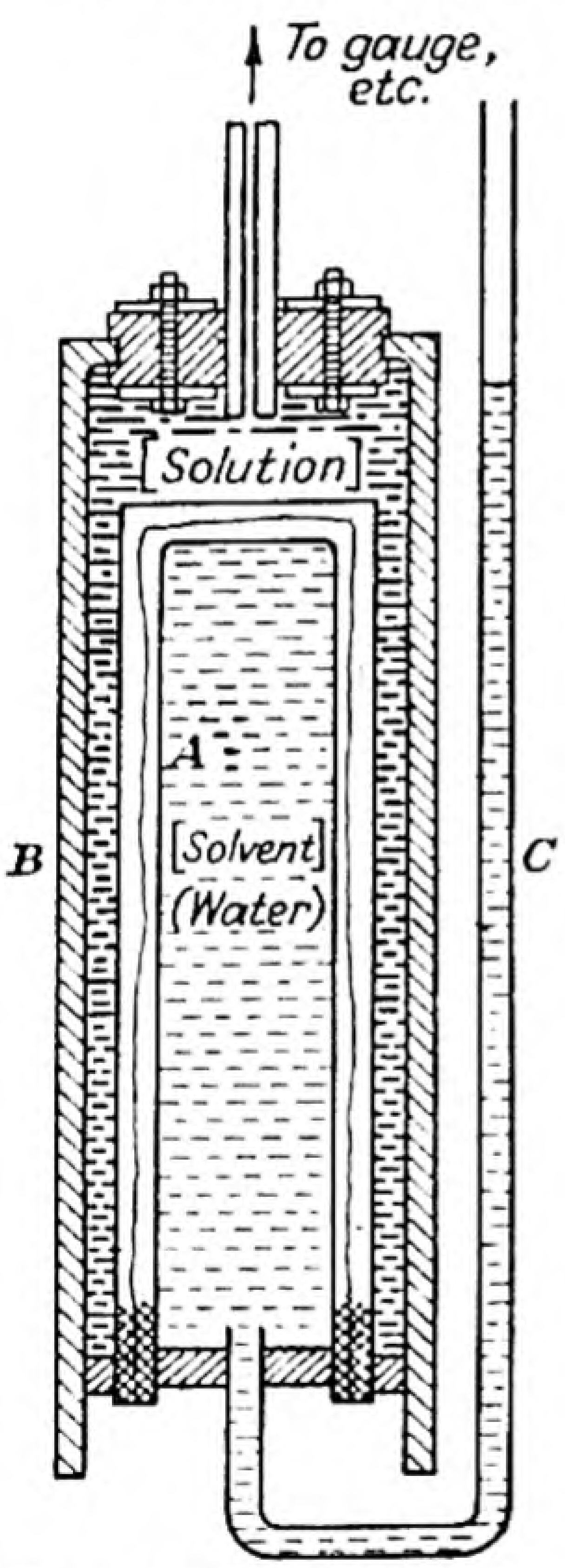

FIG. 13·10.—The principles of Morse's osmometer (as modified by Frazer and Myrick).

In these later investigations the clay cell, A, Fig. 13·10, was enclosed within a bronze cylinder, B, to which a manometer was attached. The solution was placed in the space between A and B while A itself contained the solvent (water). The tube C maintains A filled with solvent on which the pressure is atmospheric. Under these conditions water passes from the cell A into the solution, and as the solution is in a confined space a pressure difference is soon established.

Fig. 13·10 shows only the principles of the method; in the actual apparatus leaks were prevented by the use of packing glands. The pressure was measured by means of an electrical resistance gauge. Two similar coils of wire made from an alloy resembling manganin were used; one was at atmospheric pressure and the other was immersed in oil in direct contact with the solution whose osmotic pressure was to be determined. The difference in electrical resistance between the two coils was measured with the aid of a Carey Foster bridge in the usual way. The strength of such a gauge is practically unlimited. In the earlier gas-filled

manometers there is a considerable increase in volume due to entry of solvent into the solution and this diminishes the concentration of the dissolved substance; the only change in volume suffered in the resistance type of manometer is due to the compressibility of the liquids. In this way the dilution of the solution by the entry of solvent is greatly reduced and equilibrium is reached more quickly.

The actual cell, too, had several distinct advantages. In the first place cells are able to withstand external pressures rather than internal ones and, secondly, the membrane itself is made stronger, for an increase in pressure tends to drive it into the pores of the porous pot and thereby compress it; this action renders it less liable to rupture.

The results of osmotic pressure measurements.—For aqueous solutions of cane-sugar, some of the results obtained are shown in Fig. 13·11(*a*). The agreement between different workers is remarkable and the graph reveals at once the departure of the osmotic pressure of a strong solution from that calculated according to the ideal gas equation. The departure is quite noticeable even when the concentration is 0·1 mole.lit.$^{-1}$.

The manner in which the osmotic pressure varies with temperature for sugar solutions of several different concentrations is shown in Fig. 13·11(*b*). The range of temperature is naturally somewhat restricted but straight lines drawn to pass through O and to lie evenly among the appropriate points indicate that the ideal gas laws are only applicable, strictly speaking, to weak solutions. [The numbers near each straight line indicate the concentration in grams per 1000 gm. of water.]

DIFFUSION

The diffusion of salts in aqueous solutions.—The process of diffusion consists of the wandering of the molecules or ions in a solution from a region of high concentration to one of low concentration. It is therefore one resembling the flow of heat in a conductor when the steady state has been reached. Moreover, it is an irreversible process; a portion of the pure solvent having once become contaminated with a solute never returns to its initial state unless assisted by an external agent. Although it is well established that for strong electrolytes in aqueous solution there is complete dissociation, yet it is convenient to consider the phenomenon of diffusion as if the molecules of the solute remained undissociated.

In 1850 GRAHAM published his first paper on the diffusion of salts in solution, and in 1882 a further study was made by SCHEFFER. In principle the apparatus they used is shown in Fig. 13·12(*a*). A small glass cylinder, A, rests on two horizontal glass rods supported inside a larger glass vessel, B. A is nearly filled with the

solution under investigation, and a cork, C, floats centrally on the liquid. A vertical knitting needle attached to this cork can move upwards in a narrow glass tube, D, held in position by a clamp and

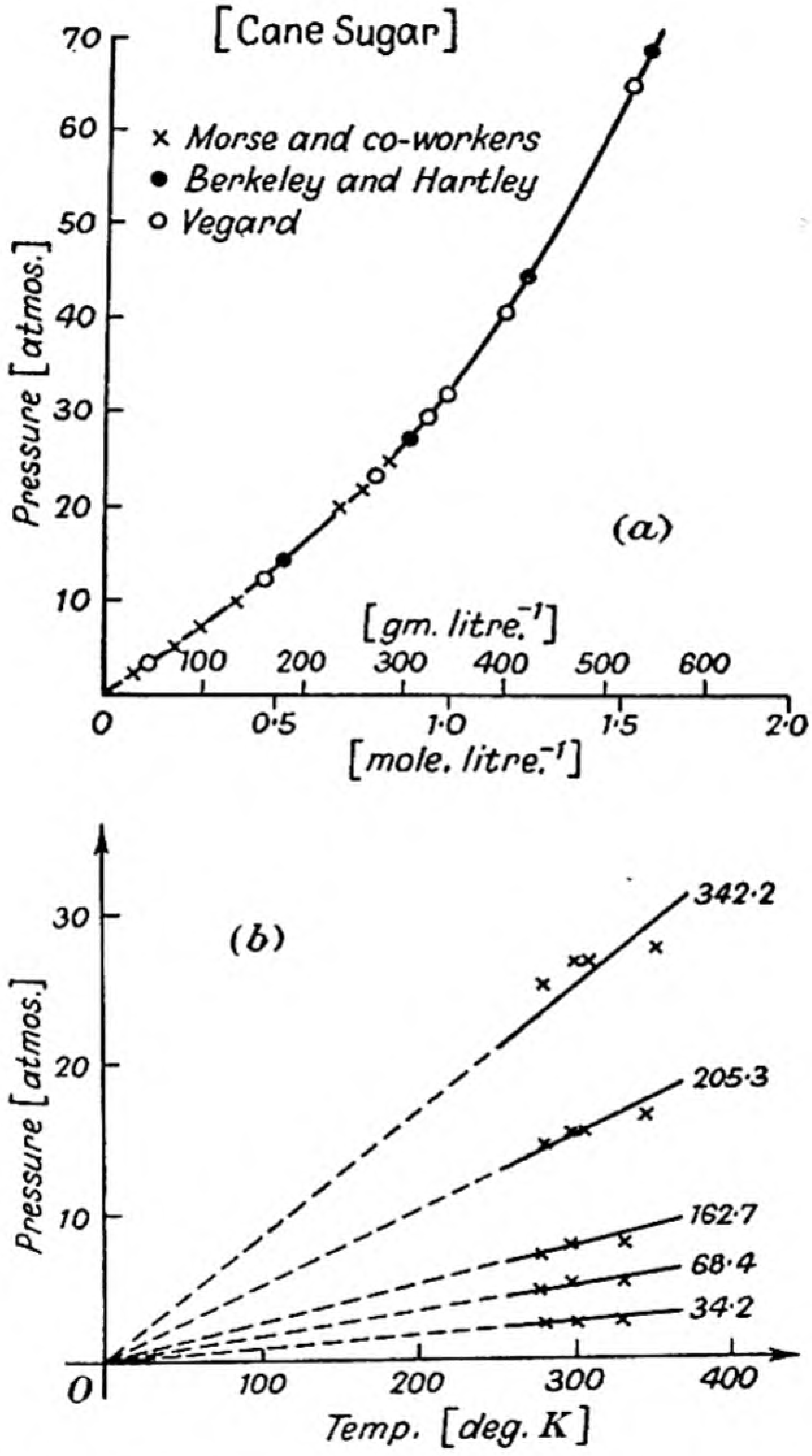

Fig. 13·11.—Some results of osmotic pressure measurements.

stand (not shown). By this means the cork is kept in a central position. Water is contained in the dropping funnel, E, and it is allowed to drop on to the top of the cork, which has been thoroughly wetted, at the rate of about three drops per second. A layer of water soon appears on top of the solution, and when the cork is clear

of the solution, it may be removed, and the vessel, A, completely filled with water. The whole of A is then surrounded by water as in Fig. 13·12(*b*). The temperature is kept constant to avoid convection currents. At first there is a distinct boundary between the solution and the water. As a result of the process called diffusion this well-defined boundary soon disappears. By determining the

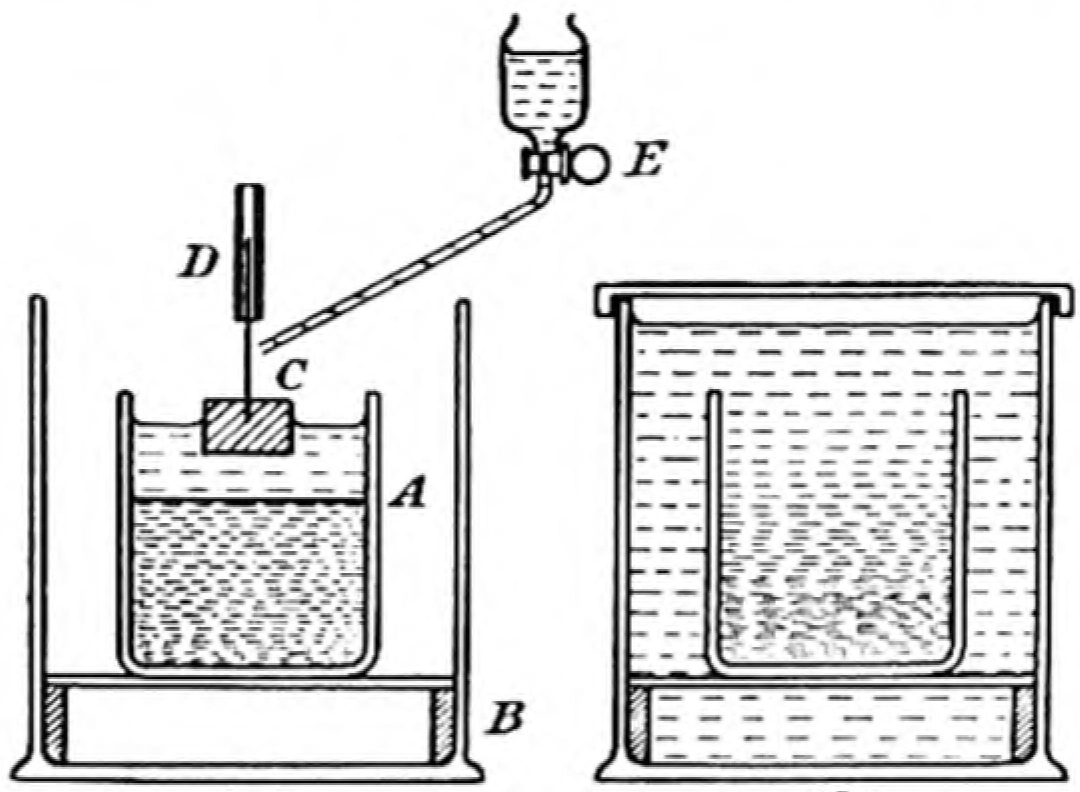

FIG. 13·12.—The diffusion of salts in aqueous solution.

amount of solute which had escaped from the inner vessel into the outer one, it was found:—

(a) the rate of diffusion depends on the nature of the dissolved substance, so that the ratio of the amounts of two substances present in a solution may alter on account of diffusion,

(b) the rate of diffusion is directly proportional to the concentration of the dissolved substance,

(c) a rise in temperature augments the rate at which diffusion takes place.

Fick's law.—Four years after the publication of Graham's first paper on diffusion, FICK, guided by Fourier's work on the conduction of heat, enunciated the following law. ***The mass, m, of a substance in solution passing across an area, A, per second is directly proportional to the rate at which the concentration, n, of the dissolved substance diminishes in a direction at right angles to the plane of the area A.*** In symbols

$$\frac{m}{\mathrm{A}} = -\mathrm{D}\frac{\partial n}{\partial x},$$

where D is a constant and $\frac{\partial n}{\partial x}$ is the ***concentration gradient,*** i.e. the rate at which the concentration n increases with the distance x.

Usually, n is expressed in grams per unit volume, e.g. gm.cm.$^{-3}$ and the equation only applies if the temperature remains constant. D is termed the ***coefficient of diffusion*** of the solute in a solution of concentration n gm.cm.$^{-3}$. The experiments of the early workers in this subject only gave mean values for the diffusion coefficient, for it is now known that the coefficient is not a constant for a given solute but that it depends upon its concentration in the solvent.

Fick's law as a differential equation.—Consider a tall jar of constant cross-sectional area A containing a solution. Let $x = 0$ define the base of the jar, and at all points in a plane at height x let the concentration be n. Then the mass of dissolved substance crossing this plane per second is

$$-\mathrm{DA}\,\frac{\partial n}{\partial x}.$$

Across a plane at height $x + \delta x$, where the concentration is $n + \delta n$, the mass which passes per second is given by

$$-\mathrm{DA}\left\{\frac{\partial n}{\partial x} + \frac{\partial}{\partial x}\left(\frac{\partial n}{\partial x}\right)\delta x\right\}.$$

Between these two planes the volume of solution is A δx and the mass of dissolved substance which enters this volume per second is

$$-\mathrm{DA}\,\frac{\partial n}{\partial x} - \left[-\mathrm{DA}\left\{\frac{\partial n}{\partial x} + \frac{\partial}{\partial x}\left(\frac{\partial n}{\partial x}\right)\delta x\right\}\right] = \mathrm{DA}\,\frac{\partial^2 n}{\partial x^2}\,\delta x.$$

Since A δx is the volume of the element considered, the rate at which the concentration is increasing in this element, viz. $\dfrac{\partial n}{\partial t}$, is

$$\mathrm{DA}\,\frac{\partial^2 n}{\partial x^2}\,\delta x \div \mathrm{A}\,\delta x,$$

i.e.

$$\frac{\partial n}{\partial t} = \mathrm{D}\,\frac{\partial^2 n}{\partial x^2}\,.$$

This is Fick's law expressed as a differential equation. The equation implicitly assumes that the liquid is at rest during the operation of diffusion. Prior to 1900 it was not realized that such was not the case, for the changes which occur in the volume of the solution caused by alterations produced in its concentration as diffusion proceeds, inevitably result in a movement of the liquid, so that an extension of Fick's law is necessary if experimental facts are to be satisfied.

Fick attempted to carry out experiments when the diffusion had reached a steady state. A tube open at both ends, was cemented

vertically into a vessel containing, for example, crystals of sodium chloride so that its lower end was in contact with the salt. The tube was filled with water and the whole placed in a large bath of water and left to itself until the rate of diffusion had become steady.

In this way the concentration at the lower end of the tube is maintained constant while at the upper end it may be taken as zero. In the steady state $\dfrac{\partial^2 n}{\partial x^2} = 0$, i.e. $n = ax + b$, where a and b are constants. To test this deduction Fick weighed a small glass bulb at various depths below the surface of the liquid in the tube and thus determined the density of the solutions at these points. From the known manner in which density varies with concentration, the latter is determined. Fick found it to be in good agreement with the above formula.

The various methods by which subsequent workers have attempted to verify Fick's law may be classified into two chief groups:—

(a) those in which the concentration at the upper end of the diffusion vessel is maintained zero at all times—these belong to the so-called 'bath-method'.

(b) those in which the concentration-gradient at the upper end or 'mouth' of the vessel is always zero. Here we have the so-called 'jar-method'. In this method the mouth of the vessel is in contact with the atmosphere so that the flow of solute across it is zero at all times, i.e.

$$-\mathrm{D}\frac{\partial n}{\partial x} = 0, \qquad \text{or} \qquad \frac{\partial n}{\partial x} = 0.$$

General theory of the 'bath method'.—In this method the aqueous, or other, solution is contained in a cylindrical vessel, A, Fig. 13·13. The axis of the cylinder is vertical and initially the

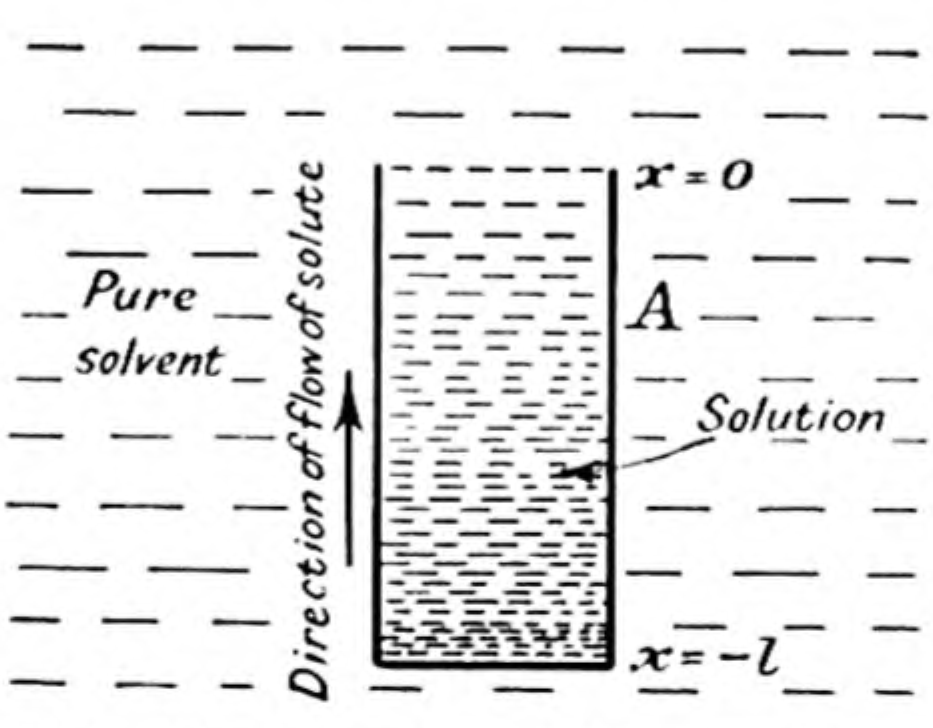

Fig. 13·13.—Theory of the 'bath' method for determining a coefficient of diffusion.

vessel is completely filled with a solution of uniform concentration N; it is surrounded by a large volume of the pure solvent. The experiment consists in determining the rate at which the dissolved substance diffuses from the vessel. The general theory is as follows.

Let $x = 0$ define the open end of the cylindrical portion of the vessel while its lower end is $x = -l$. The equation to be solved, which is common to all work on diffusion, is

$$\frac{\partial n}{\partial t} = \mathrm{D}\frac{\partial^2 n}{\partial x^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

A solution to this equation, cf. p. 39, is

$$n = a_0 + \{\exp(-m^2\mathrm{D}t)\}(a\cos mx + b\sin mx) \quad . \quad . \quad \text{(ii)}$$

The conditions of the experiment are such that

$$n_{t=0} = \mathrm{N}, \text{ everywhere,}$$

$$n_{t=\infty} = 0, \text{ everywhere,} \qquad \therefore\ a_0 = 0,$$

$$n_{x=0} = 0 \text{ at all times,} \qquad \therefore\ a = 0,$$

and

$$\left(\frac{\partial n}{\partial x}\right)_{x=-l} = 0,$$

since no dissolved substance passes across this boundary.

This last condition implies that $b\cos ml = 0$, at all times.

$$\therefore\ m = \frac{2p-1}{2l}\pi, \quad \text{where } p = 1, 2, 3 \ldots$$

$$\therefore\ n = \sum_{p=1}^{\infty}\exp\left\{-\frac{(2p-1)^2}{4l^2}\pi^2\mathrm{D}t\right\}.b_p\sin\left(\frac{2p-1}{2l}\right)\pi x. \quad . \quad . \quad \text{[iii]}$$

Putting $t = 0$ in the above equation, remembering that $[n]_{t=0} = \mathrm{N}$ everywhere, multiplying throughout by $\sin\dfrac{2p-1}{2l}\pi x\,dx$, and integrating from $x = -l$ to $x = 0$, in the usual manner for finding the coefficients in a Fourier's series, we get

$$\mathrm{N}\int_{-l}^{0}\sin\frac{2p-1}{2l}\pi x\,dx = b_p\int_{-l}^{0}\sin^2\left(\frac{2p-1}{2l}\right)\pi x\,dx.$$

$$\therefore\ \mathrm{N}\left[\frac{-2l}{(2p-1)\pi}\cos\frac{2p-1}{2l}\pi x\right]_{-l}^{0} = b_p\left(\frac{l}{2}\right).$$

$$\therefore\ b_p = -\frac{4\mathrm{N}}{(2p-1)\pi}.$$

Hence

$$n = -\frac{4N}{\pi}\sum_{p=1}^{\infty}\frac{1}{(2p-1)}\exp\left\{-\frac{(p-1)^2\pi^2}{4l^2}Dt\right\}\left[\sin\left(\frac{2p-1}{2l}\right)\pi x\right].$$

If $\dfrac{dM}{dt}$ is the rate at which the dissolved substance leaves the top of the cylinder, i.e.

$$\frac{dM}{dt} = -DA\left(\frac{\partial n}{\partial x}\right)_{x=0},$$

then
$$M = -DA\int_0^t\left(\frac{\partial n}{\partial x}\right)_{x=0} dt,$$

where M is the mass leaving in t seconds and A is the constant cross-sectional area of the jar.

Now

$$\left(\frac{\partial n}{\partial x}\right)_{x=0} = -\frac{4N}{\pi}\sum_{p=1}^{\infty}\left[\epsilon^{-(\ldots)t}\cdot\frac{1}{2p-1}\cdot\frac{(2p-1)\pi}{2l}\cos\frac{2p-1}{2l}\pi x\right]$$

$$= -\frac{2N}{l}\sum_{p=1}^{\infty}\epsilon^{-(\ldots)t}.$$

$$\therefore\ M = \frac{2DAN}{l}\int_0^t\sum_{p=1}^{\infty}\exp\left\{-\frac{(2p-1)^2\pi^2 Dt}{4l^2}\right\}dt$$

$$= \frac{2DAN}{l}\left[\sum_{p=1}^{\infty}\left\{\frac{-4l^2}{(2p-1)^2\pi^2 D}\epsilon^{-(\ldots)t} + \frac{4l^2}{(2p-1)^2\pi^2 D}\right\}\right]$$

$$= 2ANl\left[\frac{4}{\pi^2}\sum_{p=1}^{\infty}\left\{\frac{1}{(2p-1)^2} - \frac{1}{(2p-1)^2}\exp\left(-\frac{\overline{2p-1}^2\pi^2 Dt}{4l^2}\right)\right\}\right]$$

$$= M_0\left[1 - \frac{8}{\pi^2}\sum_{p=1}^{\infty}\frac{1}{(2p-1)^2}\exp\left\{-\frac{(2p-1)^2\pi^2 Dt}{4l^2}\right\}\right]$$

since

$$\sum_{p=1}^{\infty}\frac{1}{(2p-1)^2} = \frac{\pi^2}{8}, \qquad \text{[cf. p. 25]}$$

where M_0 is the mass of salt originally placed in the cylinder. The above series converges very rapidly so that in practice it is sufficient to use the formula

$$M = M_0\left[1 - \frac{8}{\pi^2}\exp\left(-\frac{\pi^2 Dt}{4l^2}\right)\right],$$

and it is from this formula that a value for D is usually found.

General theory of the 'jar' method for investigating diffusion in liquids.—Initially a tall cylindrical jar of uniform cross-section is filled with solvent (water); the solution is then introduced below the water so that the initial appearance of the two liquids is as shown in Fig. 13·14. Let $x = 0$ define the upper end of the jar, while $x = -l_1$ defines the plane of demarcation between the solvent and the solution initially, and $x = -l$ is the bottom of the liquid column.

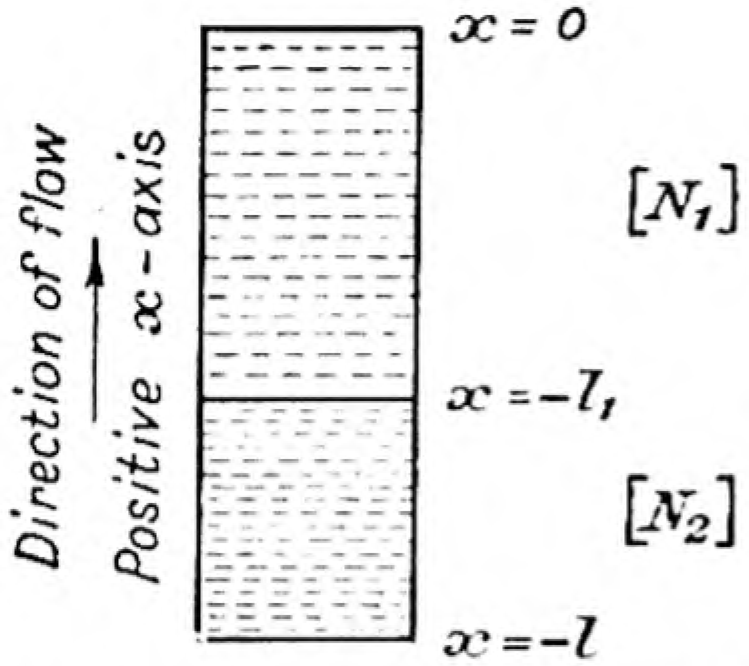

FIG. 13·14.—Theory of the 'jar' method for determining a coefficient of diffusion.

Now Fick's law is

$$\frac{\partial n}{\partial t} = \mathrm{D}\frac{\partial^2 n}{\partial x^2} \quad . \quad . \quad . \quad \text{(i)}$$

A solution to this equation is

$$n = a_0 + [a \cos mx + b \sin mx] \times \exp(-m^2\mathrm{D}t).$$

Under the conditions of the experiment in which, for the sake of generality, it will be assumed that N_1 and N_2 are the initial concentrations of the solute for the regions for which $-l_1 < x < 0$ and $-l < x < -l_1$, respectively, $\left(\frac{\partial n}{\partial x}\right)_{x=0}$ is always zero, since no solute ever passes across this boundary. But

$$\frac{\partial n}{\partial x} = [-ma \sin mx + bm \cos mx] \exp(-m^2\mathrm{D}t).$$

Hence $b = 0$.

Also, for a similar reason, $\left(\frac{\partial n}{\partial x}\right)_{x=-l}$ is always zero.

$$\therefore\ ml = p\pi, \qquad \text{where } p = 1, 2, 3, \ldots$$

$$\therefore\ n = a_0 + \sum_{p=1}^{\infty} a_p \cos\frac{p\pi x}{l} \exp\left(\frac{-p^2\pi^2\mathrm{D}t}{l^2}\right).$$

When $t = 0$,

$$n_{t=0} = a_0 + \sum_{p=1}^{\infty} a_p \cos\frac{p\pi x}{l},$$

and this enables us to find a_0. For we have

$$\int_{-l}^{-l_1} \mathrm{N}_2\, dx + \int_{-l_1}^{0} \mathrm{N}_1\, dx = \int_{-l}^{0} a_0\, dx + \int_{-l}^{0} \sum_{p=1}^{\infty} a_p \cos\frac{p\pi x}{l}\, dx,$$

since either side of this equation expresses at time $t = 0$ the total amount of solute in a column of liquid of unit area and extending from the top to the bottom of the jar.

This gives

$$N_2(-l_1 + l) + N_1(l_1) = a_0(l) + 0.$$

$$\therefore\ a_0 = \frac{N_1 l_1}{l} + \frac{N_2(-l_1 + l)}{l} = \frac{N_1 l_1 + N_2 l_2}{l}, \qquad \text{if } l_2 = -l_1 + l.$$

To find a_p we proceed as follows:—We have

$$n_{t=0} = a_0 + \sum_{p=1}^{\infty} a_p \cos \frac{p\pi x}{l}.$$

Multiply throughout by $\cos \frac{p\pi x}{l}\,dx$ and integrate. We get

$$\int_{-l}^{-l_1} N_2 \cos \frac{p\pi x}{l}\,dx + \int_{-l_1}^{0} N_1 \cos \frac{p\pi x}{l}\,dx$$

$$= a_0 \int_{-l}^{0} \cos \frac{p\pi x}{l}\,dx + \int_{-l}^{0} \sum_{p=1}^{\infty} a_p \cos^2 \frac{p\pi x}{l}\,dx.$$

$$\therefore\ N_2 \left[\frac{\sin \frac{p\pi x}{l}}{\frac{p\pi}{l}}\right]_{-l}^{-l_1} + N_1 \left[\frac{\sin \frac{p\pi x}{l}}{\frac{p\pi}{l}}\right]_{-l_1}^{0} = 0 + \int_{-l}^{0} a_p \cos^2 \frac{p\pi x}{l}\,dx = a_p\left(\frac{l}{2}\right).$$

$$\therefore\ a_p = \frac{2(N_1 - N_2)}{p\pi} \sin \frac{p\pi l_1}{l}.$$

Hence the general solution is

$$n = \frac{N_1 l_1 + N_2 l_2}{l} + \sum_{p=1}^{\infty} \frac{2(N_1 - N_2)}{p\pi} \sin \frac{p\pi l_1}{l} \cos \frac{p\pi x}{l} \exp\left(-\frac{p^2\pi^2 Dt}{l^2}\right).$$

If $N_1 = 0$, $N_2 = N$ and $l_1 = l_2 = \frac{1}{2}l$, we get

$$n = \frac{N}{2} - \sum_{p=1}^{\infty} \frac{2N}{p\pi} \sin \frac{p\pi}{2} \cos \frac{p\pi x}{l} \exp\left(-\frac{p^2\pi^2 Dt}{l^2}\right)$$

$$= \frac{N}{2} - \frac{2N}{\pi}\Bigg[\sin \frac{\pi}{2} \cos \frac{\pi x}{l} \exp\left(-\frac{\pi^2 Dt}{l^2}\right)$$

$$+ 0 + \tfrac{1}{3} \sin \frac{3\pi}{2} \cos \frac{3\pi x}{l} \exp\left(-\frac{9\pi^2 Dt}{l^2}\right)$$

$$+ 0 + \tfrac{1}{5} \sin \frac{5\pi}{2} \cos \frac{5\pi x}{l} \exp\left(-25\frac{\pi^2 Dt}{l^2}\right) + \dots\Bigg],$$

and at the plane $x = -\frac{1}{2}l$,

$$\left(\frac{\partial n}{\partial x}\right)_{x=-\frac{1}{2}l} = -\frac{2N}{l}\left[\exp(\alpha t) + \exp(9\alpha t) + \exp(25\alpha t) + \dots\right],$$

if $\alpha = -\dfrac{\pi^2 D}{l^2}$.

If M is the mass of dissolved substance leaving the lower part of the jar and entering the upper part in t seconds,

$$\mathrm{M} = -\mathrm{AD}\int_0^t \left(\frac{\partial n}{\partial x}\right)_{x=-\frac{1}{2}l} dt, \text{ where A is the cross-section of the jar,}$$

$$= \frac{2\mathrm{ADN}}{l}\cdot\frac{1}{\alpha}[\exp(\alpha t) + \tfrac{1}{9}\exp(9\alpha t) + \tfrac{1}{25}\exp(25\alpha t) + \dots$$

$$- (1 + \tfrac{1}{9} + \tfrac{1}{25} + \dots)]$$

$$= \frac{2\mathrm{DAN}}{l}\cdot\frac{l^2}{\pi^2\mathrm{D}}\left[(1 + \tfrac{1}{9} + \tfrac{1}{25} + \dots) - \exp\left(-\frac{\pi^2\mathrm{D}t}{l^2}\right)\right],$$

if small terms are omitted.

$$\therefore\ \mathrm{M} = \frac{2\mathrm{DAN}}{l}\cdot\frac{l^2}{\pi^2\mathrm{D}}\left[\frac{\pi^2}{8}\left\{1 - \frac{8}{\pi^2}\exp\left(-\frac{\pi^2\mathrm{D}t}{l^2}\right)\right\}\right].$$

$$= \frac{\mathrm{AN}l}{4}\left[1 - \frac{8}{\pi^2}\exp\left(-\frac{\pi^2\mathrm{D}t}{l^2}\right)\right].$$

A laboratory method for determining the coefficient of diffusion of a salt in aqueous solution.—This method is a 'jar' method and is due to STEPHENS and RAMSAY but in the form here described it is only applicable to coloured solutions. The reason for this limitation is that in order to determine the concentration in a given plane after the lapse of a known time the reduction in the intensity of a beam of light passing through the solution is measured with the aid of a photoelectric cell. From the theory of the 'jar' method just developed the concentration in a plane x at time t is given by

$$n = \frac{\mathrm{N}}{2} - \frac{2\mathrm{N}}{\pi}\left[\cos\frac{\pi x}{l}\exp\left(-\frac{\pi^2\mathrm{D}t}{l^2}\right)\right],$$

if the time is so large that all terms, except the first, may be omitted.

Two different methods of correlating the above equation with experimental observations immediately suggest themselves but it is important to remember that it is necessary to arrange the equations so that negative terms do not appear. The reason for this is that the logarithm of a negative number is imaginary, for

$$-1 = \cos\pi = \cos\pi + j\sin\pi = \exp(j\pi),$$

so that $$\ln(-1) = j\pi.$$

(a) In the first method a fixed plane is selected and values of the concentration n at different times are noted. Now the equation

showing how n varies with x after the lapse of some considerable time may be written

$$\left(\frac{N}{2} - n\right)\frac{\pi}{2N} = \cos\frac{\pi x}{l}\exp\left(-\frac{\pi^2 Dt}{l^2}\right).$$

If $-\frac{1}{2}l < x < 0$, i.e. x defines a point in the upper half of the jar so that, in practice, $n < \frac{N}{2}$ at all times, then each term in the above equation is positive, so that

$$\left(\frac{N}{2} - n\right) = B\exp\left(-\frac{\pi^2 Dt}{l^2}\right),$$

where B is positive. Under the conditions for which this equation is valid, it follows that a plot of $\ln\left(\frac{N}{2} - n\right)$ against t should yield a straight line whose slope is $-\frac{\pi^2 D}{l^2}$.

If, however, $-l < x < -\frac{1}{2}l$, i.e. x defines a point in the lower half of the jar, then $n > \frac{N}{2}$ at all times in practice, and we have

$$\left(n - \frac{N}{2}\right)\frac{\pi}{2\pi} = \left(-\cos\frac{\pi x}{l}\right)\exp\left(-\frac{\pi^2 Dt}{l^2}\right)$$

$$= \cos\frac{\pi(l - x)}{l}\exp\left(-\frac{\pi^2 Dt}{l^2}\right).$$

All terms are now positive and a plot of $\ln\left(n - \frac{N}{2}\right)$ against t can be made as before.

(*b*) In the second method, x is considered as the variable and n is a constant. Our fundamental equation is

$$\cos\frac{\pi x}{l}\exp\left(-\frac{\pi^2 Dt}{l^2}\right) = \left(\frac{N}{2} - n\right)\frac{\pi}{2N}.$$

If $-\frac{1}{2}l < x < 0$, $n < \frac{N}{2}$, and since $\cos\alpha = \cos(-\alpha)$, each term is positive so that

$$-\frac{\pi^2 Dt}{l^2} + \ln\cos\frac{\pi x}{l} = \text{a real constant}.$$

If $-l < x < -\frac{1}{2}l$, $n > \frac{N}{2}$, and we must write

$$\left(n - \frac{N}{2}\right)\frac{\pi}{2N} = \left[\cos\frac{\pi(l - x)}{l}\right]\exp\left(-\frac{\pi^2 Dt}{l^2}\right),$$

so that real quantities occur when logarithms are taken in order to obtain a linear relation.

Thus depending on the value of n selected, it is necessary to plot $\ln \cos \dfrac{\pi x}{l}$ or $\ln \cos \dfrac{\pi(l-x)}{l}$ against t, in order to obtain a linear graph from the slope of which D may be obtained.

Whatever method is adopted it must be remembered that the diffusion coefficient, D, depends upon the concentration so that the value of D deduced only has an approximate meaning.

Apparatus and experimental procedure.—The 'jar' used is a small glass vessel of rectangular section (5 cm. $\times$ 1 cm.) and is about 10 cm. deep. It is placed on a rigid stone pillar, P, Fig. 13·15,

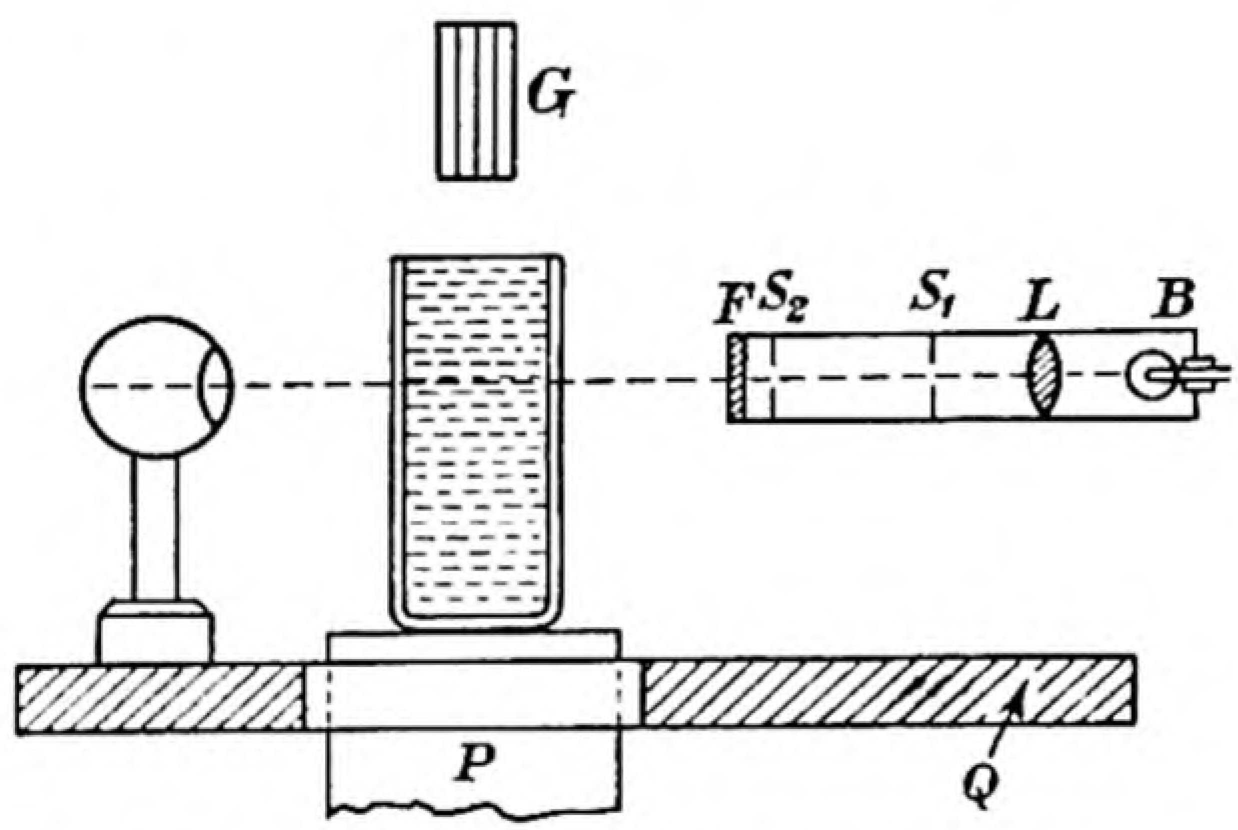

FIG. 13·15.—Coefficient of diffusion apparatus. [Mechanism for raising Q not shown.]

while the photoelectric cell and the light source are fixed on a horizontal platform, Q, which may be raised with the aid of a rack and pinion. In this way a narrow horizontal beam of light from the bulb B, an automobile headlight, may be directed through any horizontal layer of the liquid. The bulb is mounted in a suitable housing carrying a converging lens, L, which is used to obtain a parallel beam of light. Two narrow slits, S_1 and S_2, limit the width of this beam. The photoelectric cell is connected in series with a sensitive galvanometer, a high-tension battery and a suitable high resistance. A red filter, F, is employed to utilize the greater sensitivity of the caesium type photocell to the longer wavelengths.

Since the observations extend over a number of days precautions must be taken to ensure that the candle power of the light source shall be constant. The current through the lamp B is adjusted to an approximately constant value. The light is then passed through a pile of glass plates, G, which acts as an invariable absorbing

medium; the current through B is finally adjusted so that with G in position, a constant galvanometer deflexion is obtained.

In order to correlate the deflexion obtained when the cell is in use with the concentration, it is necessary to fill the cell with solutions

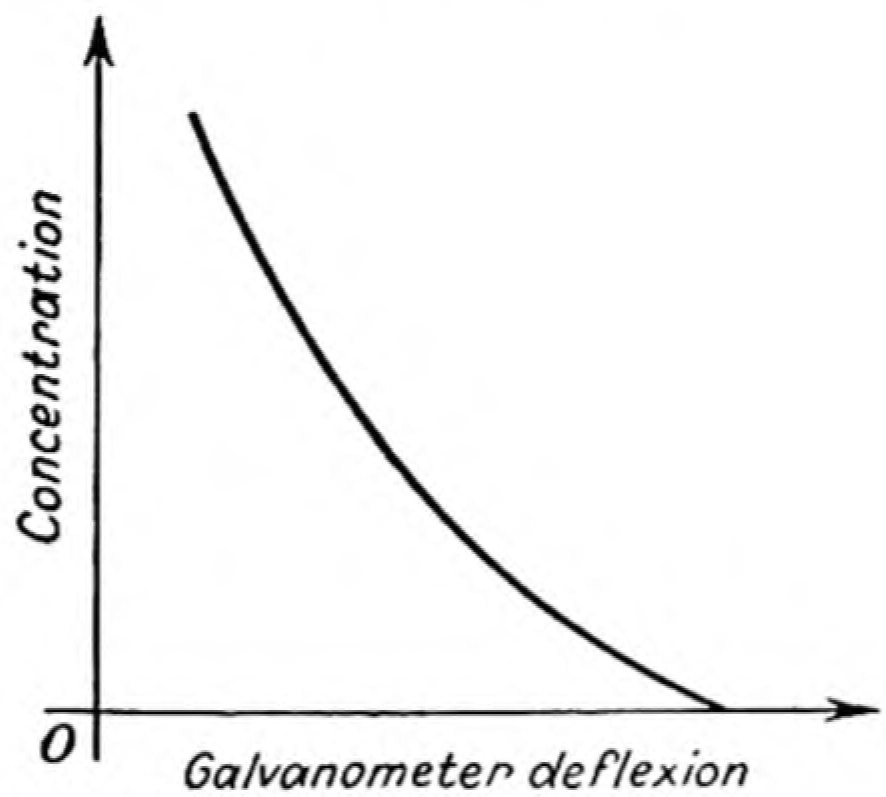

FIG. 13·16.—Calibration curve for use with the apparatus shown in Fig. 13·15.

of known uniform strength and obtain a calibration curve as shown in Fig. 13·16. External light is excluded by enclosing in a wooden box the source, the diffusion cell and the light detector.

Fig. 13·17 shows a typical set of curves, the horizontal dotted line

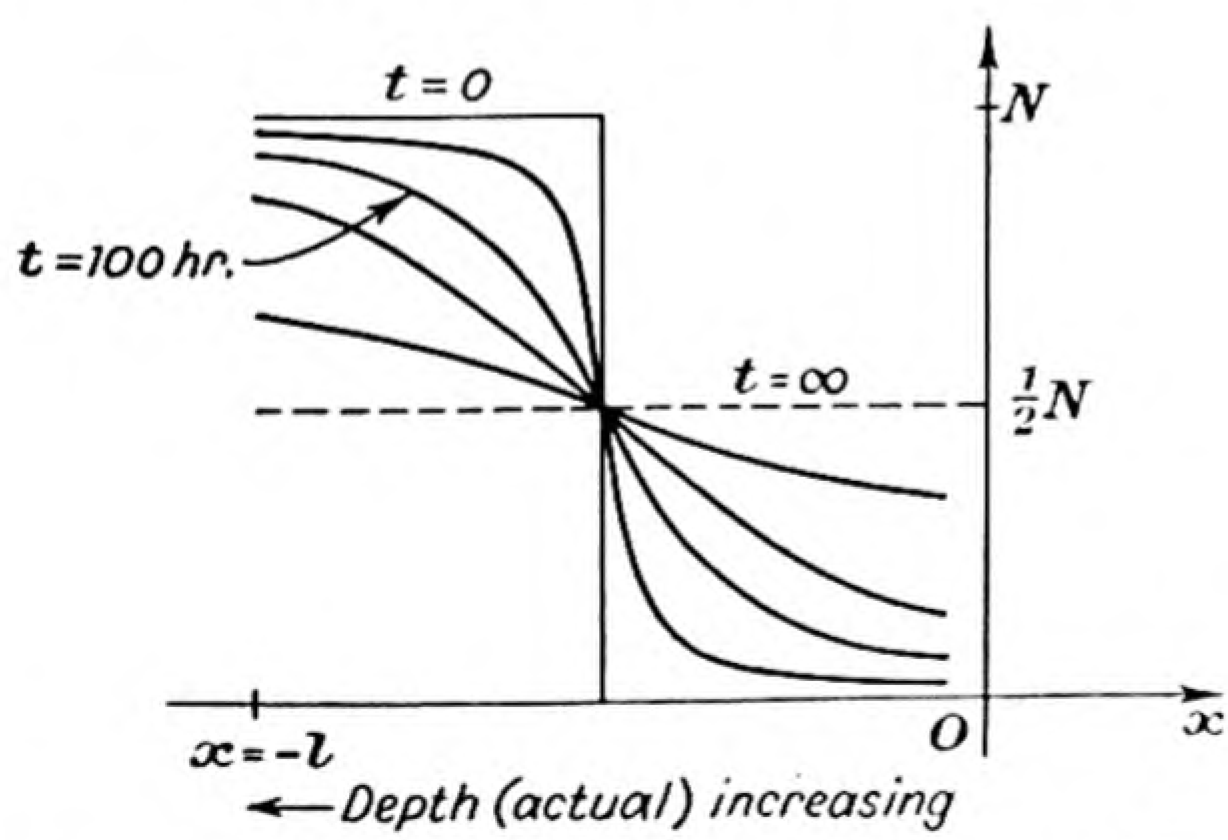

FIG. 13·17.—Typical curves obtained with Stephens and Ramsay's diffusion apparatus.

indicating the final concentration when complete mixing has been effected, i.e. the state of affairs when $t \to \infty$. From these curves a linear graph may be constructed in the manner already explained; the graph is then used to find a value for the diffusion coefficient.

Einstein's theory of diffusion.—The ordinary process of diffusion must arise as a consequence of the Brownian motions of individual particles. With this in view and the aid of probability theory EINSTEIN was able to show that $\overline{\Delta[x^2]}$, as defined on p. 655, is related to the diffusion coefficient D by the equation

$$\overline{\Delta[x^2]} = 2Dt,$$

where t is the time for which the mean value of $\Delta[x^2]$ is evaluated.

Since $$\overline{\Delta[x^2]} = \frac{RTt}{3\pi a\eta N}, \qquad \text{[Cf. p. 655]}$$

where N is Avogadro's constant, we have

$$N = \frac{RT}{6\pi a\eta D}.$$

Now for an aqueous sugar solution at 17° C., $D = 2{\cdot}9 \times 10^{-6}$ cm.^{2}sec.$^{-1}$. If it is assumed that sugar molecules behave like rigid spheres of radius $4{\cdot}15 \times 10^{-8}$ cm. and the viscosity of water at 17° C. is 0·010 gm.cm.$^{-1}$sec.$^{-1}$, we have, if R is taken as $8{\cdot}3 \times 10^7$ erg.deg.$^{-1}$C.mole.$^{-1}$,

$$N = \frac{8{\cdot}3 \times 2{\cdot}9 \times 10^{7+2}}{6\pi \times 4{\cdot}15 \times 1{\cdot}0 \times 2{\cdot}9 \times 10^{-8-2-6}}$$

$$= 10{\cdot}6 \times 10^{23} \text{ mole.}^{-1}.$$

Since accurate methods for the measurement of Avogadro's constant give $N = 6{\cdot}02 \times 10^{23}$ mole.$^{-1}$, the approximate nature of the above analysis is revealed.

EXAMPLES XIII

13·01. Define ***osmosis*** and ***osmotic pressure,*** and explain how the osmotic pressure of a solution may be measured. How do you account for the fact that the osmotic pressure of a dilute solution of cane sugar is roughly half that of a solution of potassium chloride and of the same molecular concentration?

13·02. Explain the terms osmosis and osmotic pressure.

Upon what factors does the osmotic pressure of a solution depend?

Derive an expression for the osmotic pressure at 0°C. of a dilute non-electrolytic solution, V ml. of which contain N grammes of a solute of molecular weight M. $\left[\dfrac{273\ RN}{VM}\right]$.

13.03. Give some account of the phenomena of diffusion and osmosis.

13·04. Give an account of the methods which have been used to measure *either* (*a*) osmotic pressure *or* (*b*) coefficients of diffusion. (S)

13·05. Write a short account of the phenomenon of ***osmosis*** and of the significance and measurement of ***osmotic pressure.*** (G)

13·06. Write a short essay on the Brownian movement and indicate how suitable observations of the Brownian movement enable an estimate to be made of Avogadro's constant (number of molecules in a gramme molecule). (G)

13·07. A suspension of particles in a liquid is said to have properties similar to those of an ideal gas. Illustrate this statement by discussing the effect of a uniform gravitational field on the distribution in height of the suspended particles.

Explain how the result of this investigation has been applied to obtain Avogadro's constant, giving as much experimental detail as you can. (S)

13·08. A track is formed in a cloud chamber t seconds after the passage of an ionizing particle. The images of ions are distributed in accordance with the formula

$$f(x, t) = \frac{C}{\sqrt{t}} \exp\left(-\frac{x^2}{4Dt}\right), \qquad \text{(S)}$$

where f is the number of images per unit of area, x is the distance from the track axis and C and D are constants. Show that this distribution can be accounted for, by assuming that the ions diffuse away from the track at a rate determined by the equation

$$\frac{\partial f}{\partial t} = D \frac{\partial^2 f}{\partial x^2}.$$

Give a brief account of the assumptions involved in applying a diffusion equation to the motion of the ions.

CHAPTER XIV

VACUUM PRACTICE

Historical note.—The year 1879 saw the advent of the carbon-filament electric lamp and with this there was aroused an interest in the production of low pressures which has not only continued until today but remains a matter of considerable importance. Previously only a few physicists had really considered the problem of the production of a high vacuum. In England CROOKES, and in Germany HITTORF, examined the passage of electricity through gases at low pressures and, to produce these, mechanical pumps were used. With such pumps the lowest attainable pressure was about 0·25 mm. of mercury and it was only by using a manually operated Toepler pump that a pressure of the order 10^{-3} mm. of mercury could be obtained with comparative ease; the lowest pressure ever obtained in this way would appear to be about $0{\cdot}05 \times 10^{-3}$ mm. of mercury.

Improvements in the design of electric lamps and the ever-present urge to prolong their 'life' needed better pumping apparatus and GAEDE, in Germany, was among the first to design a rotatory oil pump capable of reducing the pressure inside a piece of glass apparatus to about 10^{-4} mm. of mercury. In 1905 Gaede introduced a rotatory mercury pump and when this was backed by an oil pump a pressure of 10^{-5} mm. of mercury could be obtained although the rate of pumping was slow. Ten years later Gaede designed a so-called 'diffusion' pump and this was followed in the next year by LANGMUIR's 'condensation' pump; in each type mercury vapour was the 'working agent' and with the aid of these pumps pressures as low as 10^{-7} mm. (or 10^{-4} micron) of mercury were obtained. The result of this was at once manifested in the much better performance of thermionic valves and X-ray tubes of the Coolidge (or hot filament) type.

In 1928 BURCH found that the mercury of a condensation pump could be replaced by a high boiling-point derivative of petroleum and in 1930 HICKMAN replaced this somewhat indefinite compound by a synthetic phthalate or sebacate. The chief advantage gained by using such substances in place of mercury lies in the fact that for all of them the pressure of the saturated vapour at room-temperature is about 10^{-6} mm. of mercury and thus no 'trap' is required between the pump and the system to be exhausted. Such a trap

consists of a tube inserted in the main 'pumping-line' and maintained at a low temperature so that the vapour is frozen out. The use of such a trap always reduces the pumping speed, i.e. the rate at which the process of exhaustion can be carried out. The desirability of not being compelled to use a trap is at once apparent if it is a question of producing a high vacuum only. It must be remembered, however, that the vapour is present everywhere so that films of condensed liquid do tend to form and these can easily contaminate a metallic film which may have been placed in the vacuum in order, for example, to study its optical properties. In such instances it becomes imperative to use a trap in spite of the low vapour pressure of the substance used to make the pump function.

General remarks on vacuum pumps.—In comparing vacuum pumps one with the other the following factors must always be kept under review:—

(*a*) ***Exhaust pressure:*** This is defined as the pressure which must be maintained on the 'exit-side' of a pump in order that the pump may be operated. In general, the higher the degree of vacuum required on the 'intake-side' of a pump, the lower must be the exhaust pressure. The requisite exhaust pressure for any given pump is obtained by using one or more 'rough' pumps arranged in series; these are the so-called 'backing-pumps' and the first member of such a series must exhaust directly into the atmosphere.

(*b*) ***Degree of vacuum obtainable:*** This is defined as the lowest pressure which may be attained in a closed system connected to the pump. It is found that this limiting pressure depends upon the exhaust pressure and it is due to the leakage of gas through the pump. This leakage is most noticeable when the light gases such as hydrogen and helium have to be pumped out of a closed vessel.

(*c*) ***The speed of a pump:*** Much confusion still exists concerning the meaning of this term. Here two definitions will be given and then correlated. The first defines the speed of a pump as the mass of gas abstracted per second from the vessel or system which is being evacuated. This speed will be denoted by σ. The second, denoted by S, is defined as the volume of gas extracted per second, the volume being measured at the pressure then prevailing in the system. To correlate σ and S let us consider the following.

Let V be the volume of the system being exhausted—this will include part of the pump and, in general, is a constant. If m is the mass of gas enclosed in V at time t and $(m + \delta m)$ the corresponding mass at time $(t + \delta t)$, then $-\delta m$ is the mass removed in time δt, so that

$$\sigma = -\frac{dm}{dt}.$$

If p and $(p + \delta p)$ are the corresponding pressures of the gas in the system at times t and $(t + \delta t)$, and $\mathscr{R}$ is the appropriate gas-constant per unit mass $\left[\mathscr{R} = \dfrac{\mathrm{R}}{\mathrm{M}}\right.$ where R is the universal gas-constant and M the molecular weight of the gas$\left.\right]$, then

$$p\mathrm{V} = m\mathscr{R}\mathrm{T},$$

so that $$\mathrm{V}\frac{dp}{dt} = \mathscr{R}\mathrm{T}\frac{dm}{dt}. \qquad [\mathrm{V},\ \mathscr{R} \text{ and } \mathrm{T} \text{ are constants.}]$$

Hence $$\sigma = -\frac{dm}{dt} = -\frac{\mathrm{V}}{\mathscr{R}\mathrm{T}}\frac{dp}{dt}.$$

Now let Ω be the volume occupied by a mass m of gas at pressure p; then

$$p\Omega = m\mathscr{R}\mathrm{T}.$$

If it is assumed that the temperature is everywhere constant, we have

$$p\,\delta\Omega = \delta m\mathscr{R}\mathrm{T},$$

which equation gives the volume $\delta\Omega$ of gas at pressure p whose mass is δm. Hence

$$\therefore\ \mathrm{S} = -\frac{d\Omega}{dt} = -\frac{\mathscr{R}\mathrm{T}}{p}\frac{dm}{dt}$$

$$= -\frac{\mathscr{R}\mathrm{T}}{p}\frac{\mathrm{V}}{\mathscr{R}\mathrm{T}}\frac{dp}{dt} = -\frac{\mathrm{V}}{p}\frac{dp}{dt} = -\mathrm{V}\frac{d}{dt}(\ln p).$$

This gives an expression for the instantaneous value of the pumping speed. If it is justifiable to assume that S is a constant and that the pressure falls from p_1 to p_2 in time t_1 to t_2 $(t_2 > t_1)$, then

$$\mathrm{S}\int_{t_1}^{t_2} dt = -\mathrm{V}\int_{p_1}^{p_2}\frac{dp}{p}.$$

$$\therefore\ \mathrm{S} = \frac{\mathrm{V}}{t_2 - t_1}\ln\frac{p_1}{p_2} = 2{\cdot}303\,\frac{\mathrm{V}}{t_2 - t_1}\log\frac{p_1}{p_2}.$$

Manufacturers of high-speed vacuum pumps seem to prefer this formula, since for all modern pumps it has a high numerical value; the pump speed in terms of mass and time is more definite and equally large numbers can be obtained by taking as the unit of mass the micromilligram, i.e. 10^{-9} gm.

Langmuir's formula for pumping speed.—Langmuir has drawn attention to the fact that as p_1 approached p_u, the ultimate pressure attainable in a given system, so does the value of S tend to

zero. This led Langmuir to introduce yet another quantity, S*, which he called the 'speed of the pump'; it is defined by the equation

$$S^* = \frac{V}{t_2 - t_1} \ln\left(\frac{p_1 - p_u}{p_2 - p_u}\right).$$

This equation may be obtained theoretically as follows. Let p be the pressure of the gas at any instant within a vessel of constant volume V. This pressure may be considered as the sum of two partial pressures P and p_u, where P refers to the instantaneous pressure of the molecules, which will eventually be extracted by the pump from the vessel in the process of being exhausted, and p_u is the partial pressure of the molecules which remain in the vessel. Then m, the mass of gas in the vessel at any instant but finally removed from it, is given by

$$PV = m\mathscr{R}T,$$

so that $$\sigma = -\frac{dm}{dt}, \qquad \text{and} \qquad S^* = -\frac{d\Omega}{dt},$$

where $\delta\Omega$ is the volume at pressure P which the mass δm of the gas occupies. Then, as before,

$$S^* = -\frac{V}{P}\frac{dP}{dt},$$

and if S* is constant

$$S^*(t_2 - t_1) = -V\int_1^2 \frac{dP}{P}.$$

But $P = p - p_u$, where p is the pressure of the gas in V which is measured by a gauge, and hence $\delta p = \delta P$.

$$\therefore\ S^* = \frac{V}{t_2 - t_1} \ln\left(\frac{p_1 - p_u}{p_2 - p_u}\right).$$

If $p_u = 0$, S = S* but in all other instances where p_u is finite, the speed S decreases as the pressure approaches p_u and, as stated above, becomes zero when the limiting pressure is reached.

'Static' and 'kinetic' vacuum systems.—Vacuum systems are usually classified as 'static' or as 'kinetic' systems. The former system comprises all systems which are 'tight', i.e. sealed hermetically except for the exit from the pump and an essential feature of such a system is the extremely low value of the pressure ultimately attained. Typical of this class is a system designed for the evacuation of a small wireless valve. Here the system, i.e. the valve and the connecting 'line' to the pump, consists entirely of glass from which all leaks have been eliminated and all the metal and

glass parts have been freed from adsorbed gas. In the second, or 'kinetic' system, very fast pumps are used to deal with a continuous evolution of gas from the different parts of the apparatus and sometimes with the gas entering the system through a small but intentional leak necessary to form a source of ions, as in Thomson's positive ray apparatus. Mass spectrographs, cyclotrons and all large pieces of apparatus in which a vacuum must be maintained are only practicable if continuous pumping is employed; they are the 'kinetic' systems.

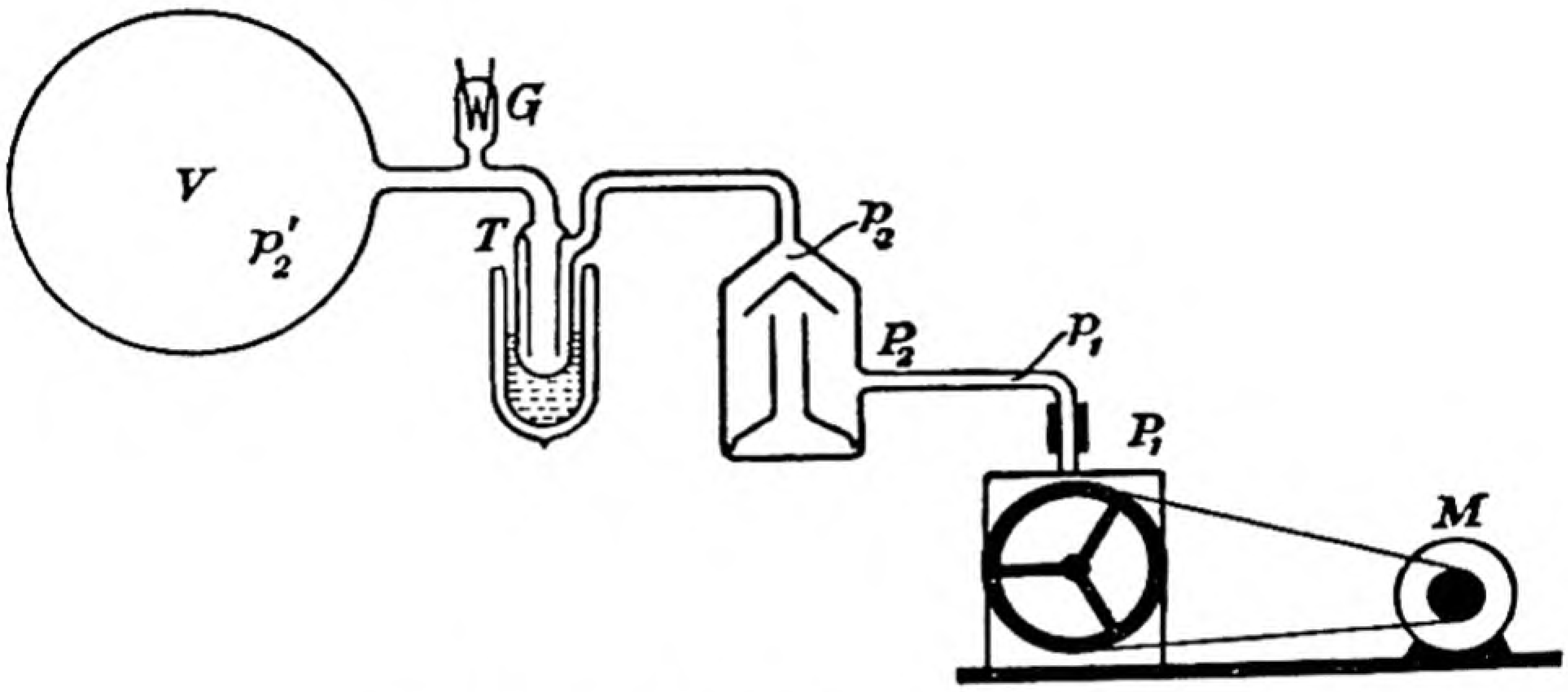

FIG. 14·01.—A typical pumping system.

The essential components of a vacuum system are shown in Fig. 14·01. P_1 is the first or backing-pump and, operating against atmospheric pressure, reduces the pressure to a value sufficiently low for a second pump, P_2, to operate. This is usually a 'condensation' pump and when the working substance is mercury, the mercury vapour is prevented from reaching the system to be exhausted by inserting a mercury-vapour trap, T, between the pump and the system. This trap is shaped as shown and is kept at the temperature of liquid-air, so that the mercury is condensed and frozen out here. Finally V denotes the apparatus in which a high vacuum is required. When the degree of vacuum is as low as can be produced within the system it does not follow that the pressure is constant in all parts of the system on the intake-side of the pump P_2. This is because gas is continuously being evolved from the walls of V and elsewhere, and on account of the relatively large mean free path of the gas molecules at extremely low pressures, these molecules do not immediately find their way to the pumps. In other words, the connecting tubes between V and the pumps offer a resistance to the flow of gas; this resistance can be reduced considerably by using wide connecting tubes and making them as short as possible. The actual pressure attained in V is measured by some form of low-pressure

gauge, G, attached as close as is practicable to the vessel V. Each component of the above system will be described in detail in the sequel.

Experimental determination of the speed of a pump.—(*a*) ***The constant pressure (or mercury-pellet) method:*** In this method, originally due to GAEDE, the volume-rate at which gas must be supplied to a vessel to counterbalance the rate at which gas is withdrawn from it by the pump, is determined; under such conditions the pressure within the vessel is maintained constant.

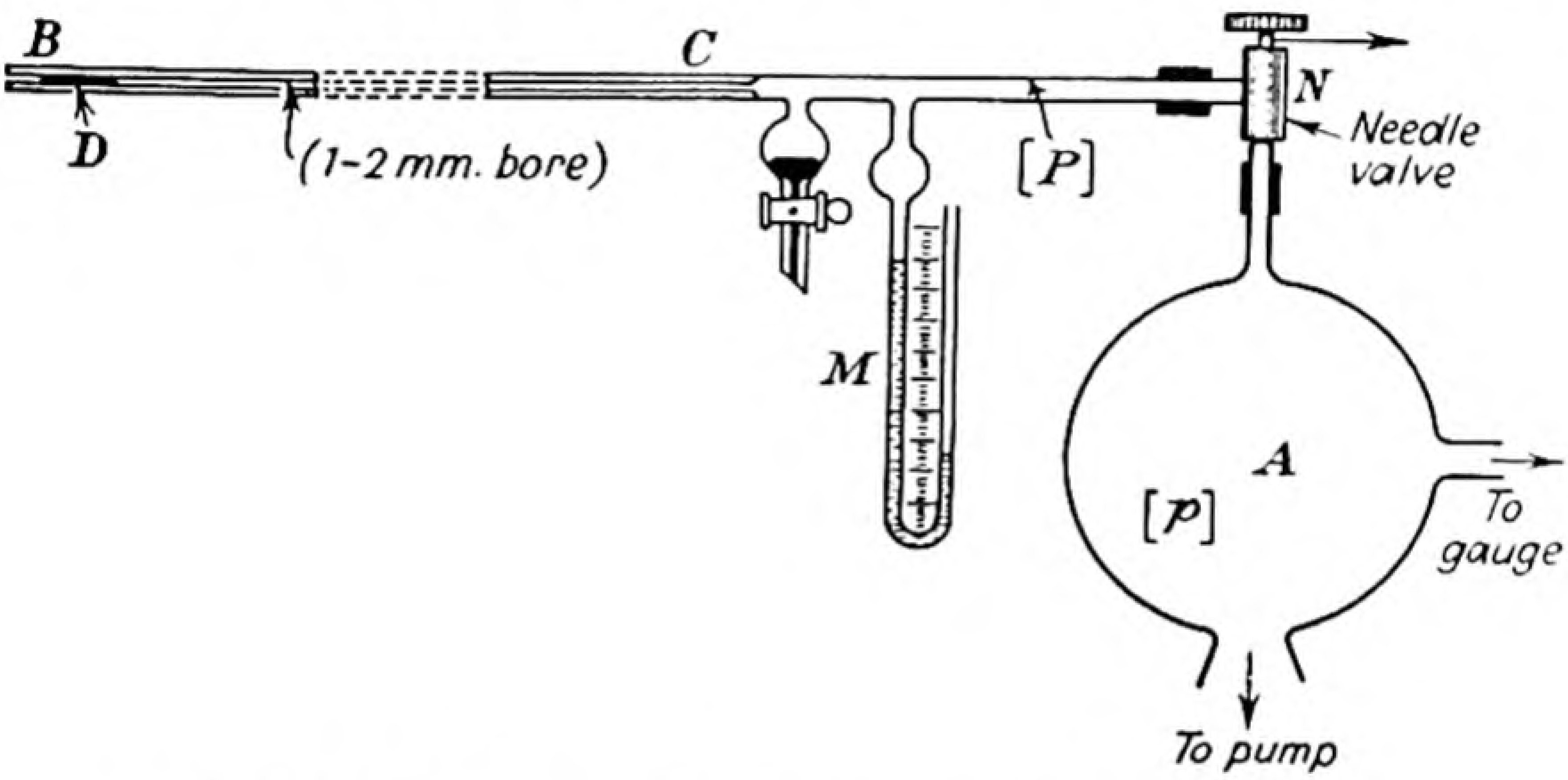

FIG. 14·02.—Mercury-pellet (*or* constant pressure) method of determining the speed of a pump.

As developed at the N.P.L. the constant-pressure method is as follows. A is a large glass bulb, about one litre capacity, mounted immediately above the pump whose speed is to be determined; it is necessary to make the connexion between A and the pump as short as possible, for the measured speed of the pump will be less than the true speed by an amount depending on the dimensions of the connecting 'link'. BC, Fig. 14·02, is a fine capillary tube connected to an oil manometer, M, and a needle valve, N. The oil used must have a low vapour pressure; tetrahydronaphthalene [$C_{10}H_{12}$, density 0·972 gm.cm.$^{-3}$ at 20° C.] is very suitable for this purpose. By suitably adjusting the valve N the admission of air to the pumping system is regulated so that the pressure in A can be adjusted to any desired value. This pressure is measured by a low-pressure gauge. The air is supplied to the control valve at a pressure slightly less than atmospheric. The difference is indicated by the oil-manometer. The rate of admission of air for each setting of the needle valve, N, is found by observing the time required for the 'advancing-end' of the mercury pellet, D, in BC to move between two fiducial marks on BC.

If ω is the volume between the fiducial marks on BC, t the time of transit of the mercury pellet, and P the pressure of the air in BC, then Ω the volume of air at pressure p, the pressure in A, which enters the pumping system in time t, is given by

$$p\Omega = \mathrm{P}\omega,$$

in so far as it is justifiable to apply Boyle's law. The speed of the pump is

$$\mathrm{S} = \frac{\Omega}{t} = \frac{\mathrm{P}\omega}{pt}.$$

This method is to be recommended when the determination of the speed of a modern high-speed pump is being made, for the speed of such a pump varies considerably with the pressure.

(*b*) ***The constant-volume method:*** It has already been shown that the speed of a pump is given by the equation

$$\mathrm{S} = \frac{\mathrm{V}}{t_2 - t_1} \ln \left(\frac{p_1}{p_2}\right).$$

The constant-volume method of determining a pumping speed is essentially a direct application of this equation. [The more correct equation, as given on p. 685, is unnecessary here, for p_u is usually vanishingly small under the conditions which prevail when observations are being made.]

The apparatus consists of a large vessel, of volume V, connected to the pump and the pressure of the gas in V is noted at various times subsequent to the opening of the vessel to the pump. If p_1 is the pressure at time $t = 0$, and p is the pressure at time t, then we have

$$\mathrm{S} = \frac{\mathrm{V}}{t} \ln \left(\frac{p_1}{p}\right) = 2{\cdot}303\mathrm{V}\left(\frac{1}{t}\right) \log \left(\frac{p_1}{p}\right),$$

or

$$t = \mathrm{C} - 2{\cdot}303 \frac{\mathrm{V}}{\mathrm{S}} \log p,$$

where $\mathrm{C} = 2{\cdot}303 \frac{\mathrm{V}}{\mathrm{S}} \log p_1$, a constant in any one experiment.

Thus a plot of t against $\log p$ is a straight line as long as S is independent of the pressure. From the slope of this line the speed, S, may be determined. It is at once clear that a McLeod gauge cannot be used successfully in this experiment on account of the time-lag caused by the resistance which the connecting tubes offer to the gas-flow.

Water-jet pumps.—A water-jet pump is a very old device for removing air from a vessel but it is still in daily use in nearly all laboratories. Originally it was made in glass and consequently

easily fractured; modern water-jet pumps, one of which is shown in Fig. 14·03(a), are usually made in metal. Water from a constant pressure supply enters the pump at A and a thick wire W, inserted across the stream, helps to keep the flow of water turbulent. The exit jet B is close to the 'choke-tube' C and a side tube D leads to the vessel to be exhausted. Small air bubbles are trapped by the water-jet as it passes from B to C and are carried away by the water stream. The pumping speed is about 20 cm.^{3}sec.$^{-1}$ and the lowest pressure attainable is about 1 cm. of mercury, i.e. a pressure slightly less than the pressure of saturated water vapour at room temperature.

Hickman's water circulator for washing photographic plates and prints.—This is a modified form of water-jet pump and it is designed to supply a rapid stream of water to a photographic

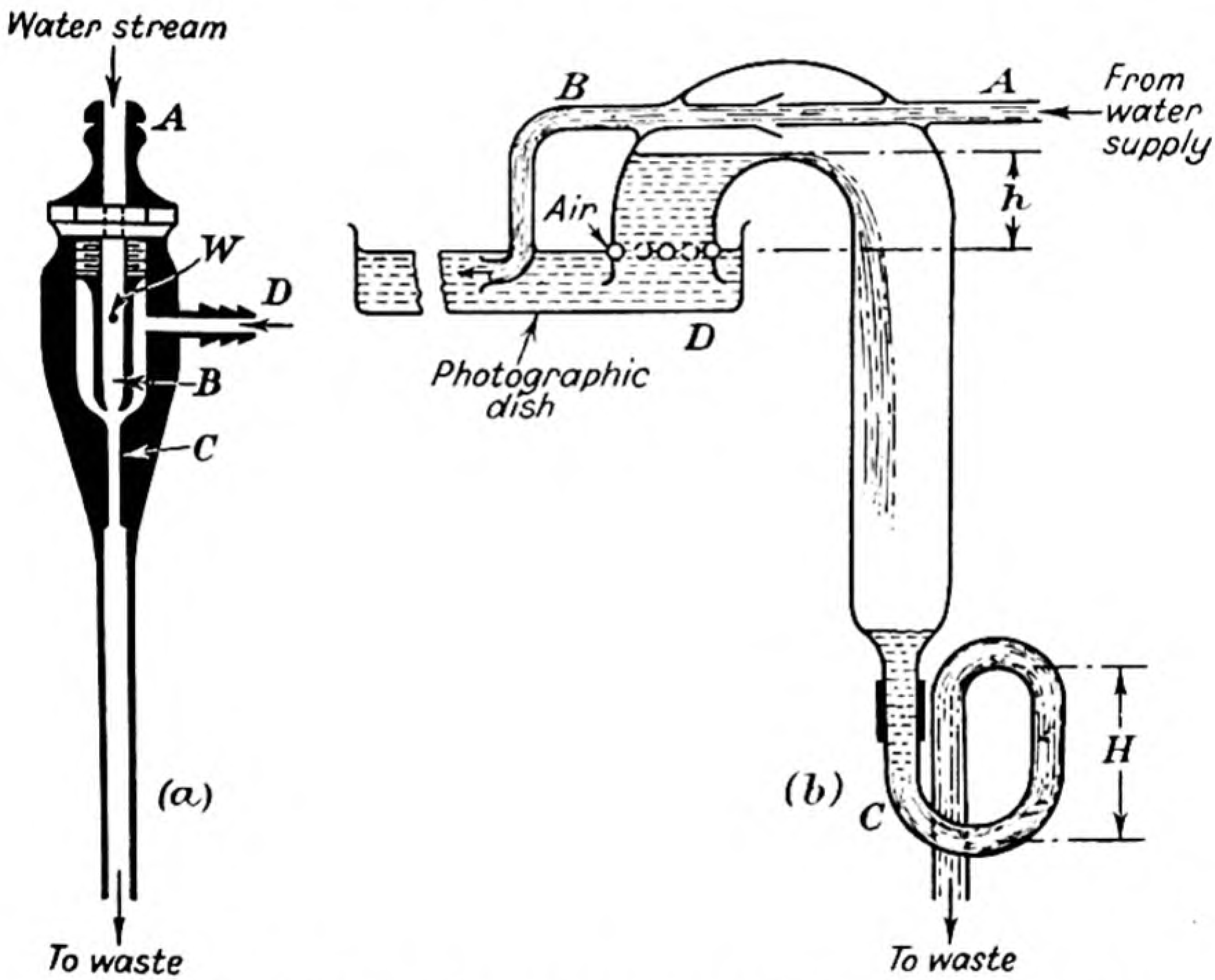

FIG. 14·03.—(a) A metal water-jet pump, (b) Hickman's rapid washer for use in photographic processes.

dish and then to remove the water at the same speed. By its means photographic plates and prints may be thoroughly washed in about five minutes. The device is made from glass or metal and its overall length is about six inches. It consists of an inverted U-tube with unequal limbs and into the bend of this tube there are fused two tubes A and B as indicated in Fig. 14·03(b). Water enters at A

and a mixture of air and water escapes through the tube B which is bent so that its exit end lies below the surface of water in a photographic dish, D. C is a piece of glass tubing, shaped as indicated, and attached to the longer limb of the inverted U-tube. This attachment must be filled with water before the pump will work. As air is removed from the U-tube its place is taken by water from the dish and, provided the height h is less than H, the action is continuous. Air vents, as indicated, permit air to enter the system provided that these vents are arranged at the general level of the water in the dish. The water from B circulates rapidly above the plate or print contained in the dish and just as rapidly it is removed so that no overflowing occurs. The device is most efficient.

The Sprengel pump.—A form of this pump which works in conjunction with a water pump is shown in Fig. 14·04. The capillary tubes in it are 0·15 cm. in diameter, the others about 0·5 cm. except where they widen out into bulbs approximately 2 cm. in diameter. The tube A leads to the vessel being exhausted. Pellets of mercury fall from the jet B and entrain bubbles of gas as they enter the fall tube below. The supply of mercury in B is replenished from the reservoir E which is in direct communication with a water pump. A capillary tube passes down the centre of this reservoir, through its base, and ends in the trough C. At the end of this tube there is a T-piece to which is attached a fine-drawn-out glass tube by means of a stout rubber tube. When the water pump is operating air is drawn in through this orifice and carries pellets of mercury with it. When this mixture arrives at the upper end of the tube the air passes to the water pump while the mercury falls into the reservoir. A clip, K, controls the rate at which air enters the apparatus.

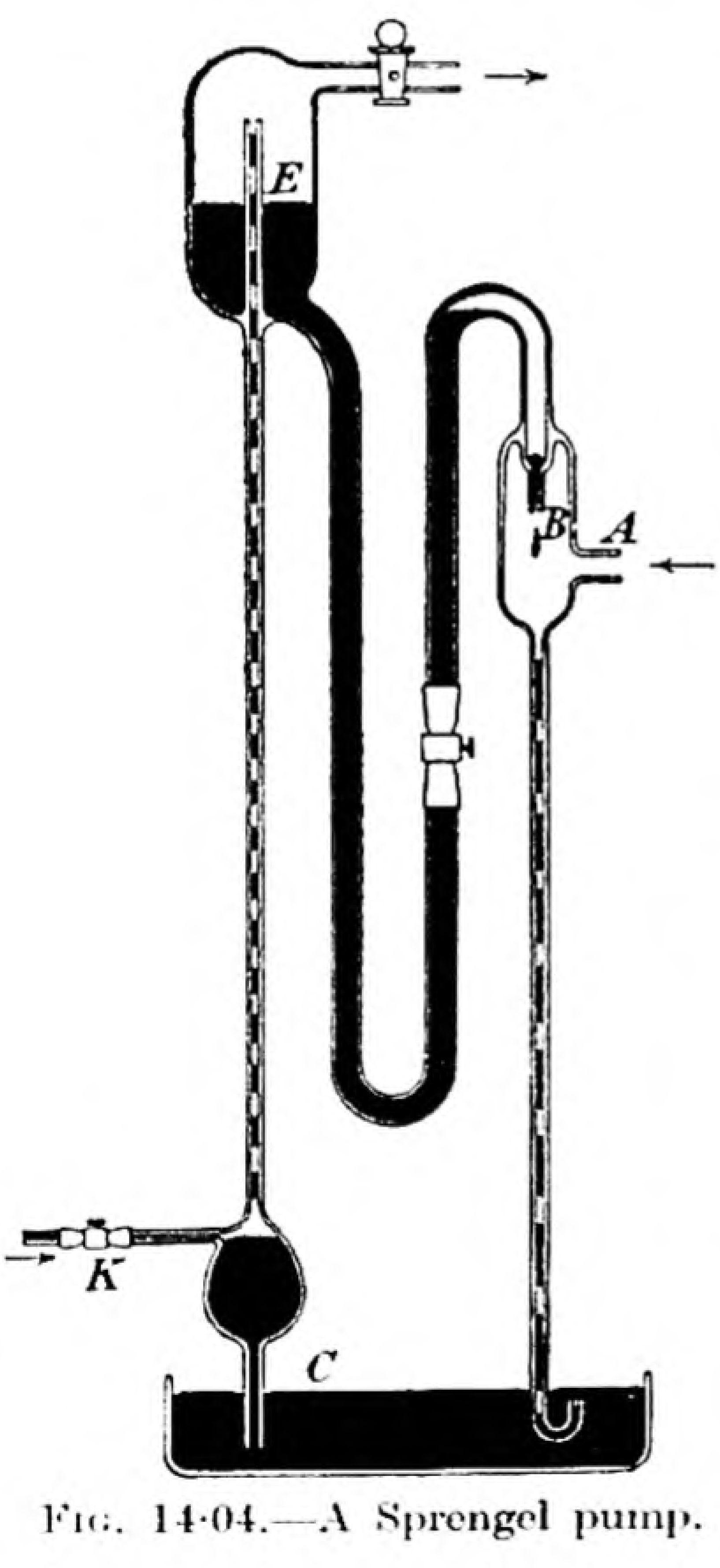

FIG. 14·04.—A Sprengel pump.

Geissler pumps.—If a lower pressure is required some other form of pump must be used; when the space to be exhausted is not

greater than 200 cm.^{3} a Geissler pump (often termed a Toepler pump) is very useful. One form of such a pump is shown in Fig. 14·05(a). It consists of a cylindrical barrel A, about 200 cm.^{3} capacity. At its upper end is a two-way capillary tap T; by turning this tap the barrel A can be put into communication, either with the tube B, which leads to the apparatus to be exhausted, or with C, which is open to the air. At the lower end of A is a smaller

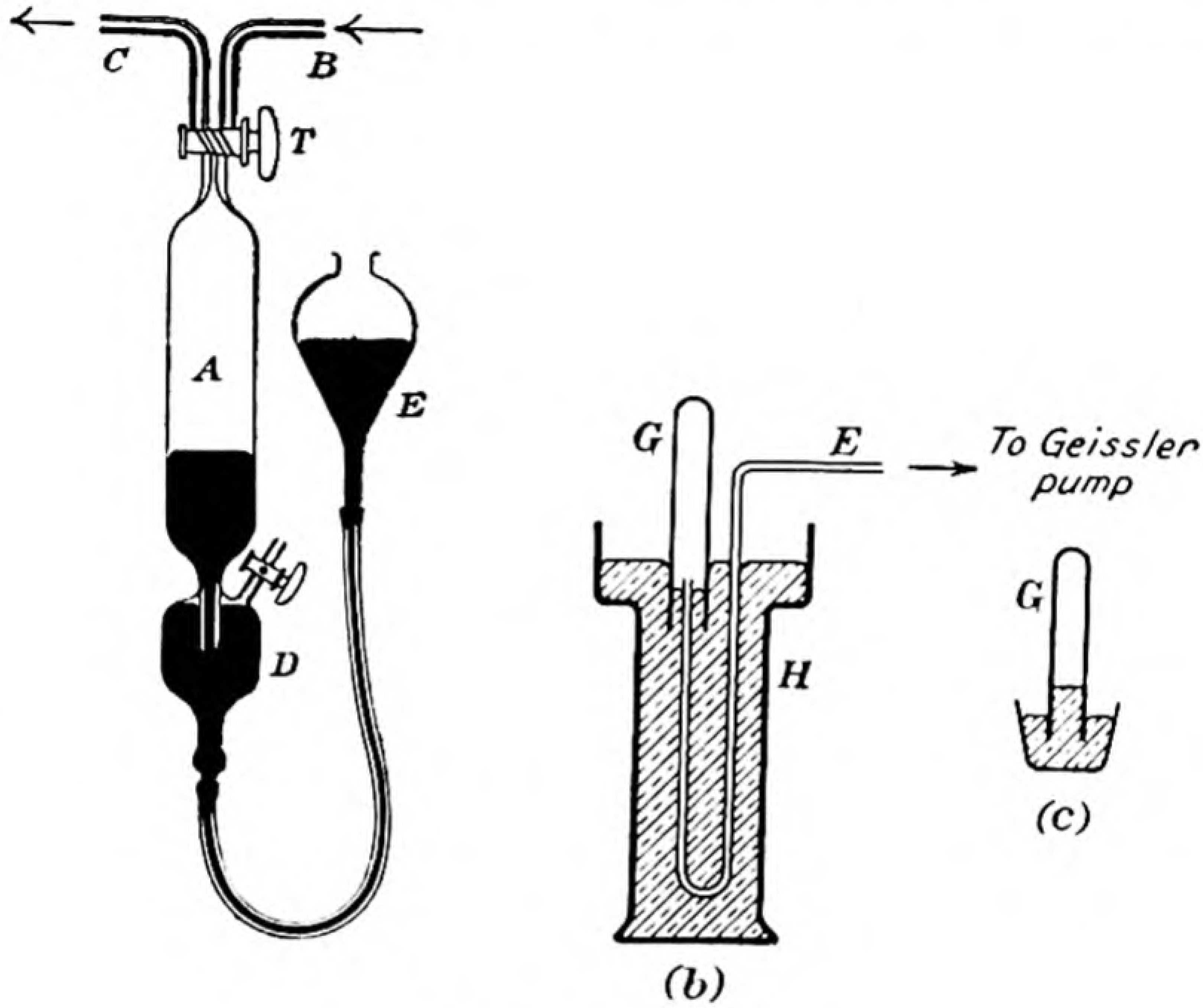

FIG. 14·05.—A Geissler pump.

barrel D, with a side tap attached. By means of pressure tubing D is connected to a mercury reservoir E. Any air entering the apparatus via the pressure tubing is entrapped in D and can be removed through the side tap.

To commence operations the reservoir E is raised, T being connected to C, so that the mercury fills the barrel A completely. T is closed; E is then lowered a little and T rotated so that B and A are in connexion. The pressure of the gas in B and the vessel to which it is attached forces the mercury downwards in A; E is lowered until A is nearly filled with the gas. T is then closed and E raised until the pressure in A is greater than atmospheric. When this is so, T is rotated until there is direct communication between A and C; gas may then be removed from A. The operation is repeated ten times or more, after which it will be found that no more

gas can be removed from the vessel which is being exhausted. When the mercury in A reaches the tap T, the sound of a good metallic click indicates that a low vacuum has been reached.

One important feature of such a pump is that with a small additional part it may be used to introduce a gas into a previously exhausted vessel and at a later stage this gas can be recovered. To carry out this operation in this way the tube C is lengthened or joined to another tube E, shaped as shown in Fig. 14·05(*b*). The portion of E resembling a U-tube dips into a thick-walled jar, H, containing mercury, the open end of the tube E being below the mercury surface. The gas to be introduced into the exhausted vessel is stored in an inverted test tube whose mouth is closed with mercury in a large crucible, cf. Fig. 14·05(*c*). This container is transferred to H and the large mouth of this vessel enables the crucible to be removed with ease. The gas container is forced down into H so that the open end of E is in the gas. The method of introducing the gas, via the pump, into a vessel to which it is attached is at once clear.

A rotary vacuum pump.—The pump shown in Fig. 14·06 is designed for the production of a high vacuum and the exhaustion of

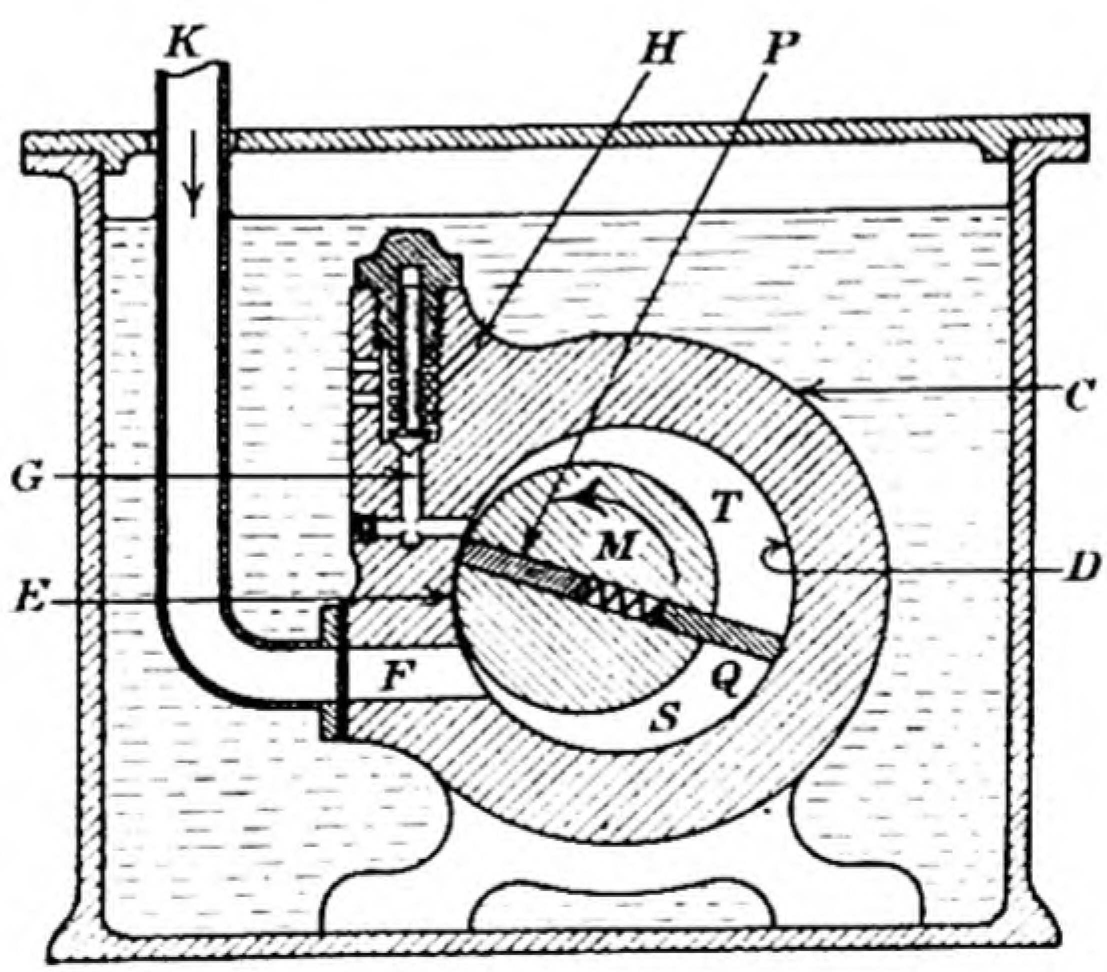

FIG. 14·06.—A rotary oil vacuum pump.

vessels of large capacity. It works directly from atmospheric pressure and, being entirely immersed in oil, the leakage of air into the high vacuum is prevented. The pump consists of an outer steel casing, C, through which is bored a cylindrical chamber, D. A shaft, M, runs through this chamber, its axis being parallel to but eccentric from the axis of the chamber. The shaft revolves about its own axis and always touches the periphery of the chamber D at the point E.

On each side of this point is a port—one an inlet, F, and the other an outlet, G, which is fitted with a spring-loaded valve, H. In the shaft M is a slot in which two plates, P and Q, are free to slide to and from the axis of the shaft. These two plates are kept apart and their extreme edges forced against the periphery of the chamber D by a series of springs placed at right angles to the axis of the shaft —one of these is shown in sectional view.

The action of the pump is as follows. Let us consider the position shown in the diagram. The shaft M is rotating in an anticlockwise direction and the effective space between the chamber D and the shaft M is divided into two portions, S and T. As the shaft rotates, remembering that the plate Q is touching the wall of the chamber, the portion S enlarges and air is drawn in from the vessel to be exhausted through the inlet pipe K. Meanwhile the portion T gets smaller and any air in it is compressed. When the pressure is sufficiently great this air escapes through the exhaust valve. Thus the pump will exhaust air from a vessel to which the inlet pipe K is connected.

Mercury-vapour pumps.—In 1915 GAEDE devised a type of pump for producing high vacua and, with modifications, this high-speed pump has practically displaced all others. Pumps of this

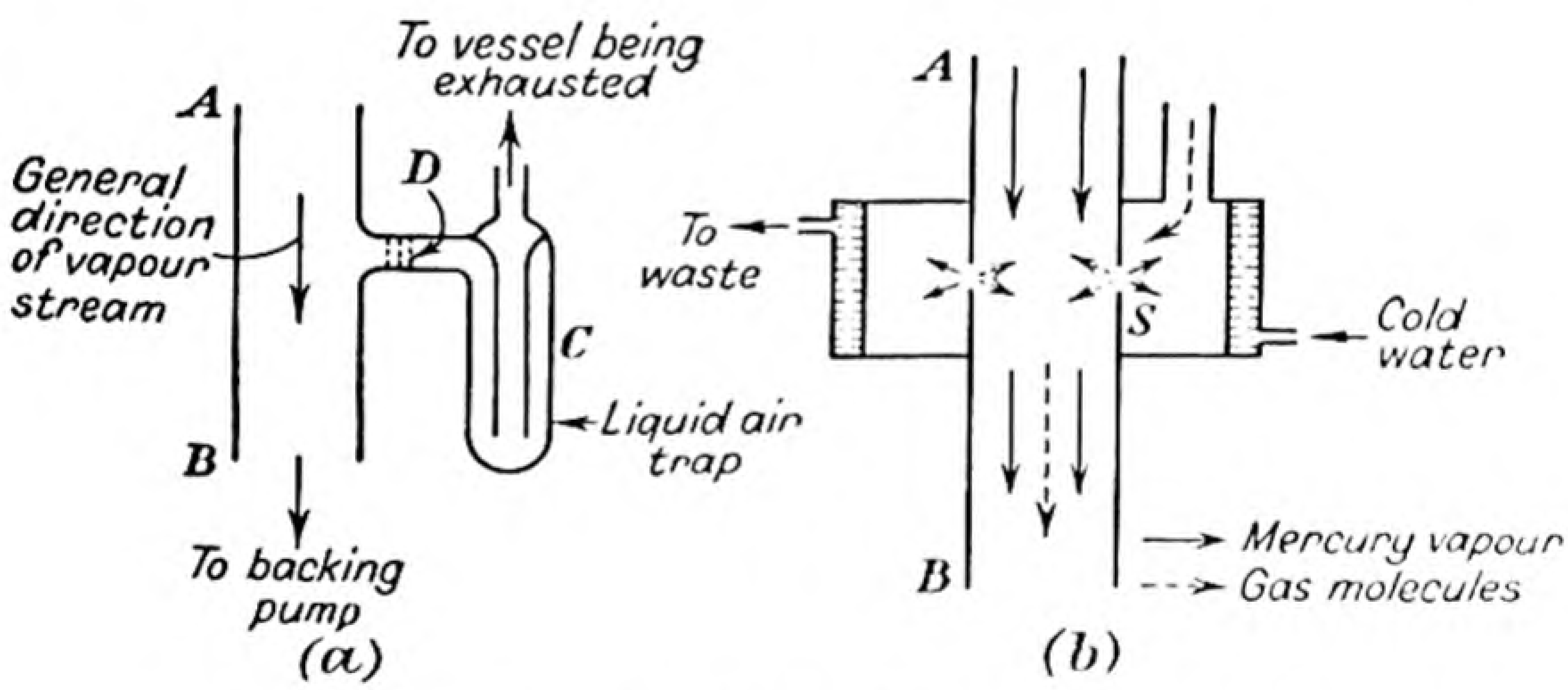

FIG. 14·07.—The principle of a diffusion (Gaede) pump.

type are usually constructed of pyrex glass, fused silica or steel, and in each instance a fore-pump is required to reduce the backing pressure to a value within the range 20 mm.–10^{-2} mm. of mercury.

Gaede's original type of vapour pump is known as a ***diffusion pump***. To understand its mode of action consider the system shown in Fig. 14·07(a). AB is a glass tube through which a stream of readily condensable vapour is passing. C is a 'vapour-trap' attached to AB, as indicated, with a porous diaphragm, D, dividing

the space in AB from that in C. If C is connected to the apparatus to be exhausted, already at a low pressure, then vapour and air molecules diffuse through the diaphragm D. The vapour is condensed in C while the air molecules which have passed through D into AB are rapidly drawn away by the vapour stream. In consequence of this the gas pressure in the vessel attached to C is reduced and eventually reaches a very low value.

In an actual diffusion pump it is not necessary to have present the diaphragm or diffusion disc. For suppose that the tube, AB, Fig. 14·07(*b*), has a very narrow cylindrical slit, S. Diffusion of vapour and air molecules will occur through this slit; the gas molecules which enter AB through the slit will acquire, in virtue of collisions between them and the heavy vapour molecules, momentum with a component directed downwards. The mixture thus passes into a region where the vapour condenses and the air is carried away by the backing-pump. At the same time some vapour will pass by diffusion into the high vacuum system. By carefully controlling the density of the vapour—Gaede used mercury vapour—in the main stream and by using a very narrow slit (0·1 mm. wide) an endeavour was made to reduce this 'back diffusion' of the mercury vapour to a minimum. This was essential for on account of back diffusion the speed of the pump is reduced and, furthermore, it imposes a limit on the ultimate pressure attainable with a given pump. The effects of back diffusion were also made less objectionable by using a water-cooling jacket as indicated; for still lower pressures, i.e. pressures less than the vapour pressure of mercury at room temperature, Gaede found it necessary to interpose a liquid-air trap between the intake of the pump and the vessel to be exhausted.

Gaede showed theoretically and then confirmed his conclusions experimentally that for a pump of this nature to attain its maximum speed the width of the slit S must be of the same order of magnitude as the mean free path of the molecules in the back-streaming vapour. It is frequently stated that Gaede's condition for the successful operation of a diffusion pump is that the mean free path of the vapour (mercury) molecules shall be of the same order as the width of the orifice. This misconception may be due to the fact that in his 1915 paper Gaede did not make it clear that the important factor is the mean free path of the gas molecules in the back-streaming vapour; he clarified the matter in 1923.

The great advantage which a diffusion pump possesses over all types of mechanical pump is that, theoretically, there is no limit to the degree of vacuum that can be attained. As long as there is a difference between the partial pressures of the air on the two sides of the slit, diffusion will take place until the pressure on the intake

side of the pump is zero. In actual practice some molecules are driven back through the slit so that the ideal is never reached. Among its disadvantages, two must be mentioned; the speed of pumping is relatively slow and the maximum speed can only be attained under carefully controlled conditions of the density of the mercury vapour as it passes by the slit S.

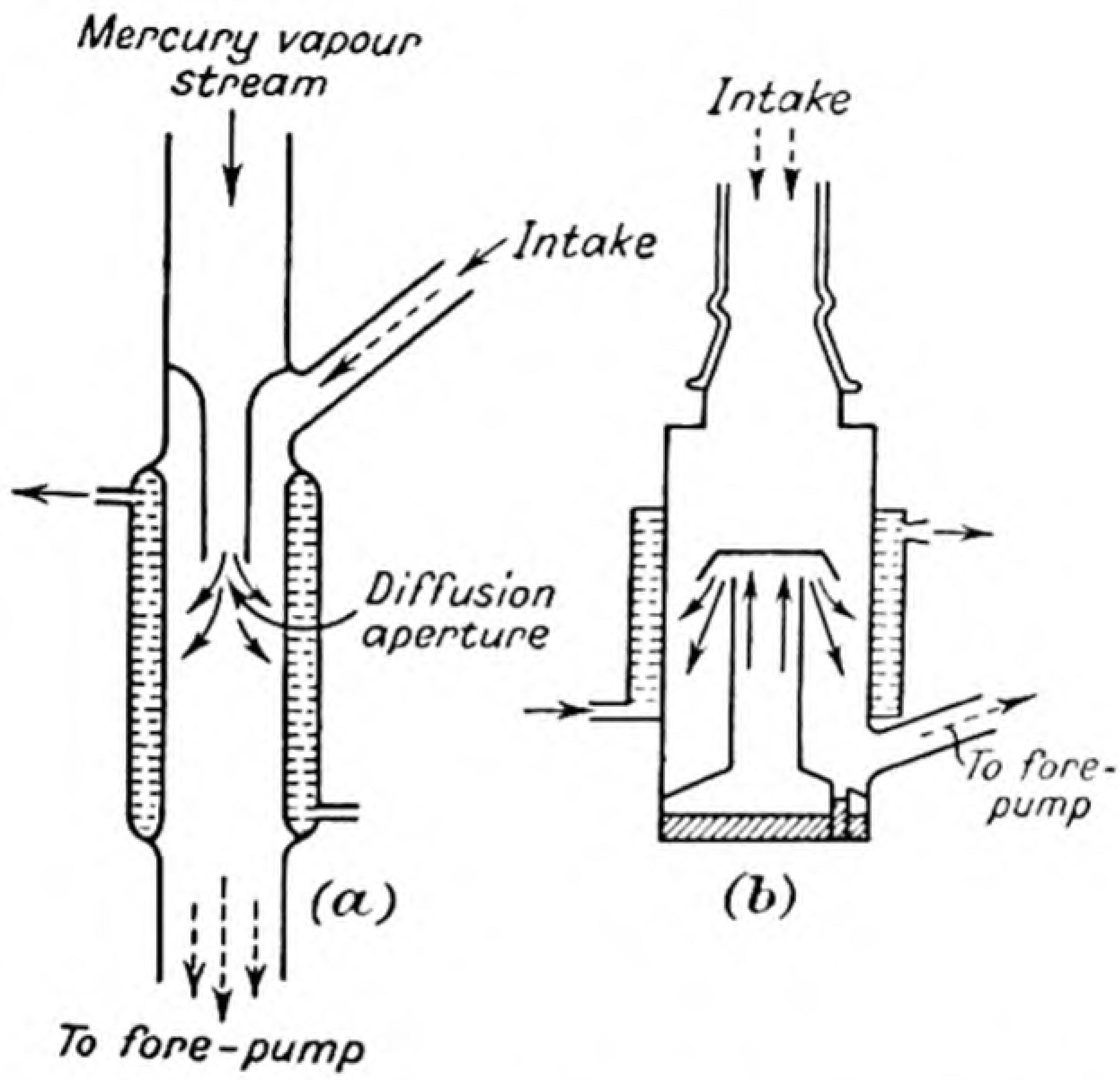

FIG. 14·08.—The principle of Langmuir's condensation pump. (a) glass type, (b) metal type.

In 1916 LANGMUIR, by introducing the salient features of an ejector pump into the design of a diffusion pump, succeeded in eliminating the two objections just indicated. Langmuir improved the vapour pump in three distinct ways:—

(a) By raising the speed of the vapour stream and making the speed of the pump almost independent of the actual pressure of the mercury vapour, i.e. it is no longer necessary to adjust within narrow limits the temperature of the mercury in the boiler where the stream of vapour is generated.

(b) The mercury vapour stream is directed so as to procure a more efficient entrainment of the air molecules.

(c) By doing away with the narrow slit used by Gaede the speed of the pump is increased many times.

The essential features of a Langmuir pump—called a 'condensation pump' to distinguish it from Gaede's diffusion pump—are shown in Fig. 14·08. Mercury vapour molecules issue from an

aperture and constitute a high-speed mercury vapour stream. Air molecules diffuse into this stream and, on account of molecular bombardments between the two types of molecules present, the air molecules are driven downwards and removed by the backing pump.

In an ideal condensation pump the mercury molecules would move in straight lines from the diffusion aperture or nozzle to the walls of the pump and exhibit no 'back-streaming'. This back-streaming is caused if mercury is present near the nozzle and at a sufficiently high temperature for it to form by evaporation a sufficiently large quantity of mercury vapour below the nozzle and with a random distribution of molecular velocities. To avoid this back-streaming Langmuir endeavoured to condense the vapour as soon as it reached the walls of the pump and so render less the disparity between an ideal and an actual pump.

To work with a rather high backing pressure (e.g. backed by a water-jet pump) the speed of the vapour stream must be increased and the diffusion aperture reduced in cross-section. For work with lower backing pressures the cross-section of the aperture may be increased and hence, if a high vacuum is required with a somewhat inefficient backing pump, then two or more condensation pumps must be used in series. Under such conditions each pump or 'stage' acts as a backing pump for the one which is next nearer the intake end of the pumping system. Either gases or vapours may be removed by means of these pumps.

Simple forms of mercury vapour pumps.—(*a*) ***The Waran pump:*** In 1923 Waran described two mercury vapour pumps which, in spite of their very simple construction, are most efficient in producing a high vacuum in vessels which are not too large in size. The pumps should be made in pyrex glass and their structures are at once apparent from the details given in Fig. 14·09. The first model, Fig. 14·09(*a*), is provided with a convergent jet and a choke-tube and, when backed by a simple water-jet pump, produces a pressure of about 0·1 mm. of mercury. When the pump is provided with a parallel jet about 8 mm. in diameter, cf. Fig. 14·09(*b*), and an efficient backing-pump is used, very high vacua can be obtained quickly.

In the original models the water cooling-jacket was fitted in position with the aid of rubber bungs but if the jacket is fused in position, although it may be a little more difficult for an amateur to construct, it is much less troublesome in action as all water leaks are impossible. The author also finds it convenient to put a 'U' in the tube along which the mercury returns to the boiler; this is done so that there is a certain amount of 'spring' in the glass framework when the final joint is made at A.

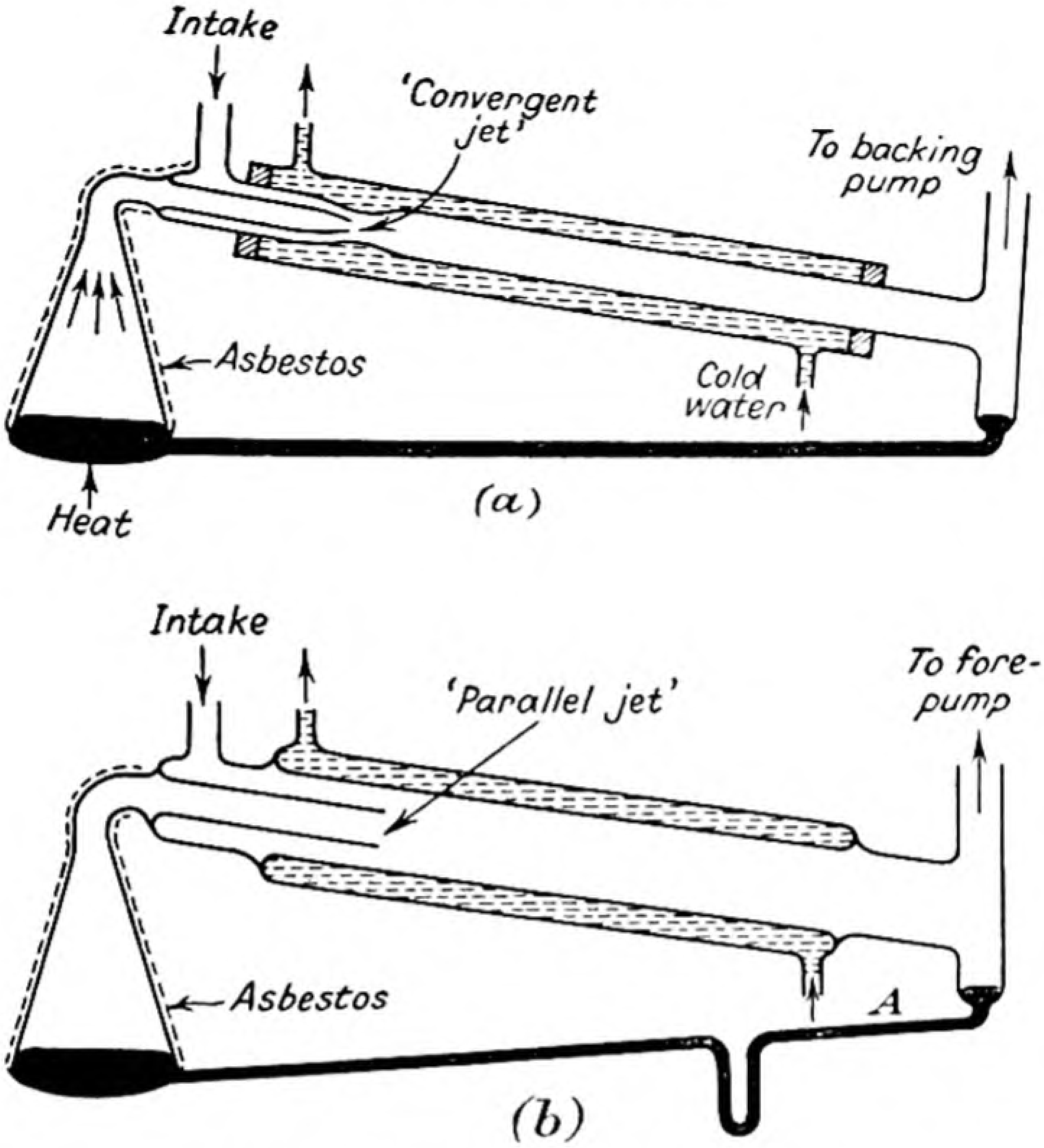

FIG. 14·09.—Waran's mercury vapour pumps.

(*b*) ***The Kaye-Backhurst annular steel-jet pump:*** This pump was developed at the National Physical Laboratory and consists essentially of steel. The structure is very simple and its main features are shown in Fig. 14·10. It is a single-stage pump and its high speed is due to its generous dimensions.

The action of the pump is as follows: mercury vapour from the boiler passes up the central tube, A, and then through holes at the top into the annular space between two deflectors D_1 and D_2. From this it issues as a jet. Gas molecules which diffuse through the annular space between the outer casing and the upper deflector are entrained by the mercury jet. The mixture is cooled, the mercury condenses and the air is carried away by the backing pump. The condensed mercury returns to the boiler.

The oil diffusion pump.—In 1928 BURCH described the preparation of several high-boiling-point petroleum distillates which could replace the mercury generally used till then in a vapour pump. Compared with mercury these oils have low vapour pressures so that in many types of vacuum operation the use of a freezing trap

may be dispensed with, and this allows the maximum speed of the pump to be utilized. Now although the pressure of the saturated vapour of any one of these distillates is low yet if the operation is a prolonged one minute drops of oil do appear in the high vacuum system; these are fatal to metallic films, activated filaments and the cathodes of photoelectric cells. In such circumstances a freezing trap, with its consequent reduction of pump speed, must be employed.

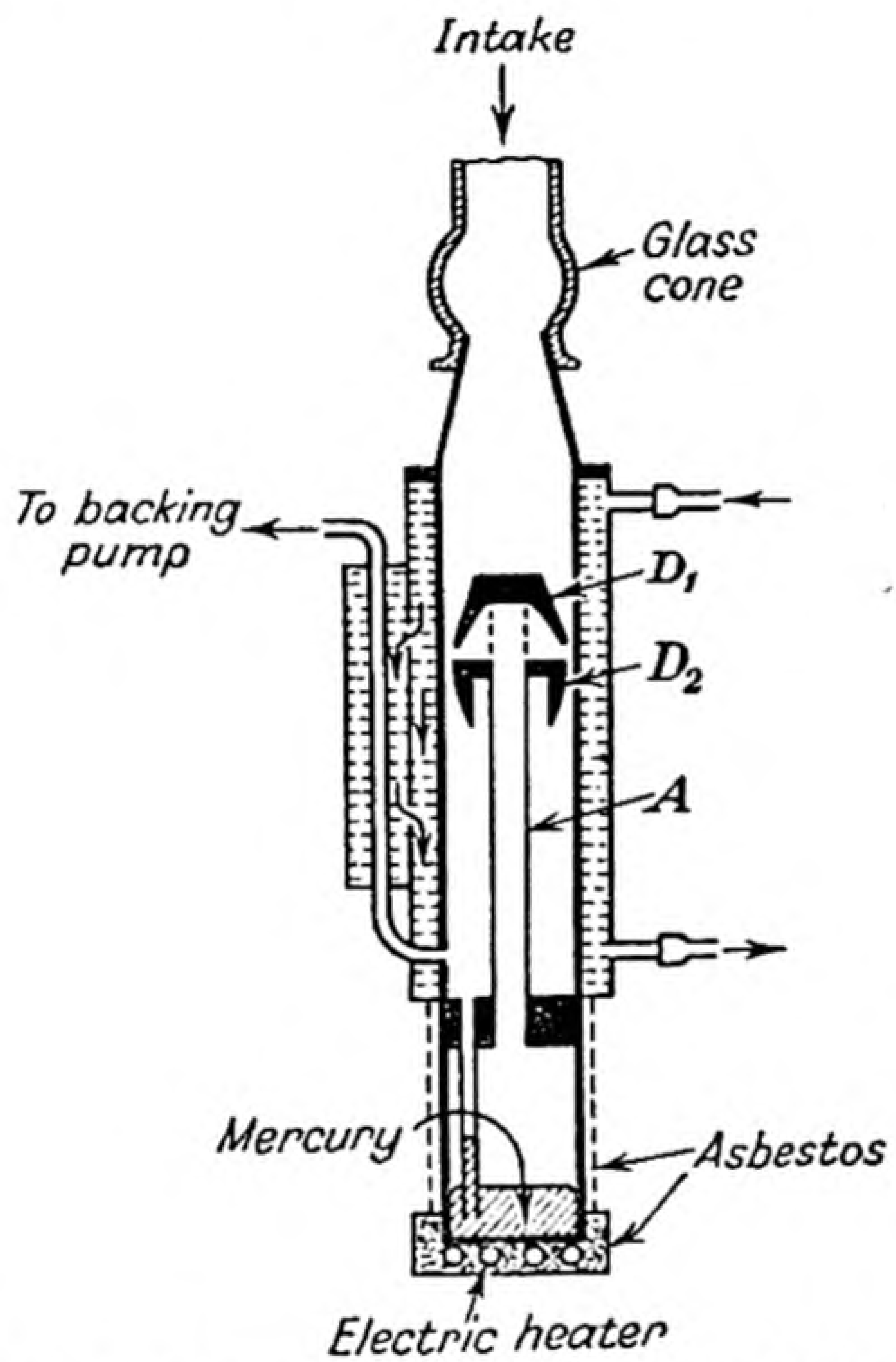

FIG. 14·10.—Principle of Kaye's annular-jet vapour pump.

An important advantage of a pump operating with such an oil is that, size for size, much greater speeds are obtainable than with mercury vapour pumps. This is because the molecular mass of the oil exceeds that of mercury; the molecular volume is many times greater. Unfortunately, however, whereas mercury does not dissolve gases and vapours, the organic oils behave in the opposite manner and must therefore be subjected to frequent cleansing.

The pioneer work of Burch in this connexion was soon followed by that of HICKMAN, who used synthetically prepared pure organic compounds such as n-dibutyl phthalate [$C_6H_4(COOC_4H_9)_2$, M = 278], n-dibutyl sebacate [$C_8H_{16}(COOC_4H_9)_2$, M = 314] and i-diamyl sebacate [$C_8H_{16}(COOC_5H_{11})$, M = 343]. The first of these compounds has a vapour pressure one hundred times smaller than that

of mercury at room temperature and this divergence increases as the temperature falls. A simple form of glass pump, designed by Hickman, for use with any one of these organic compounds is shown in Fig. 14·11(a). With tubes 1·5 cm. in diameter or less, efficient cooling is obtained by using a stout copper wire twisted into spirals at frequent intervals and wrapped, in close formation, round the

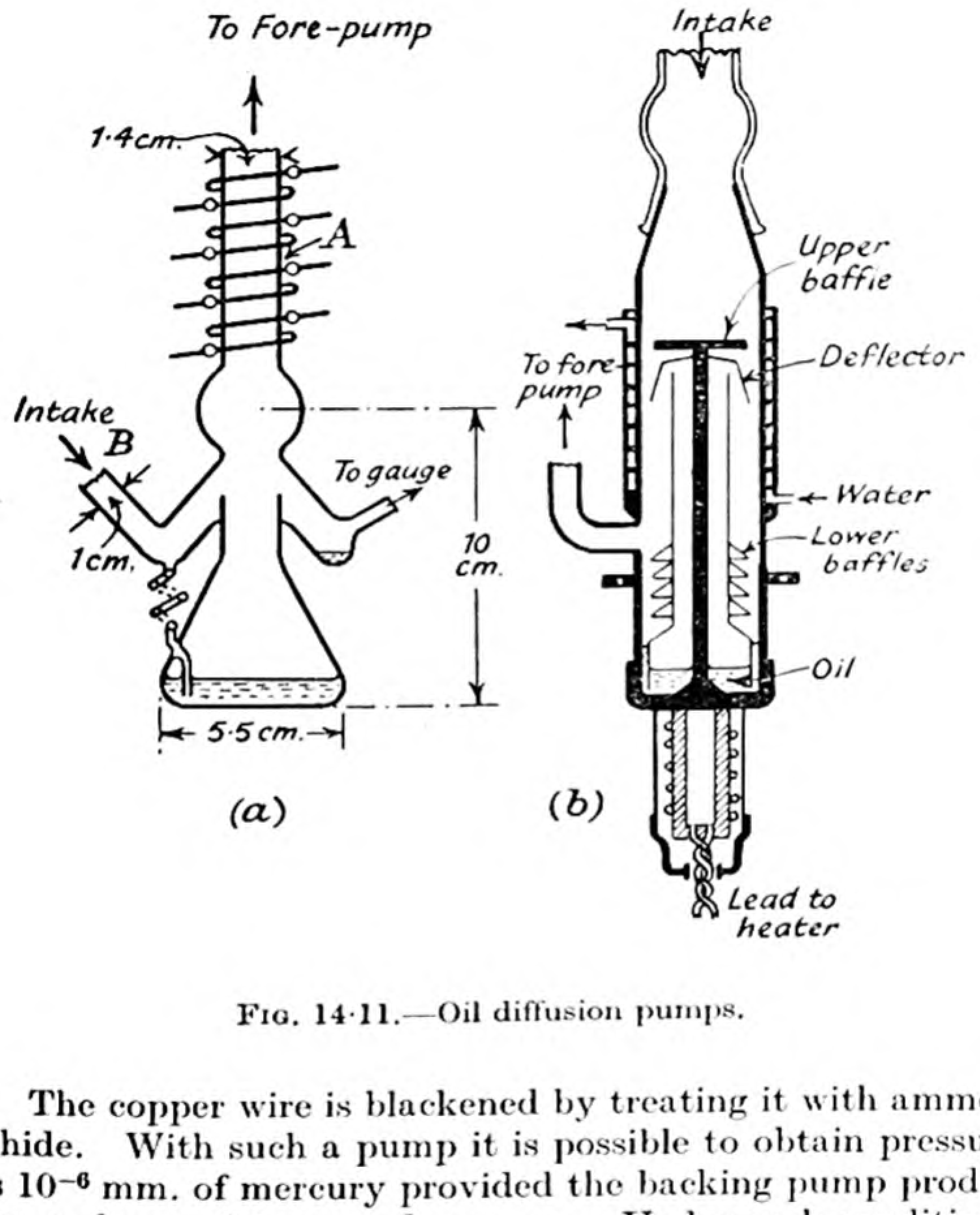

FIG. 14·11.—Oil diffusion pumps.

stem. The copper wire is blackened by treating it with ammonium disulphide. With such a pump it is possible to obtain pressures as low as 10^{-6} mm. of mercury provided the backing pump produces a vacuum as low as 0·1 mm. of mercury. Under such conditions the speed is about 10^4 cm.^{3}sec.$^{-1}$.

An all-metal pump employing oil as the working substance is shown in Fig. 14·11(b). It was designed at the Metropolitan-Vickers Electrical Company and uses an oil known as 'Apiezon B'. The outer casing is made of steel but the jet system is made of copper on account of its superior thermal conductivity [cf. below]. The oil is heated by means of an electric heater, for the heat supply must be capable of being controlled within narrow limits, since

a good pumping speed is only maintained if the heat supply is correct; the temperature is more critical than with mercury diffusion pumps. It is also found that the temperature of the oil must never be such that its vapour pressure exceeds 1 mm. of mercury, for at higher temperatures rapid oxidation sets in and as a result of this an oil with a large vapour pressure and a high viscosity is formed.

It will be noticed that in this pump there is one deflector plate and two sets of baffles; the upper one is to prevent vapour from entering the high vacuum system and the other to retard the entry of vapour into the fore pump.

In general, diffusion pumps designed for use with mercury are not suitable for use with organic liquids. The chief reasons for this are as follows:—(*a*) in the mercury pump the gap between the deflector plate and the outer casing may be so small that it is easily blocked by a film of oil, (*b*) there is a danger that if oil is used in such a pump the oil will be overheated in an attempt to obtain an adequate supply of vapour, the supply being restricted by too narrow a tube, and (*c*) excessive vapour condensation may occur near the deflector, since its temperature is not high enough to keep the oil as a vapour.

De-gassing and 'gettering'.—The removal of air (or other gases) from a leak-tight vessel is accomplished by one or other of the pumps already described. With the aid of a high-speed pump pressures as low as 10^{-6} mm. of mercury are not unusual and, if the vessel is pumped continuously, this pressure is low enough for most researches or industrial operations involving vacuum technique. [Under these conditions the mean free path is about 6×10^3 cm.] If, however, the vessel must be sealed off from the pump as is done with electric lamps, X-ray tubes, radio-valves, cathode-ray tubes, etc., the pressure must never rise above 10^{-4} mm. of mercury, for at 10^{-3} mm. pressure the mean free path is about 6 cm. and this distance is comparable with the dimensions of the apparatus. Should the gas pressure reach such a value then when the filament is heated the emitted electrons will be capable, if they have sufficient energy, of removing some of the residual gas and this will upset the characteristics of the tube. In order that the pressure shall not reach an undesirable magnitude after the vessel has been isolated from the pump it is necessary to de-gas the walls of the vessel and the metal parts therein, i.e. the adsorbed gas on the glass and metal must be released and carried away by the pumps.

Several methods of dealing with the problem of adsorbed gas are available and among them we have the following:—

(*a*) the whole vessel is baked in an electric or gas oven at a temperature some 50 deg. C. below the lowest softening point of any

material present in the vessel—generally this is the glass of the envelope;

(*b*) all the metal parts are made red hot by the method of eddy-current heating at radio-frequencies;

(*c*) in this method a 'getter' is used. While the pumps are in operation the vessel is 'well-baked', i.e. the glass and metal parts are de-gassed. Well-baked glass and de-gassed metals are, to a certain extent 'gas-hungry' until an equilibrium pressure in the space to which they are exposed is established. Materials which are exceptionally 'gas-hungry' are known as ***getters***. The electro-positive metals magnesium and barium, or an alloy made from them, are the most common materials used as getters. A small pellet of the material is introduced into the apparatus and by heating dispersed on the glass walls, a large area of which is covered with the getter. This operation should be performed before the vessel is finally sealed off, so that occluded gases may be removed.

The deposited layer is capable of absorbing a large amount of gas so that it may be regarded as a type of high speed pump which remains continuously in action after the vessel has been sealed off; moreover, it needs no further attention.

The use of 'getters' was first proposed by Soddy in 1907 and as a getter he used calcium. Water vapour and the gases CO, CO_2, NH_3 and SO_2 are all readily cleaned up with calcium.

THE MEASUREMENT OF LOW PRESSURES

U-tube manometers.—A simple mercury manometer may be used to measure pressures which are less than about 100 mm. of mercury and greater than 5 mm. When used in conjunction with a cathetometer pressures as low as 0·5 mm. of mercury may be measured with this type of manometer.

Such manometers possess several disadvantages:—

(*a*) The vacuum in the closed limb is gradually destroyed by gases which creep between the mercury and glass surfaces.

(*b*) If a leak suddenly develops in the apparatus the mercury is forced rapidly into the closed limb and the impact is sufficient to cause a fracture of the manometer.

(*c*) The instrument is not sensitive at low pressures.

(*d*) The mercury tends to stick to the glass so that it becomes difficult to observe the true pressure.

The first two disadvantages can be minimized by the use of a device due to WARAN. A small glass reservoir R, Fig. 14·12, is joined by means of capillary tubing to the usual form of manometer. The whole is filled with mercury as before. When the pressure upon the free surface of the mercury is diminished, at some stage

in the process the mercury recedes from the point A. If at this stage the instrument is tapped gently, the continuous thread of mercury in the capillary tube is broken and the mercury assumes the position shown in the diagram. The capillary tube space is then an almost perfect void, so that the height BC is a true representation of the pressure at C.

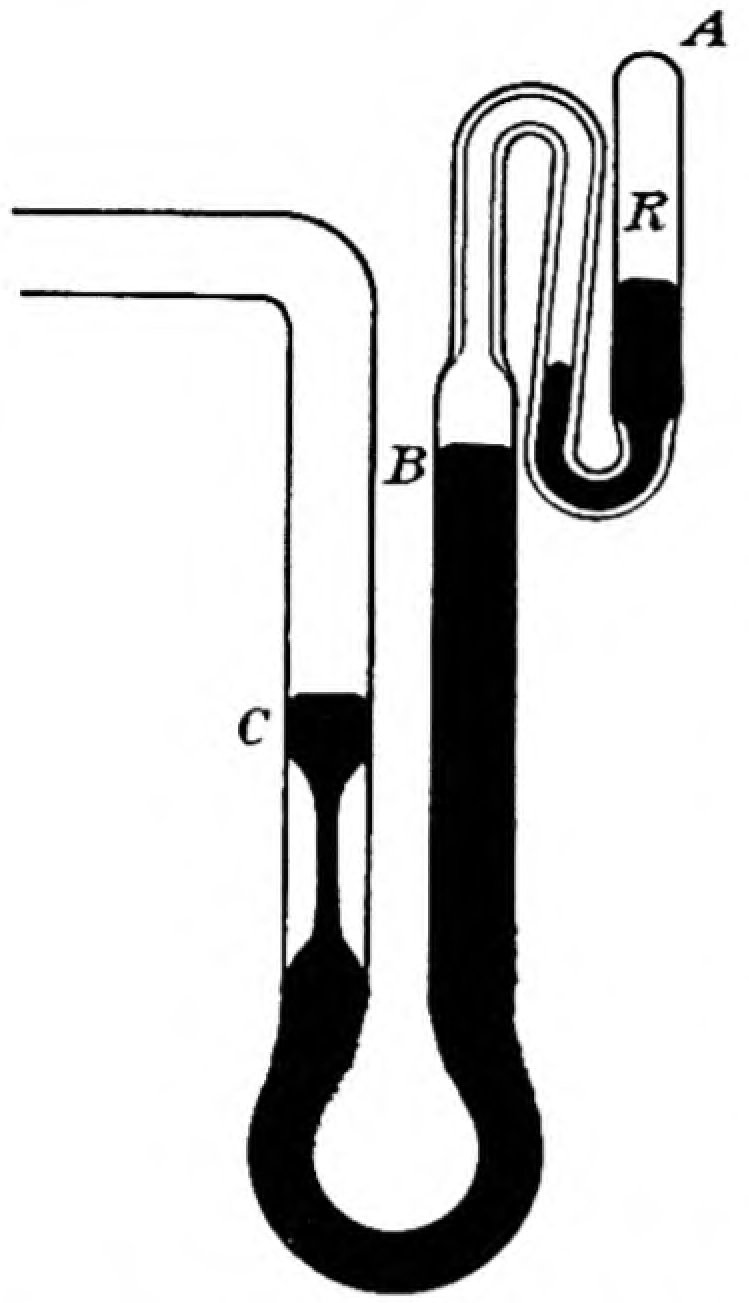

FIG. 14·12.—Manometer with regenerative vacuum device.

After some time gases may make their appearance in the capillary; they are removed by subjecting the manometer to atmospheric pressure, thereby forcing the gases into R.

By constricting the open limb of the U-tube as shown in the diagram, the motion of the mercury is retarded so that a fracture from the causes mentioned above becomes a very remote possibility.

Gauges for pressures below 5 mm. of mercury.—There exists a number of physical properties of a gas which vary with pressure and some of these, as listed in the table below, are useful for determining the pressure within a high vacuum system.

Physical property	Range of pressure (mm. of mercury)
(*a*) Electrical discharges	$10–10^{-2}$
(*b*) Compressibility, i.e. the volume variation with pressure	$10^{-1}–10^{-5}$
(*c*) Transfer of thermal energy	$10^{-1}–10^{-4}$
(*d*) Ionization effects	$10^{-2}–10^{-7}$

Approximate estimation of pressure by means of a discharge tube.—By means of a simple discharge tube, in direct communication with the system in the process of exhaustion, an approximate idea of the pressure within the vacuum system may be obtained. Fig. 14·13 indicates the appearance of the discharge in air for a series of pressures, but it must always be remembered that the actual appearance of the discharge is modified by the geometric form of the tube and of the electrodes as well as by the wave-form of the potential difference across the tube; the discharge is usually unidirectional but the wave-form of the applied voltage is seldom really steady.

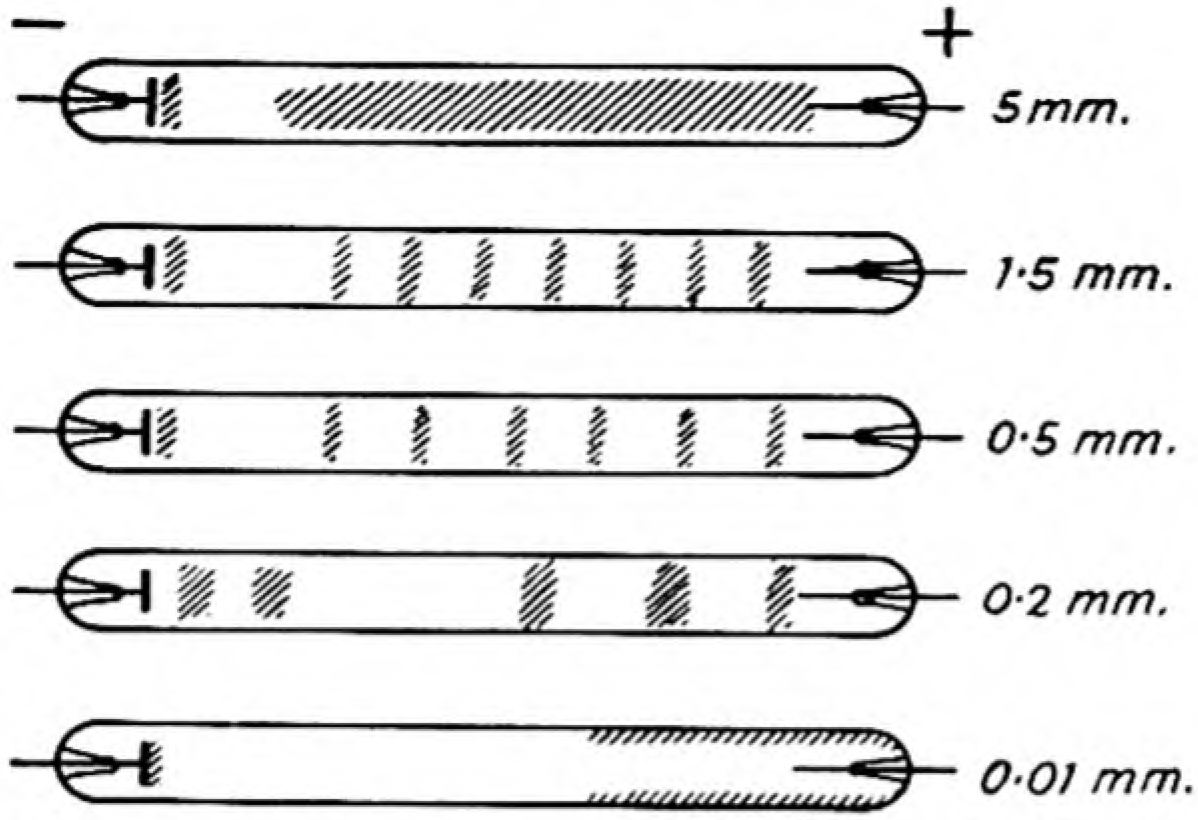

FIG. 14·13.—Approximate estimation of pressure by means of a discharge tube.

BURROWS (1943) designed a discharge tube suitable for use with metal systems. Its structure is shown in Fig. 14·14 and it is operated from a spark coil giving in air at atmospheric pressure a spark about 1 cm. long. According to Burrows the following approximate relations exist between the pressure and the nature of the discharge.

Nature of the glow discharge	Pressure (mm. of mercury)
0·5 cm. diameter column of discharge	5
First visible striations	1·5
Striations separated by 1 cm.	0·5
Fluorescence on the inside walls	10^{-2}
Black-out	10^{-3} or less

The colour of the discharge in the tube enables us to estimate the nature of the gas in the tube. When the colour is a deep pink air is present, while a greenish-grey colour indicates the presence of carbon dioxide, which frequently arises from the decomposition, in vacuo, of traces of oil. In the same way a faint blue colour indicates the presence of water vapour.

The McLeod gauge.—This was the first really low pressure gauge and it was designed in 1874 by McLEOD. Today it is used extensively in all high-vacuum work and by it other gauges may be calibrated. Its essential features are shown in Fig. 14·15 and the mode of action is as follows. V is a glass vessel and to its upper end a capillary tube is sealed—the whole volume is V. The volume per unit length of AB must also be known. CD is a side tube of sufficient diameter to ensure that the gas pressure in V is equal to that in the high-vacuum system to which CD leads. E is a mercury reservoir connected by rubber tubing to the main part of the gauge. When

E is in the position indicated and the pumps are in operation, the pressure of the gas in V is equal to that in the high-vacuum system. Let this pressure be p. When E is slowly raised the gas in V is isolated from the pumping system as soon as the upper level of the

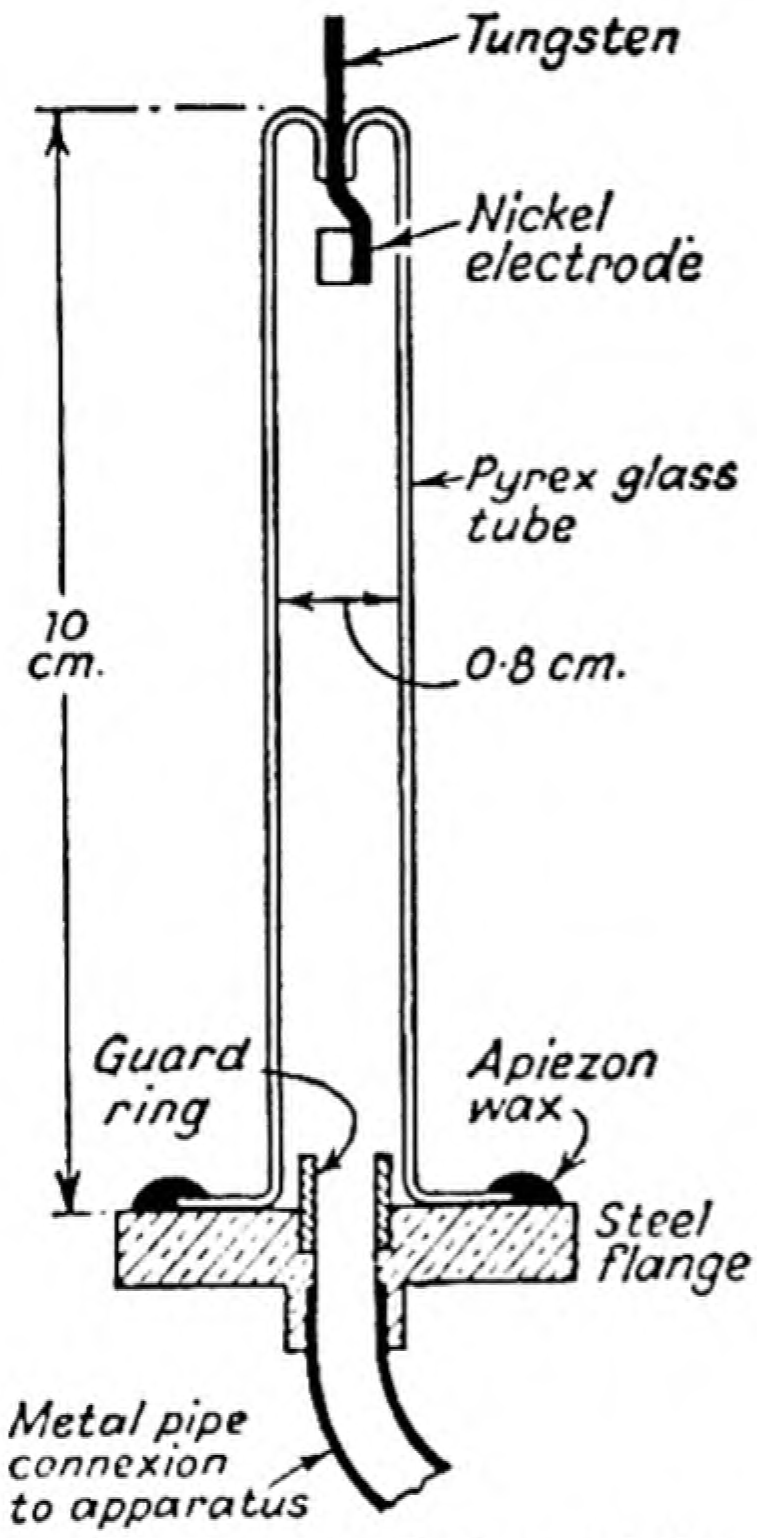

FIG. 14·14.—A discharge tube for use with a pumping system.

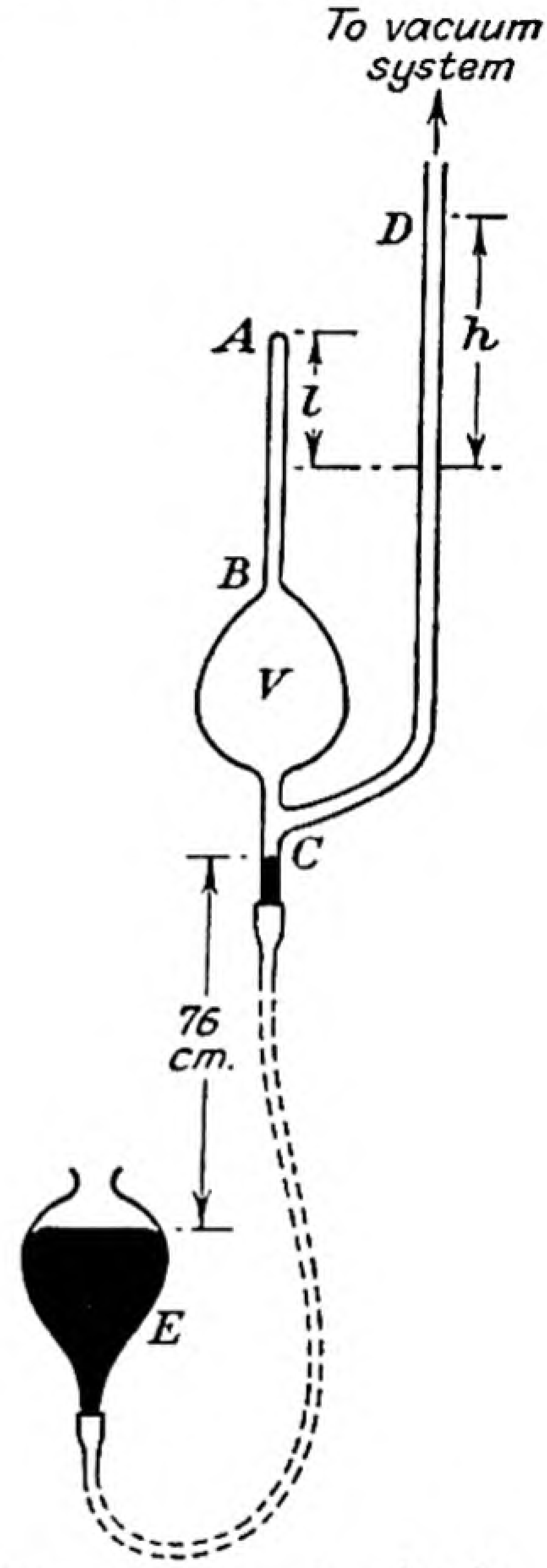

FIG. 14·15.—A simple form of McLeod gauge.

mercury passes beyond C; as E continues to be raised the trapped gas in V is compressed into the capillary tube AB.

Suppose that after raising E to its final steady position the portion of air in AB occupies a length l of that tube and that the free surface of the mercury in D is at a height h above the mercury meniscus in the capillary tube. Then the pressure of the gas in AB is h cm. of mercury [strictly speaking $(h + p)$ cm. but $p \to 0$]. If v is the volume per unit length (cm.) of AB, a direct application of Boyle's law gives

$$pV = (vl)h,$$

so that p may be determined.

Two methods of operating the gauge are in common use. In the first of these the mercury is always brought to a fixed mark B on the tube AB and the excess pressure needed to do this is indicated by the column of mercury h, Fig. 14·16(a). In this instance

$$p = \left(\frac{vl}{V}\right)h,$$

so that we have a uniform scale of pressures and such a scale can be placed alongside the tube D.

In the second method the mercury in the tube D is brought to such a position that its free surface is opposite the end, A, of the capillary tube as shown in Fig. 14·16(b). Let λ be the length of AB occupied by the compressed gas under these conditions. Then Boyle's law gives

$$(v\lambda)\lambda = pV, \qquad \text{or} \qquad p = \frac{v}{V}\lambda^2.$$

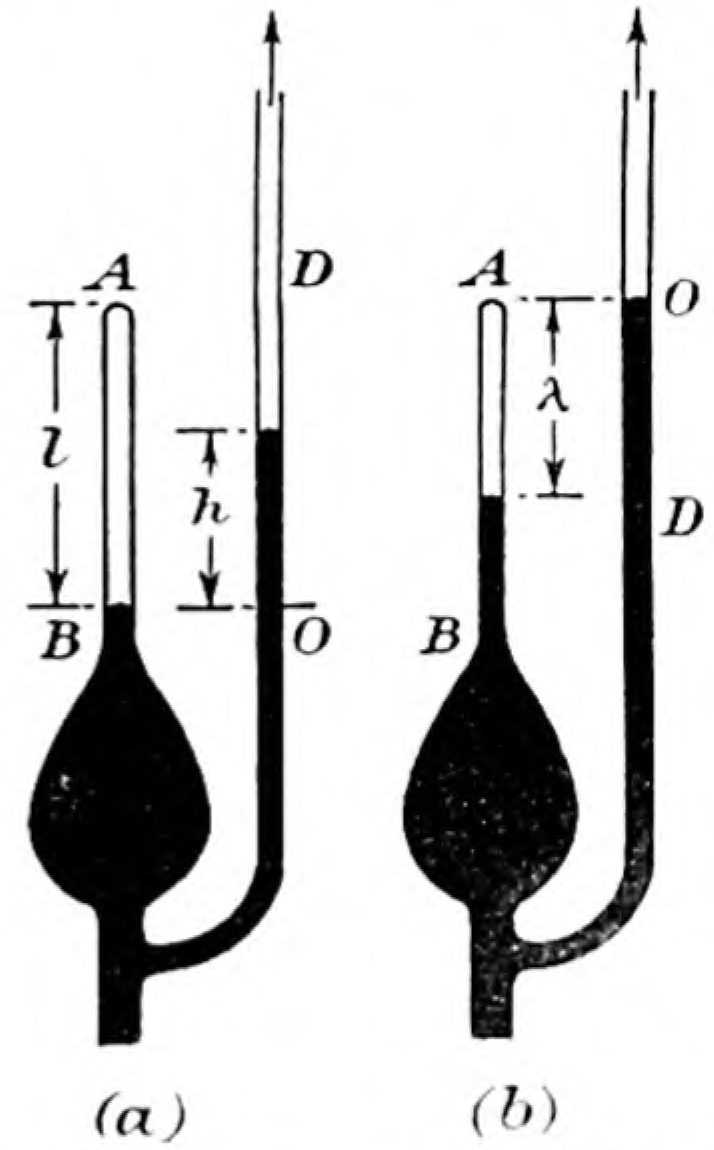

FIG. 14·16.—Theory of a McLeod gauge.

This method gives a non-uniform scale of pressure—such a scale is placed alongside AB—but the scale opens out progressively at the lower pressures; this is the main advantage of this method of using a McLeod gauge.

The McLeod gauge in practice.—One serious disadvantage of the simple form of McLeod gauge just described is that h is not a true measure of the pressure of the compressed gas. The error arises on account of the difference in the diameters of the tubes A and D and the consequential difference in capillary depression of the mercury in the two tubes. To overcome this difficulty a capillary tube, F, of the same diameter as AB, is sealed in parallel with D, as shown in Fig. 14·17, and the height λ is determined from observations on the positions of the mercury levels in the two capillary tubes.

An additional disadvantage of the original form of McLeod gauge is that it is necessary to manipulate a reservoir containing a large mass of mercury. The design shown in Fig. 14·17 overcomes this difficulty. The mercury is contained in a Woulff's bottle, W, and the gauge proper is fitted into one of its apertures. The second aperture is fitted with a rubber bung through which passes a glass tube leading to a two-way glass stop-cock, as shown. From this stop-cock a tube, K, leads to a wider glass tube packed with soda

lime and glass wool to prevent particles of this drying agent from entering the gauge. To this tube there is attached, by means of rubber tubing, a very fine capillary tube, J; a spring clip closes the

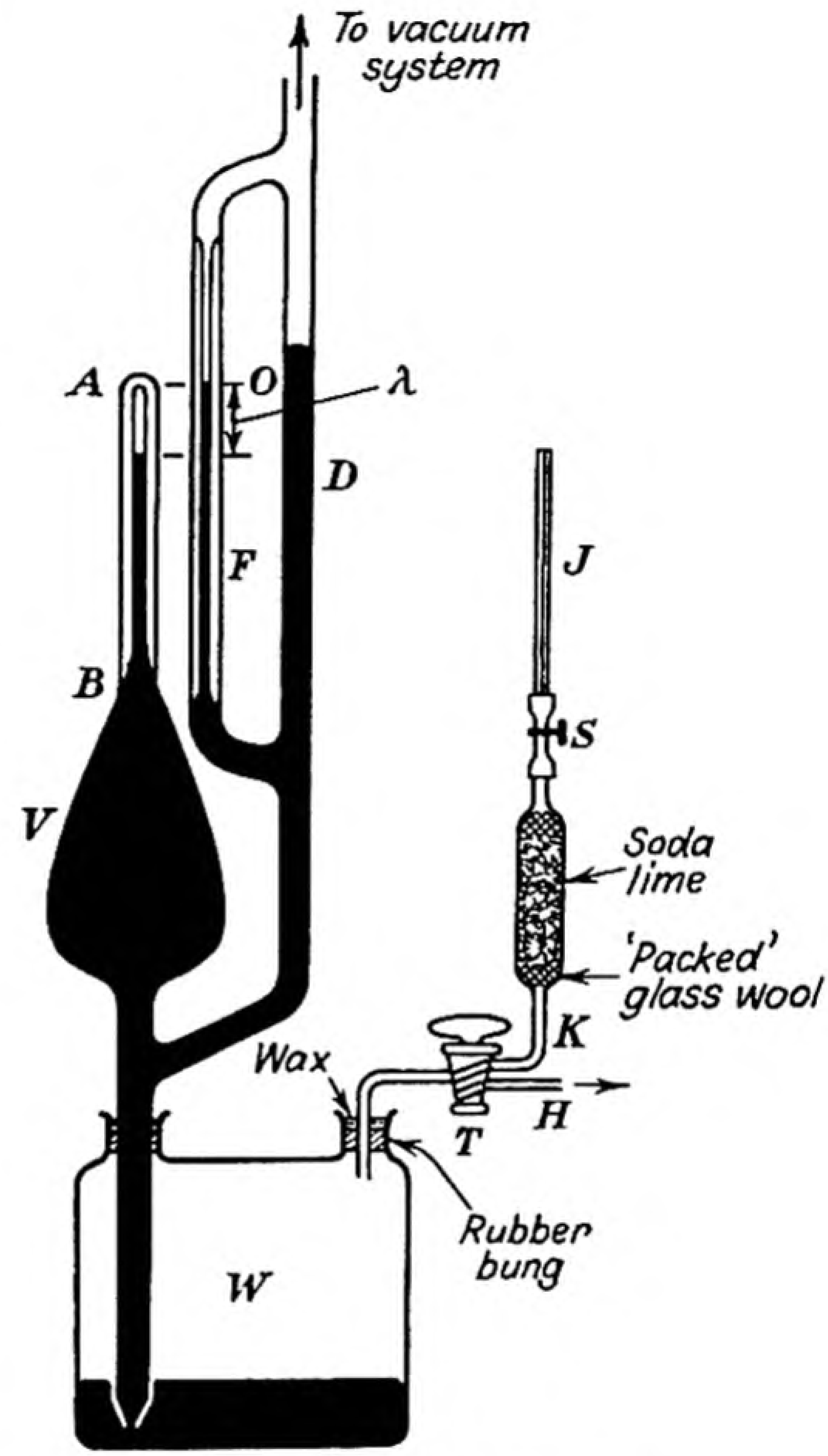

FIG. 14·17.—A McLeod gauge in practice.

rubber tube when necessary. The glass tube H is in connexion with a simple form of pump—with care in its use it can be connected to the backing pump of the main exhaust system.

When the stop-cock is opened with H in direct communication with W all the mercury in the gauge proper descends into the Woulff's bottle and the pressure in V is equal to that in the pumping system. By closing T the bottle is isolated from the backing-pump and by giving to the barrel of T a further partial rotation air is slowly admitted to the space above the mercury in W. The mercury is forced into the gauge and this is then used in one of the two ways already cited.

In addition to the actual mechanical labour saved in this way, the absence of a long length of rubber tubing is another feature of this form of McLeod gauge. Rubber tubing always permits air to diffuse slowly through its walls and this air may find its way into the gauge and, moreover, the use of rubber in contact with mercury is always objectionable since the mercury is easily contaminated by the sulphur in the rubber.

All McLeod gauges behave erratically if readily condensible vapours are present, but by interposing a liquid air trap between the gauge and the high-vacuum line all such vapours are rapidly removed.

Pirani (resistance) gauges.—It is a well-known fact, predicted by theory and amply confirmed by experiment, that at high pressures the thermal conductivity of a gas is independent of the pressure. At pressures below 0·1 mm. of mercury, when the mean free path of the gas molecules is of the same order of magnitude as the diameter of the container, the thermal conductivity of a gas is a linear function of the pressure, i.e.

$$\kappa = \alpha p,$$

where κ is the thermal conductivity and α is a constant. In 1907 WARBURG used this relation to measure low pressures. Manometers making use of this relation are not absolute, since the constant α, for a gauge based on this principle, must be determined with the aid of a McLeod gauge.

In all gauges of this type, which are really due to PIRANI, a heated metal filament is mounted in a small vessel attached to the pump line, and its mode of action depends on the variation with pressure of the rate at which heat is conducted across the gas filling the space between the heated filament and the walls of the containing vessel. The rate of heat transfer determines the temperature of the filament and hence its electrical resistance; a preliminary calibration of the gauge with the aid of a McLeod gauge enables the relation between the resistance of the wire and the pressure of the surrounding gas to be determined.

The hot filament of a Pirani gauge must have a high coefficient of increase of resistance with temperature and it is generally made of tungsten about 0·06 mm. in diameter. The actual structure is very similar to that of an ordinary cage-type incandescent filament lamp but the filament is never made red-hot. For the gauge to be accurate in its performance it is essential to keep taut the filament so that the distance between it and the walls is constant, and to keep as low as possible the heat losses along the filament supports. The bulb should preferably be kept in a thermostat—melting ice at once suggests itself—and the temperature of the filament should be raised to about 120° C.

Since every pressure gauge depending on the thermal conductivity of a gas must be calibrated by comparison with an absolute pressure gauge, in use it is only necessary to keep constant the potential difference across the filament and observe the current. The lower the gas pressure, the lower the thermal conductivity of the gas and thus the smaller is the current required to keep the excess temperature of the filament at a predetermined value. Since the calibration

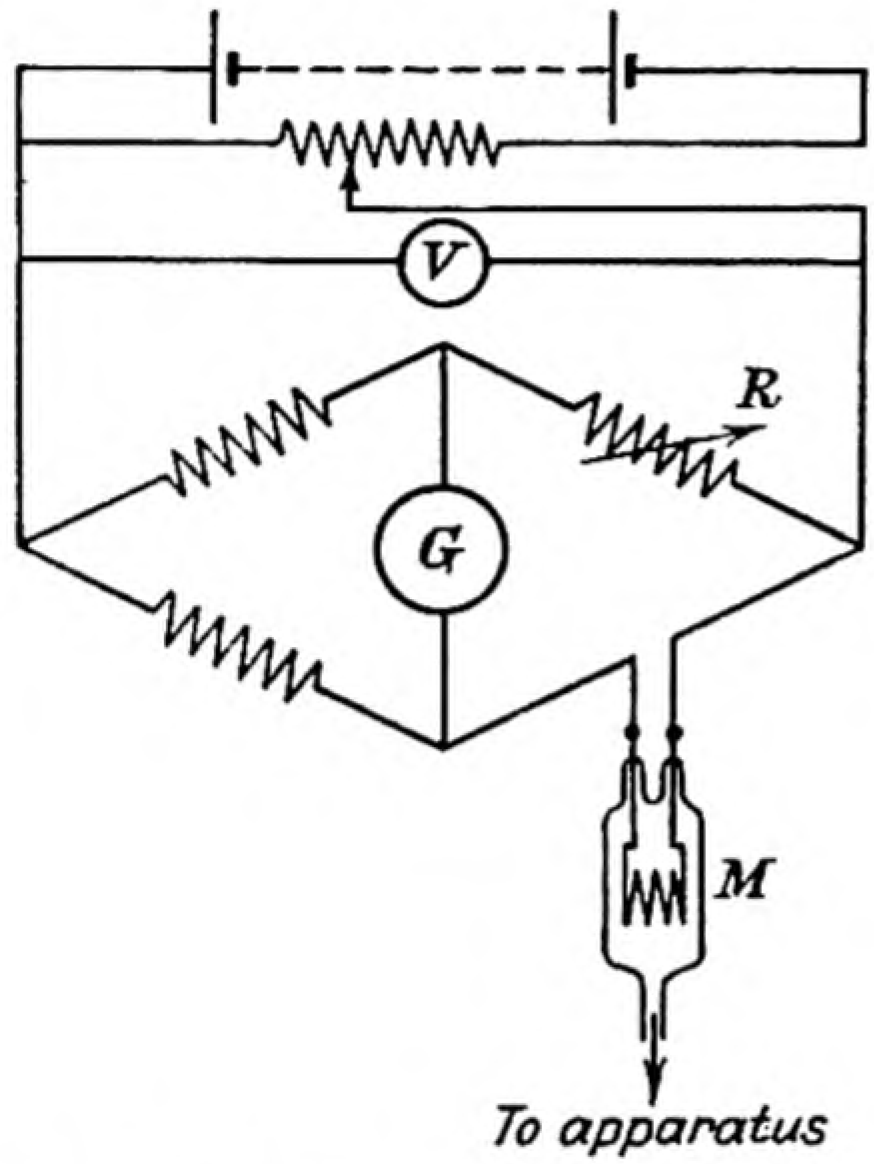

FIG. 14·18.—A Pirani gauge and bridge.

is strictly applicable only if the conditions under which the calibration is carried out are maintained, CAMPBELL recommends that the temperature of the filament should be kept constant.

To carry out this suggestion the gauge, M, is placed in one arm of a Wheatstone bridge network as shown in Fig. 14·18, all other resistances being made of manganin or 'minalpha', since each of these alloys has a temperature coefficient practically zero. The resistance R is adjustable and its value is chosen so that the galvanometer deflexion is zero, the current through M being such that the temperature of its filament is about 100° C. In Campbell's method the voltage, V, across the bridge is adjusted, as indicated, until the balance of the bridge is restored on each occasion that the pressure changes. Let θ be the constant excess temperature assumed by the filament. Then if the heat losses along the leads are small, they will be directly proportional to θ—say $\beta\theta$, where β is a constant. The heat dissipated in the filament per second is αV^2, where α is a constant and V is the voltage across the bridge, while the rate of heat loss by conduction through the gas is $f(p)$, where $f(p)$ is an

unknown function of the pressure. Hence we may write

$$\alpha V^2 = \beta\theta + f(p).$$

Let V_0 be the voltage across the bridge when $p = 0$, and θ is still the excess temperature of the wire in the gauge. Then

$$\alpha V_0^2 = \beta\theta.$$

These two equations give

$$\frac{V^2 - V_0^2}{V_0^2} = \frac{f(p)}{\beta\theta},$$

or

$$\frac{V^2 - V_0^2}{V_0^2} = \xi f(p),$$

if $\xi = (\beta\theta)^{-1}$, a constant.

Measuring the gas pressure independently by means of a McLeod gauge it is found that $f(p) = \gamma p$, where γ is a constant, i.e. $\dfrac{V^2 - V_0^2}{V_0^2}$ is directly proportional to the pressure. It is also found that the constant ξ is almost independent of the length of the filament and

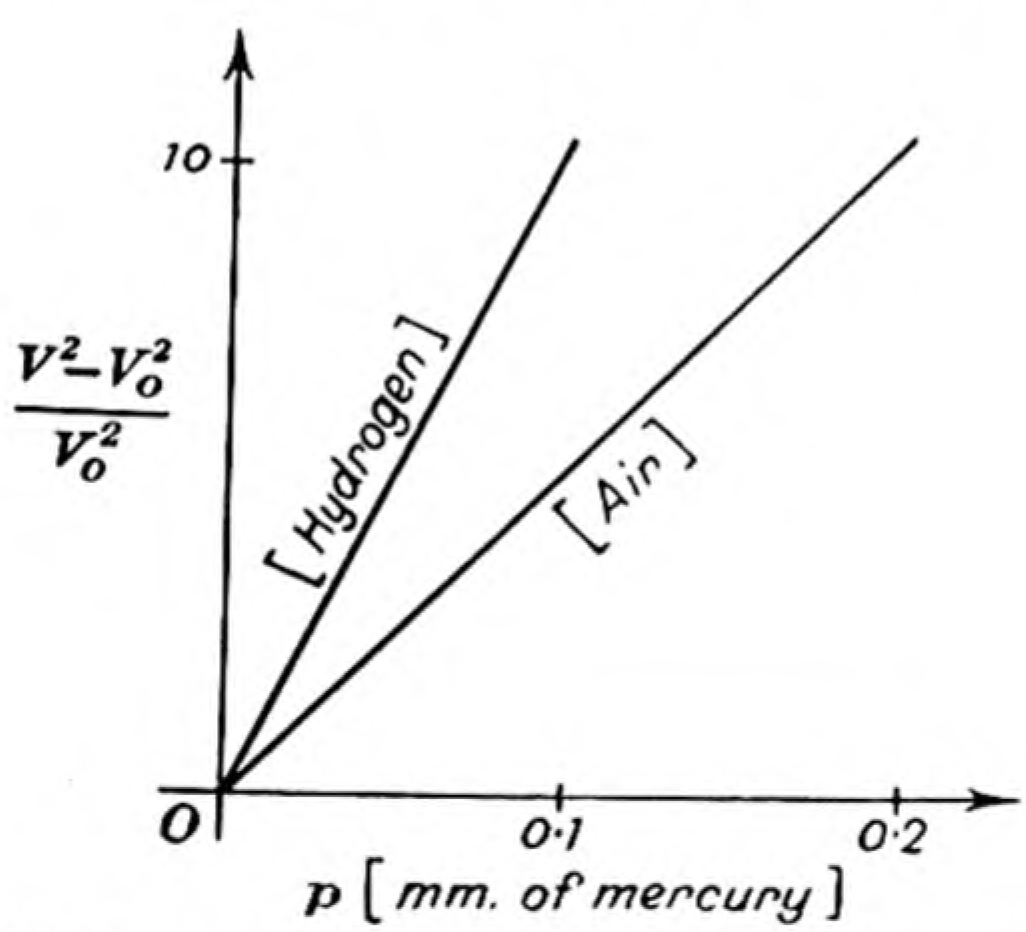

FIG. 14·19.—Calibration curves for use with a Pirani gauge.

whether it is made of tungsten or of platinum; ξ only varies with the nature of the gas.

Fig. 14·19 is a reproduction of two of Campbell's curves in which abscissae are pressures, as measured with a McLeod gauge and, ordinates are the corresponding values of the fraction $(V^2 - V_0^2)/V_0^2$.

Fig. 14·20 shows two forms of Pirani gauge—one of them is demountable.

To simplify the use of a gauge of this type it is usual not to follow the method worked out by Campbell, but to balance the bridge when

the pressure in the gauge is the lowest attainable and then, keeping the voltage across the bridge constant, to observe the out-of-balance

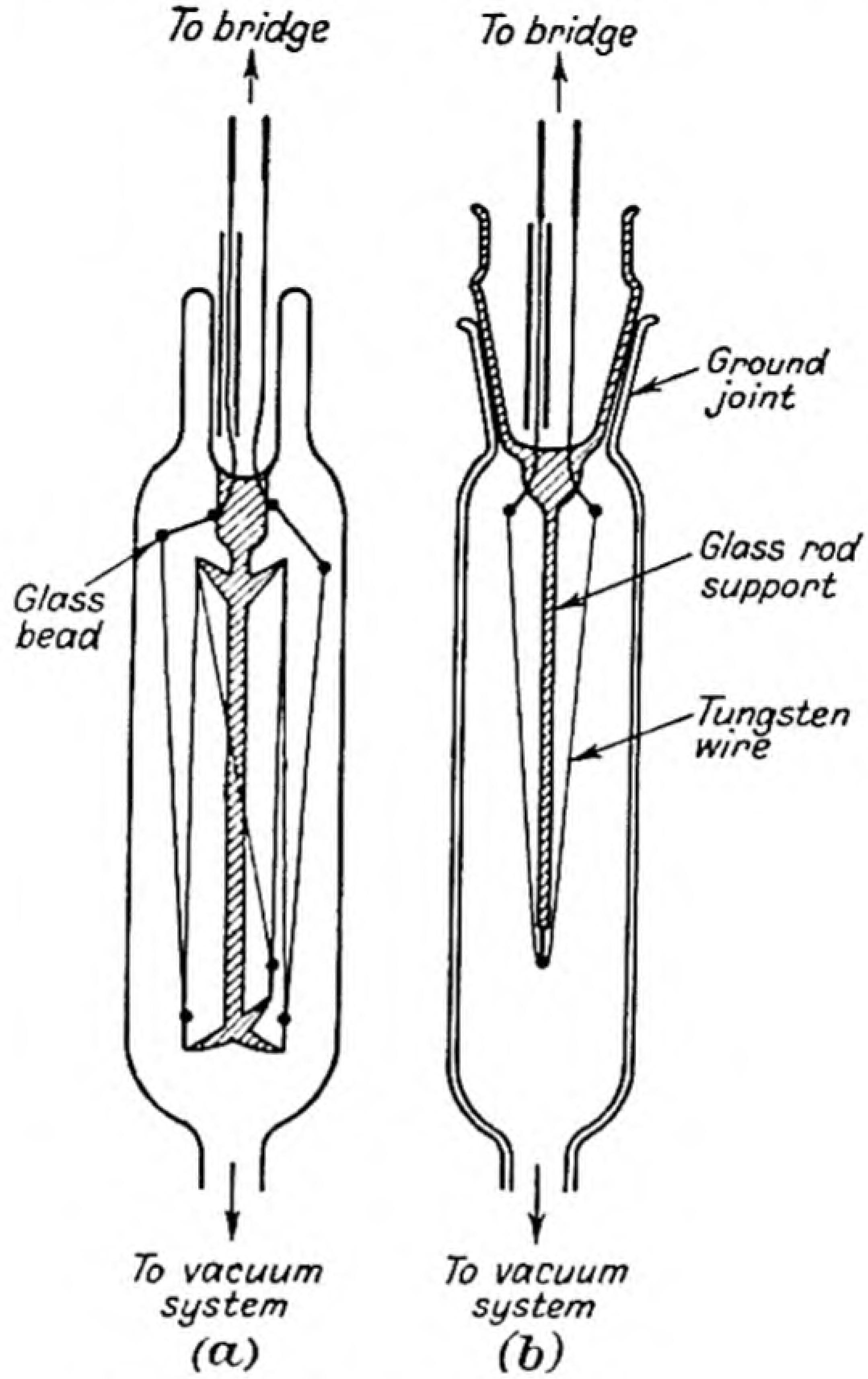

FIG. 14·20.—Pirani gauges.

current through the galvanometer, in terms of scale divisions, when the pressure is not zero. The method of calibration needs no particular comment.

Scott's modified Pirani gauge circuit.—Most gauges used to measure pressures within the range 10^{-3} to 10^{-5} mm. of mercury require some sort of manual adjustment. In 1939 SCOTT† overcame this difficulty by designing the following circuit in which a change in the resistance of the filament in a Pirani gauge is measured by feeding the voltage drop across it on to the grid of a triode valve. In this way the pressure may be determined directly on the calibrated scale of a sensitive galvanometer G, placed in the anode circuit. As the filament resistance varies in accord with changes in the pressure of the gas in the gauge, the potential difference across the grid will also vary when the corresponding

† *Rev. Sci. Inst.*, **10**, 349, 1939.

change in the plate current becomes a measure of the pressure in the gauge. Fig. 14·21(*a*) shows how the circuit is arranged. R_1

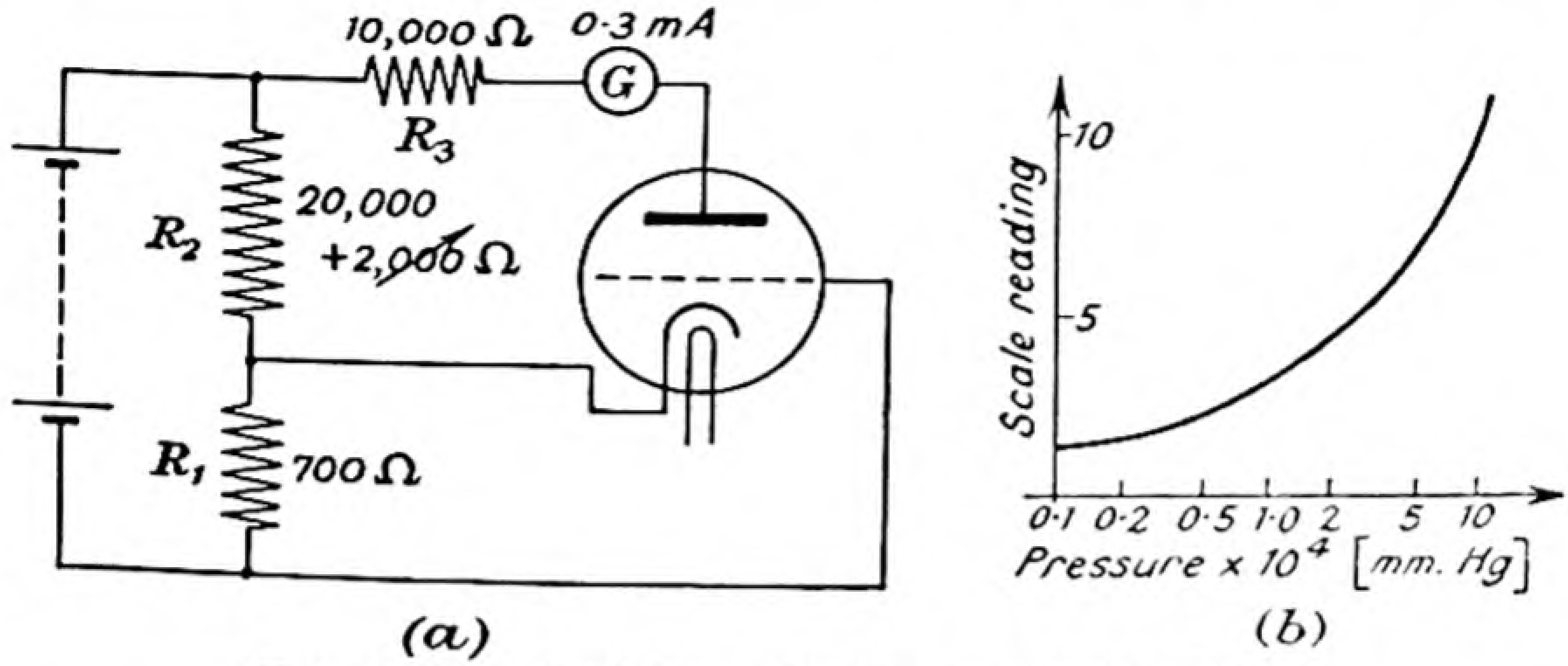

FIG. 14·21.—Scott's modified Pirani gauge circuit.

is the Pirani filament, its resistance being about 700 ohms. R_2 is a resistance of about 20,000 ohms in series with R_1—a 2000 ohms adjustable resistance is included with R_2 and a high-tension battery is placed across this combination of resistances which is arranged in conjunction with a triode valve as shown. R_3 is a suitable anode resistance to give the triode a more nearly linear characteristic for values of the grid voltage near to the cut-off value. R_2 is varied until a certain anode current is obtained for a known pressure.

When in operation the Pirani resistance, R_1, varies so that V_1 and V_2, the potential differences across R_1 and R_2, respectively, will vary in opposite directions, the sum remaining constant. When the pressure in the Pirani gauge decreases the resistance of R_1 increases, in consequence of which the grid volts increase and the plate volts decrease. Both these changes result in a lowering of the plate current. Fig. 14·21(*b*) is a typical calibration curve.

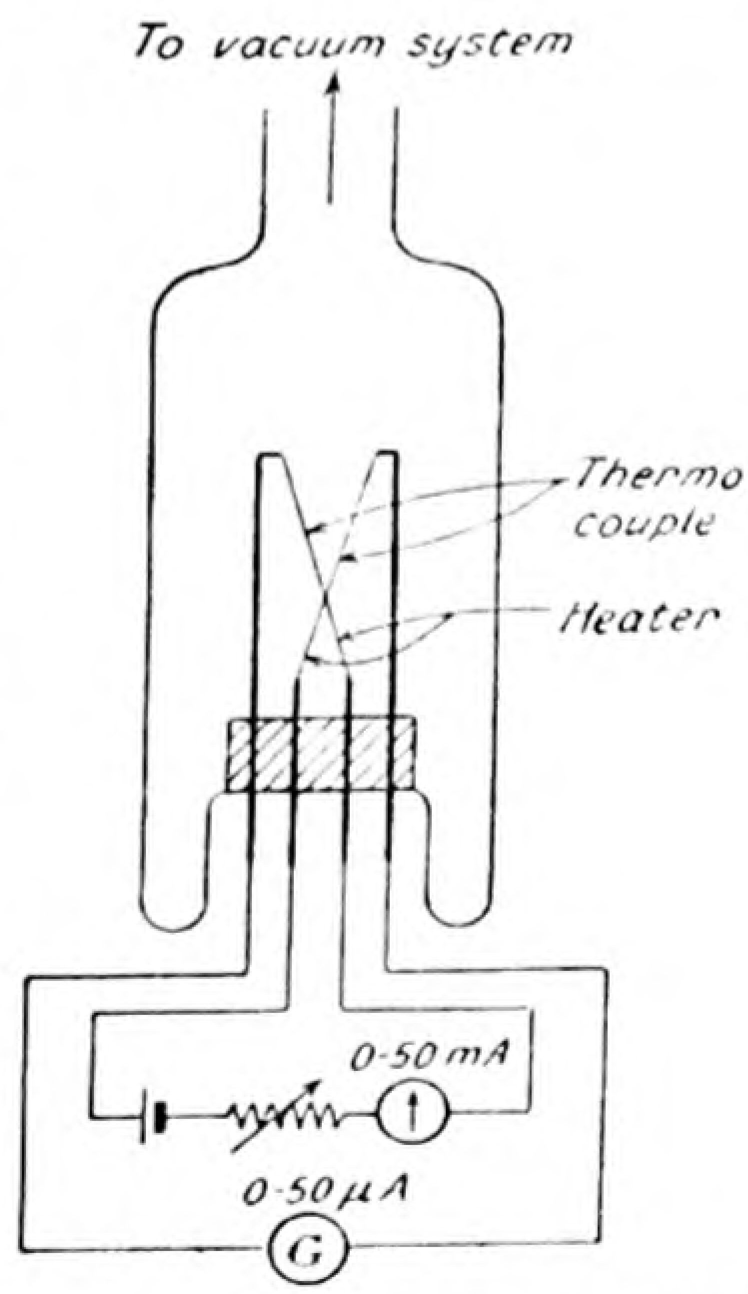

FIG. 14·22.—A thermocouple gauge for measuring low pressures.

Thermocouple gauges.—This is a variant of the Pirani gauge and its structure is indicated in Fig. 14·22. The heated filament consists of a short length of constantan in the

form of a thin flat ribbon and carrying a current of several milliamperes. One junction of an iron-constantan (or chromel-alumel) thermocouple is fixed to the middle of the hot filament. The temperature of the external part of this gauge is kept at 0° C. The temperature of the 'hot' junction of the thermocouple is determined, for a given current through the filament, by the rate at which heat is transmitted through the gas to the walls. This gauge must be calibrated with the aid of a McLeod gauge and for strictly accurate work the gas must be the same in the two instances.

Ionization gauges.—When an electron leaves a cathode at zero potential and passes to an anode at potential V, the energy equation is

$$e(0) + \tfrac{1}{2}m(0^2) = eV + \tfrac{1}{2}mu^2,$$

or

$$\tfrac{1}{2}mu^2 = -eV,$$

where m is the mass of the electron, e its charge, and u the velocity with which it strikes the anode. In setting down this equation it is assumed that the electron moves in a high vacuum so that its motion is not affected by collisions with gas molecules. Since e is essentially negative and V is positive the equation is clearly satisfied. When a gas is present and the electrons are accelerated by a potential difference exceeding a certain critical value, V_i, the so-called ionization potential of the gas in which the electrons move, collisions between the electrons and gas molecules will produce positive ions. These positive ions are produced whenever a high-velocity electron collides with a gas atom and removes an electron from its outer shell. If the positive ions thus formed are collected on an auxiliary or third electrode it is found that the positive ion current for a fixed value of the accelerating voltage in excess of V_i varies with the pressure. At low pressures it is to be expected that the current will vary linearly with the pressure, since it is unlikely that an electron will make more than one collision in passing from the cathode to the auxiliary electrode. When the grid of a triode is used as the auxiliary electrode it must be recalled that when the valve is free from gas, the grid current is exceedingly small. When gas is present in the valve and its molecules become ionized in the manner indicated, then the positive ions will move towards the grid when this is kept at a negative potential with respect to the filament.

Any normal triode valve may be used as an ionization gauge, but in order to diminish electrical leaks between electrodes it is better to use specially constructed triodes such as that shown in Fig. 14·23. The filament F is supported in the manner indicated and is surrounded by a grid coaxial with the axis of the filament. A silver deposit, P, on the wall of the valve acts as the 'plate' and electrical connexion to it is made by means of a platinum wire sealed into the glass envelope as shown. The loose glass collars,

C_1 and C_2, mounted on the glass rod supporting the filament are to prevent the deposition of metal films on the support which, if present, cause inter-electrode leaks.

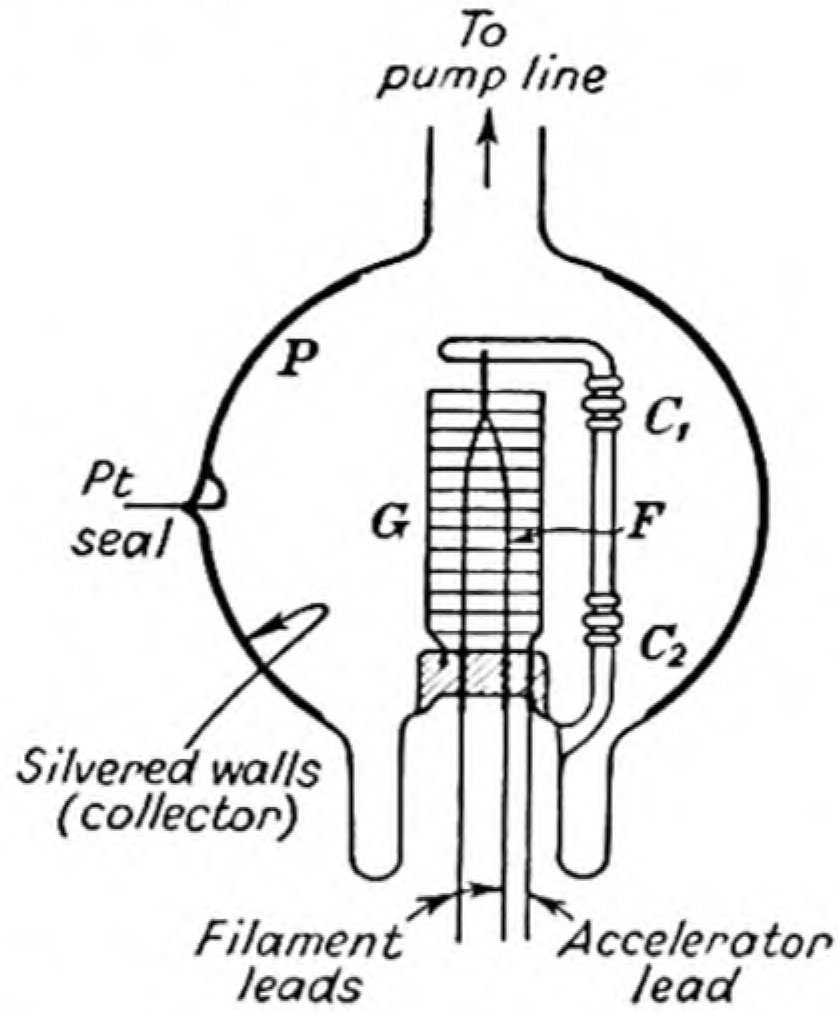

FIG. 14·23.—An ionization gauge for measuring low pressures.

The essentials of one electrical circuit required to operate this gauge are shown in Fig. 14·24(*a*). Electrons pass through the

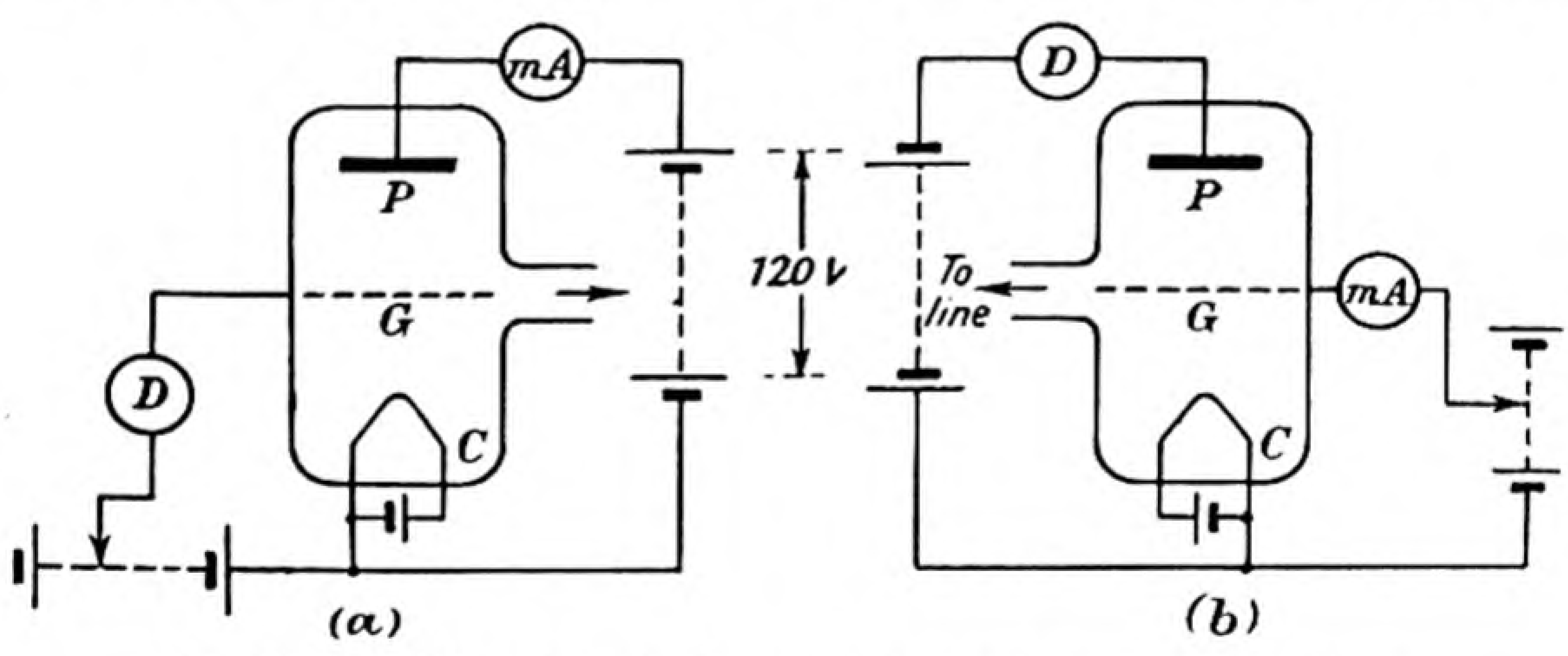

FIG. 14·24.—Two circuits for operating an ionization gauge. (*a*) internal collector, (*b*) external collector (most sensitive).

meshes of the grid and positive ions formed between the grid and anode are collected by the grid—it is necessary to measure the grid current by means of a sensitive galvanometer, D.

An alternative scheme is shown in Fig. 14·24(*b*). In this the grid is maintained at a positive potential with respect to the filament and the silver lining is made negative. The electrons from the hot filament are attracted towards the grid but some of them, in virtue

of their momentum, are carried forward through the meshes of the grid to ionize the gas between G and P, i.e. when the accelerating voltage exceeds V_i. The positive ions are collected at the plate P;

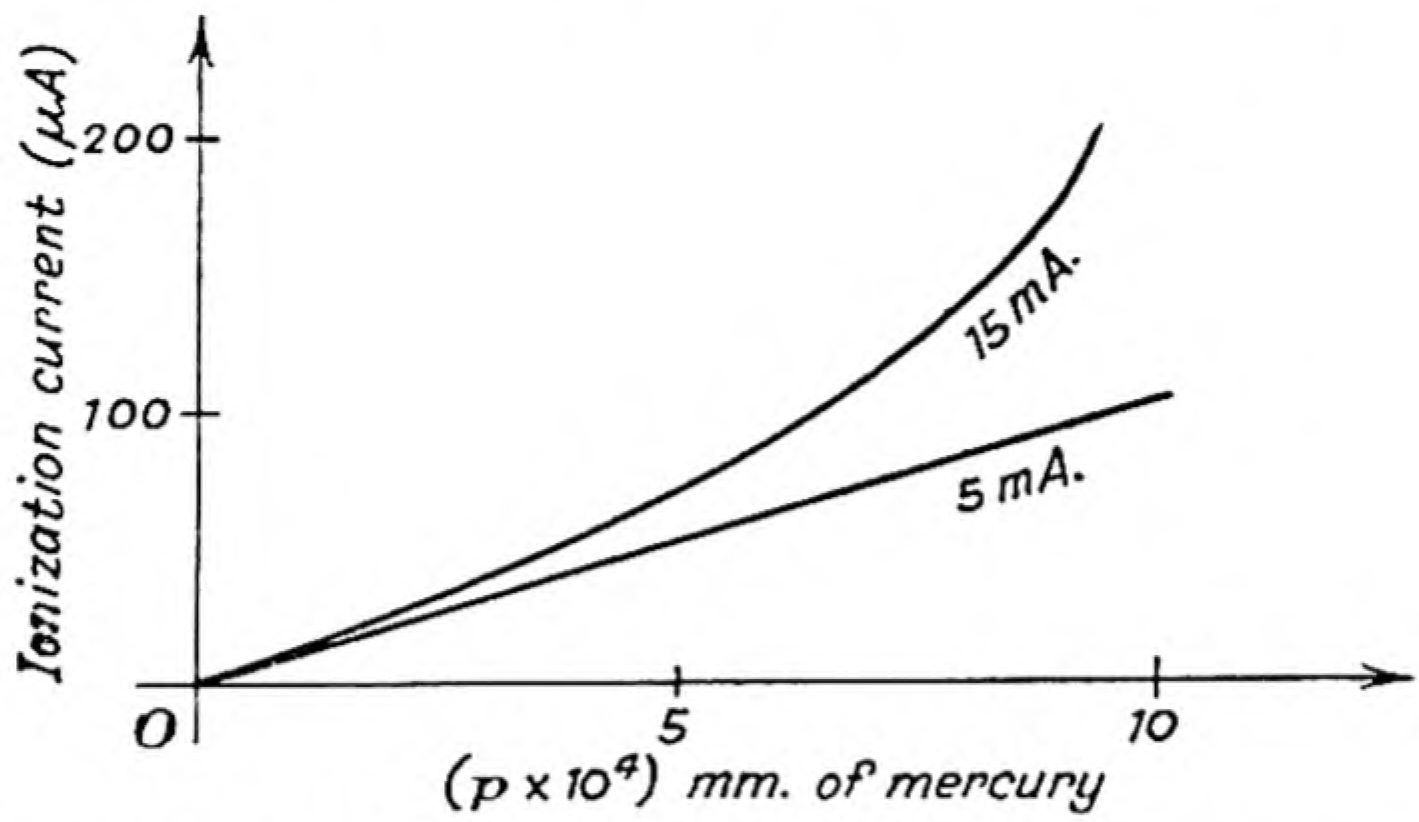

FIG. 14·25.—Calibration curves for use with an ionization gauge.

the electrons which have entered the region between G and P are repelled by the negative charge on P. The positive ion current is measured by a sensitive detector D.

Such gauges are not absolute and are usually calibrated for a series of pressures determined by means of a McLeod gauge. Unlike a

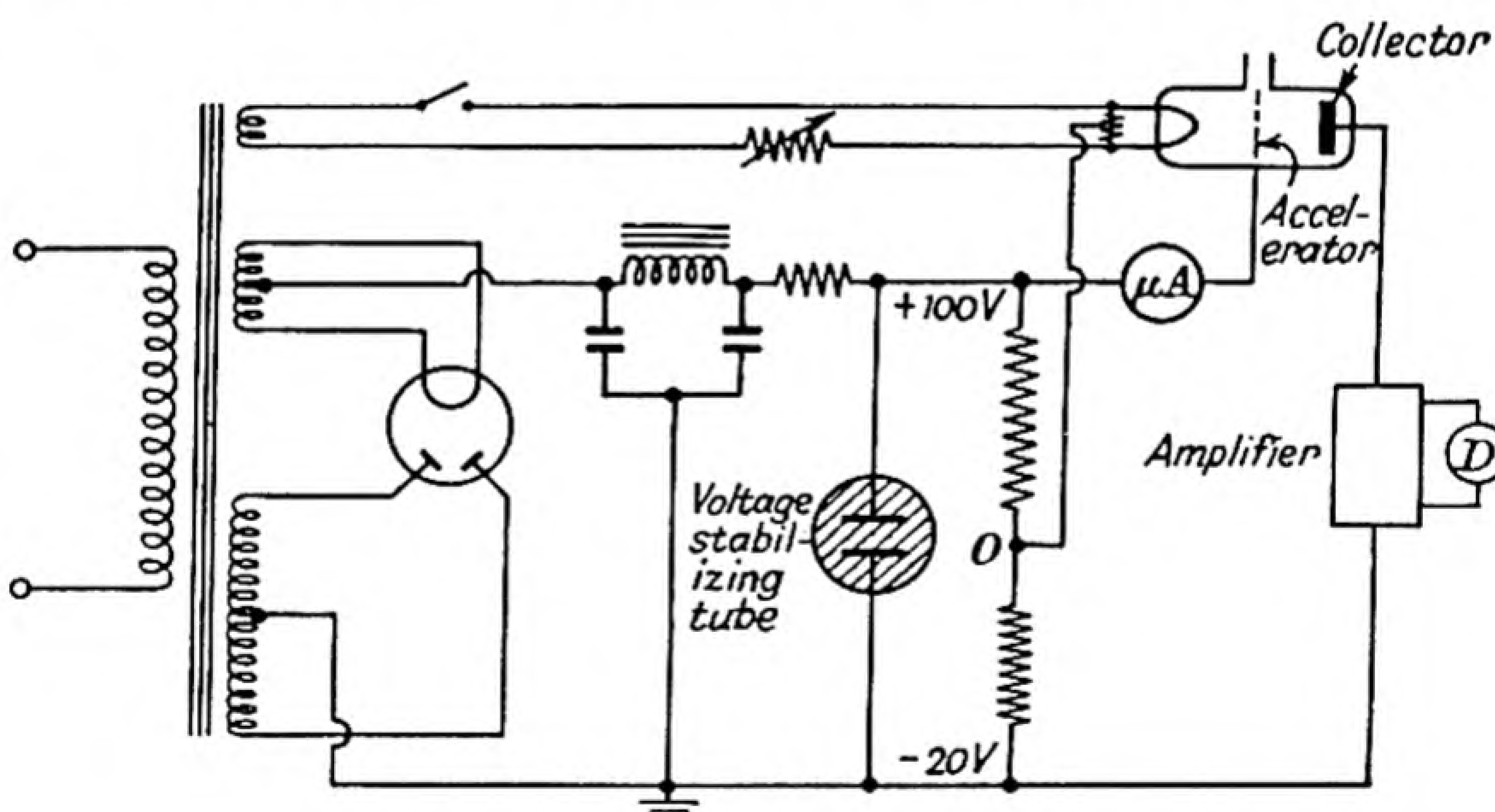

FIG. 14·26.—An ionization gauge and its auxiliary circuit.

McLeod gauge an ionization gauge measures the pressures of vapours and of gases so that before the calibration is carried out it is essential thoroughly to out-gas the system and to remove all condensible vapours.

Below 10^{-4} mm. of mercury pressure, the relation between pressure

and ionization currents is linear; at higher pressures the relation is only linear if the electron emission from the filament is relatively small and under these conditions the sensitivity of the gauge is reduced. Fig. 14·25 shows a typical calibration curve for an ionization gauge for two different values of the current due to electron emission from the filament; these values are shown alongside the curves.

Fig. 14·26 gives a convenient electrical circuit for use with an ionization gauge.

The Alpharay ionization gauge.—This is a novel type of cold-cathode ionization gauge designed by DOWNING and MELLEN in 1946, and is called the 'Alpharay' to indicate that alpha particles are the ionizing agent. It is a continuous reading gauge to cover, in three stages, the range of pressure from 10^{-3} to 1000 mm. of mercury although the relation between current and pressure is only linear up to about 10 mm. of mercury pressure. Fundamentally it works on the same principle as the hot-filament ionization gauge, described earlier in this chapter, but such a gauge can only be operated at pressures below 10^{-3} mm. of mercury. At higher pressures the life of a hot-filament ionization gauge is short on account of the bombardment of the filament by ions or its chemical reaction with the surrounding gas.

In the Alpharay ionization gauge the radioactive source is a small plaque; it is 1 cm.^{2} in area and has suitable ears so that it may be held securely in position. One side of this plaque is the active area and consists of a gold-radium alloy containing 0·4 mgm. of radium ($_{88}Ra^{226}$). It is in equilibrium with its decay products, radon ($_{86}Rn^{222}$), radium A ($_{84}Po^{218}$) and radium B ($_{82}Pb^{214}$). Since the first product is a gas and it is most desirable that this should not escape, the active layer is coated electrolytically with a layer of nickel. In this way a highly efficient α-emitter, but with low emanating power, is obtained; the nickel film also prevents contamination by mercury vapour. The losses from the plaque are so small that the instrument only needs to be calibrated once every four or five years. Precautions, however, must be taken in handling the instrument on account of harmful physiological effects which otherwise arise.

A section of this gauge and the electrical circuit required to operate it are shown in Fig. 14·27. A is the source fixed in the walls of an ionization chamber C and maintained at a potential of 40 V. above that of the inner electrode or grid G. This consists of four wires 'spreadeagled' so that the ions produced do not have to travel relatively long distances before being captured. If the distance travelled is too long excessive re-combination of ions at the upper limit of the lowest pressure range occurs and this causes the linear

relation between ionization current and pressure, which is a characteristic of this gauge at low pressures, to be destroyed.

The base of C is drilled so that gas may enter the ionization chamber freely. The small ionization current is amplified and read on a microammeter M. The gauge is calibrated by comparison with a McLeod gauge.

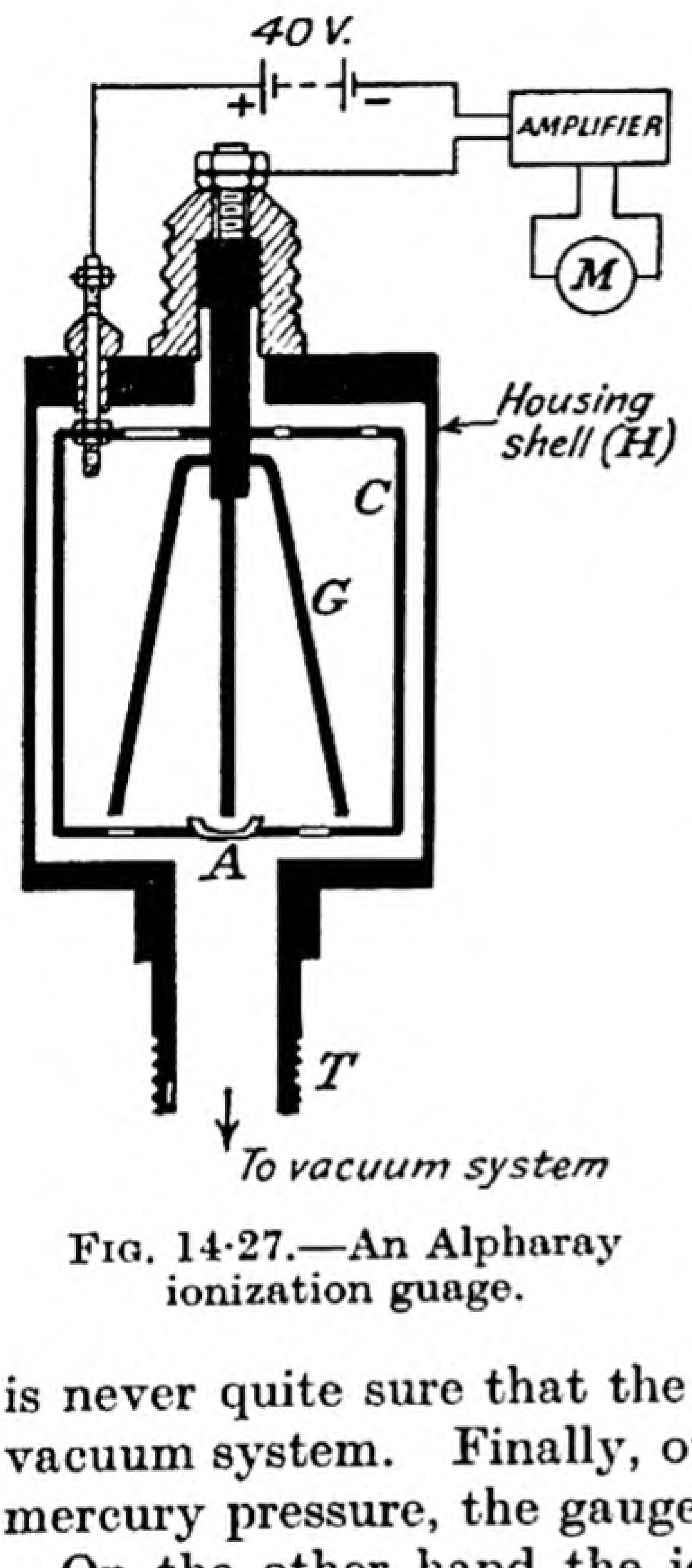

FIG. 14·27.—An Alpharay ionization guage.

Some general remarks on low-pressure gauges.—Since the heat-conducting gauge and the ionization gauge must each be calibrated with reference to a McLeod gauge it will be worth while to state why these two gauges are sometimes preferable to the McLeod gauge and to indicate some of their limitations.

The McLeod type of gauge is insensitive to condensible vapours, does not give continuous readings and the mercury vapour from it must be kept from entering the rest of the vacuum system by using a liquid-air trap to freeze out the mercury vapour. The presence of such a trap reduces the rate of pumping and one is never quite sure that the gauge is indicating the pressure in the vacuum system. Finally, over the range from 10^{-4} to 10^{-5} mm. of mercury pressure, the gauge is not quite trustworthy.

On the other hand the ionization gauge is quick in action but its use requires considerable auxiliary electrical equipment. Its sensitivity varies with the nature of the gas and if an excess of air suddenly enters the system while the filament in incandescent the latter burns out and the gauge is useless. Moreover, certain gases 'poison' the filament, i.e. they reduce considerably the electron emission from it.

The Pirani or hot-wire gauge has two main disadvantages. In the first place it is difficult to maintain a stable 'zero' so that it cannot be used to measure, with certainty, pressures below 10^{-4} mm. of mercury and secondly, at pressures below 10^{-3} mm. the loss of heat takes place mainly by radiation and not by conduction.

The detection of leaks in a vacuum system.—Work with a vacuum system, especially in its initial stages, is almost invariably delayed, sometimes for weeks, by the presence of minute leaks in the system. The more complex the design of the system is the more are

leaks likely to be present and the more difficult are they to locate.

For the detection of small leaks in apparatus made from glass the time-honoured method of using the spark discharge from a small induction coil has much to recommend it. One terminal of the secondary is connected to a piece of metal in direct communication with the system, while to the second terminal there is fixed a long length of fine copper wire. The other end of the copper wire is attached by means of an elastic band to a long glass rod, about 5 cm. of the wire projecting beyond the end of the rod. This rod serves as an insulator and with its aid the high-potential wire may be made to pass over any part of the glass where a leak is suspected. Provided the pressure within the system is sufficiently low the presence of a pin-hole leak will be revealed by the passage of a spark through the hole and a general increase in the intensity of the discharge within the system. If the glass is very thin and the strength of the electric field high, the glass may fracture by the strain produced in it; the obvious remedy is not to use too powerful an induction coil.

The above method is useless for detecting the presence of leaks in metal parts of the system. Should these be suspected then it is convenient to have some form of discharge tube attached to the system. The suspected parts are then painted with alcohol or acetone. When a leak is present vapour of the liquid used will enter the system and the colour of the discharge change from pink to pale blue, this being the colour associated with the presence of carbon compounds in the region where the discharge takes place. The use of such vapours is not always desirable since they may find their way into the oil of a backing-pump and return to the system when the oil gets hot while the pump is in action. It has been suggested that a stream of carbon dioxide should be directed on to the part of the apparatus under test; when this gas enters the system the colour of the discharge changes and the gas is readily removed.

If the leak is very small and the pressure within the system so low that a discharge cannot be maintained, then an ionization gauge attached to the system may be used to indicate the entry of vapours through a pin-hole into the vacuum line. To avoid trouble at a later date all metal parts should be tested for leaks before being assembled. To do this they are filled with compressed air and placed under water. A stream of minute bubbles rising from a point on the metal indicates a small leak at that spot.

EXAMPLES XIV

14·01. Explain, with the aid of diagrams, the action of a Geissler vacuum pump. If a flask of volume 2 litres containing air at a pressure of 3 cm. of mercury is to be evacuated by means of a pump whose bulb

has a volume of 500 ml., estimate the pressure of the air in the flask after the pump has been operated (*a*) 12 times, (*b*) 24 times. [Assume no change in temperature.] [(*a*) 0·069, (*b*) 0·0047 cm. of mercury]

14·02. Explain briefly the details of the process of producing a very high vacuum in a vessel of about 1 litre capacity starting at atmospheric pressure and explain the action of a diffusion pump as fully as you can. Illustrate both parts of your answer by diagrams. (S)

14·03. Describe and explain a method suitable for measuring a gas pressure in the region of 10^{-4} mm. of mercury. Discuss briefly (*a*) the sensitivity and (*b*) the calibration of the apparatus described. (S)

14·04. Give an account of the technique employed in the production and measurement of very high pressures. (S)

14·05. Give an account of the use of the McLeod gauge for measuring low pressures, stating its importance and limitation. Describe some other form of instrument for the measurement of very low gas pressures, giving the theory of its action. (S)

14·06. Stokes' law states that the viscous force on a sphere of radius a moving with terminal velocity u through a fluid is $6\pi a\eta u$, where η is the viscosity of the fluid. Show that this law is dimensionally correct and describe how you would test its validity experimentally.

A vessel containing oil of very low vapour pressure is in communication with a vacuum pump which reduces the pressure of the residual gases above the oil to a negligible value. A spherical bubble of air is timed as it rises through the oil and it takes 10 seconds to reach half-way to the surface of the oil. What is the total time of its journey to the surface of the oil? [Assume that the mass of air in the bubble remains constant and neglect surface tension.] [31·3 sec.]

APPENDIX

A

Owen's bar pendulum; an accurate method of using it.—Since for a uniform bar pendulum of length a and breadth b the radius of gyration κ about an axis through its centre of gravity and normal to its plane is given by $12\kappa^2 = (a^2 + b^2)$, and the period is a minimum when $r = \kappa$, it follows that the most accurate determination of gravity by means of an Owen pendulum is made from observations at the setting $r = \kappa$, for in this instance a small first-order error in the measurement of r produces only a second-order error in T. At this setting the accuracy with which gravity can be determined depends essentially on two measurements only, viz. the total length of the bar and the period. It may be shown that for a bar of length 100 cm. a departure of 0·1 cm. from the ideal setting $r = \kappa$, only affects the value of T by 3 parts in a million. The length of the pendulum is easily determined to within 0·02 cm. Since, under the conditions contemplated,

$$T = 2\pi\sqrt{\frac{2\kappa}{g}},$$

it is necessary to appreciate that if the error in the calculated value of κ does not exceed 1 part in 10,000 then the same degree of accuracy in determining T must be attained, i.e. the total duration of the timing must be about ten minutes. The well-known procedure† and a correction for any error in the watch used, which should indicate tenths of a second, is all that is required.

B

The use of metal bellows for controlling the pressure in an enclosed system.—The basis of any one-piece metal bellows is a thin seamless brass tube having a wall thickness of about 0·01 cm.; the brass contains about 80 per cent copper. By means of a longitudinal push the tube is compressed in a single operation to form a bellows; a section of a short length of such a bellows is shown in Fig. A 1(a). This particular bellows is closed at one end only and all convolutions are active.

† cf. I.P. p. 45

To use these bellows to control the pressure of the air required to blow bubbles or control the position of a liquid index, as in Ferguson's apparatus for measuring surface tension, cf. p. 497, two such bellows† are mounted as shown in Fig. A 1(*b*). It needs no description but a discussion of the relative sizes of the bellows may not be inappropriate.

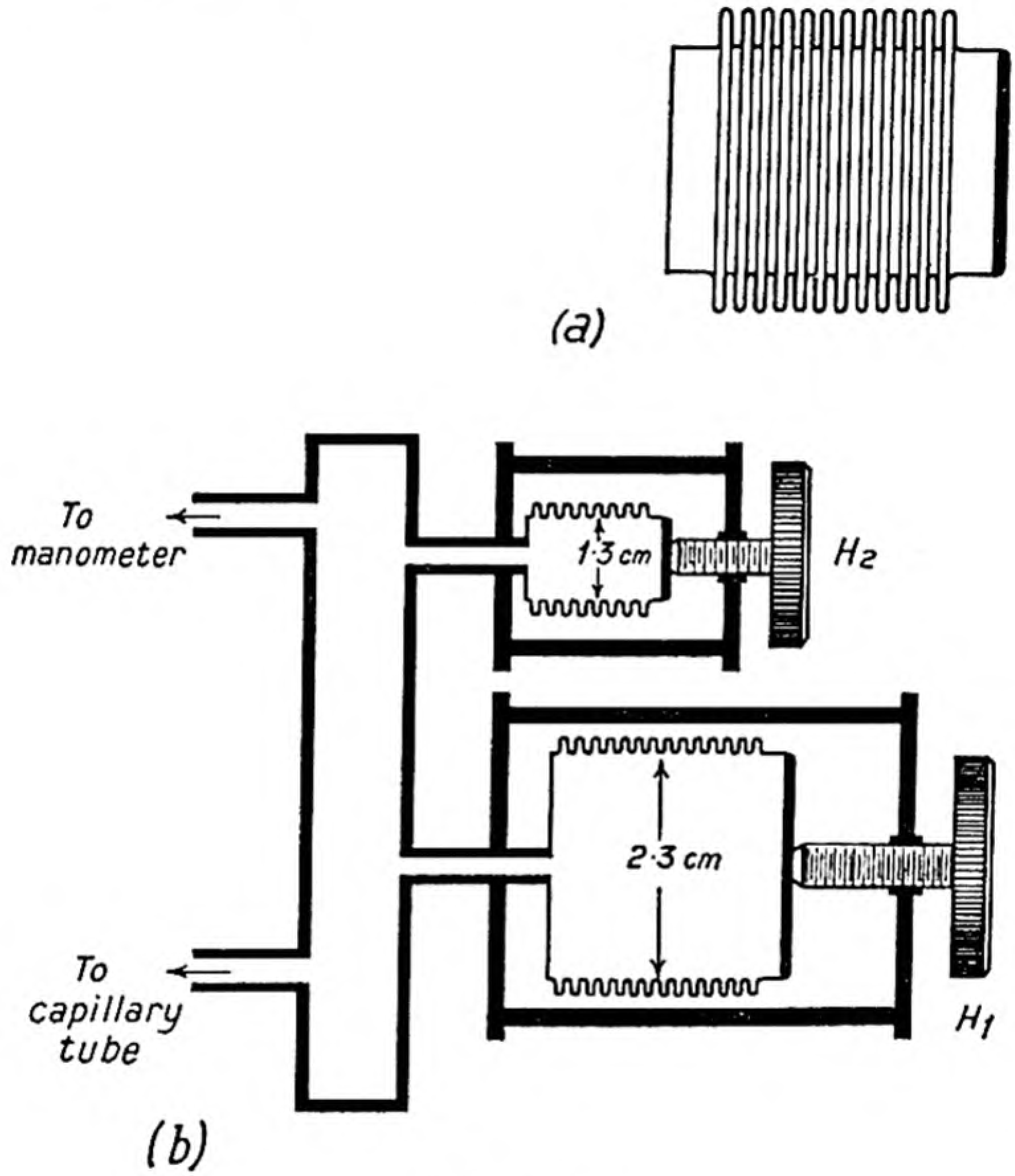

FIG. A 1.

If V is the total volume of the apparatus and p the initial pressure of the air enclosed, pV is constant, so that

$$\frac{\delta p}{p}+\frac{\delta V}{V}=0.$$

For use with Ferguson's apparatus, V may be taken as 100 cm.^{3} and p as 1300 cm. of an oil of density 0·8 gm.cm^{-3}. The large bellows have a diameter of 2·3 cm. and a change in length of 1·2 cm.

† This apparatus was designed by A. S. Edmondson, Esq.

for 10 turns of the screw-head H. Thus, if $\delta V \risingdotseq \pi(1{\cdot}2)^2 \times 1{\cdot}2 = 5$ cm.^{3}, $|\delta p| = 65$ cm. of oil. For the small bellows, diameter 1·3 cm., a change in length of 0·4 cm. is effected by 6 turns of the screw-head H_2, so that if $\delta V = 0{\cdot}5$ cm.^{3}, $|\delta p| = 6$ cm. of oil.

In a surface tension experiment $\delta p \risingdotseq 5$ cm. of oil and this corresponds to nearly one turn of the large screw. For adjustment of the excess pressure to within 0·05 cm. of oil by means of the small bellows, $|\delta p| = 1$ cm. of oil for one turn of H_2, i.e. this change in pressure may be caused by a rotation of H_2 through about 20°.

All other ranges and sensitivities may be obtained by a suitable choice of internal volumes, size of bellows, pitch of screw, etc.

C

Compressibility and its value at high pressures.—The bulk modulus of a substance is defined, cf. p. 302, by the equation

$$\beta = -V\frac{\partial p}{\partial V}.$$

Its compressibility is therefore given by

$$\kappa = -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right).$$

If ρ is the density of the substance and v the volume of unit mass, then $\rho = \dfrac{1}{v}$ and we have

$$\kappa = -\frac{1}{v}\frac{\partial v}{\partial p} = -\rho\frac{\partial}{\partial p}\left(\frac{1}{\rho}\right) = \frac{1}{\rho}\frac{\partial \rho}{\partial p}$$

Such expressions are exact provided the differential coefficients are evaluated under the conditions prevailing during an experiment. These are generally such that the temperature remains constant. In work at high pressures it is found convenient to take as the compressibility the fraction

$$-\frac{1}{V_0}\cdot\frac{\Delta V}{\Delta P},$$

where V_0 is the volume of the substance under atmospheric pressure and $-\Delta V$ is the decrease in volume when the pressure is increased by ΔP; ΔP may be several thousand atmospheres.

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